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Some Results on Common Fixed Point of Multivalued Generalized φ -Weak Contractive Mappings in Integral type Inequality

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Abstract: The aim of this study is to establish unique Common Fixed Point of Multivalued Generalized φ-Weak Contractive Mappings in Integral type inequality. This result improves the result of Nadler and Daffer-Kaneko's and references there in.

Key words: Common fixed point, contractive mapping, fixed point

INTRODUCTION

Fixed point and coincidence results are presented for multivalued generalized φ -weak contractive mappings on complete metric spaces, where ϕ: $[0, +\infty) \to [0, +\infty)$ is a lower semicontinous function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Our results extend previous results by Zhang and Song (2009), as well as by Rhoades (2001), Nadler (1969) and Daffer and Kaneko (1995). Branciari (2002) gave a fixed-point result for a single mapping satisfying an analogue of Banach's contraction principle, which is stated as follows;

Theorem 1: Let (x, d) be a complete metric space. $c \in [0, 1]$, T: X→X a mapping such that , for each $x, y \in X$.

$$\int_0^{d(T_x,T_y)} f(t)dt \le c\int_0^{d(x,y)} f(t)dt$$

Where $f: R^+ \to R^+$ is a lebesgue-integrable mapping which is summable, non-negative and such that for each $s > 0$. $\int_0^s f(t)dt > 0$ then T has a unique fixed point $z \in X$ such that, for each $z \in X$, $\lim_n T^n x = z$.

This result was further generalized by Abbas and Rhoades (2007), Alionche (2006), Gairola and Rawat (2008), Kumar *et al.* (2007) and Rhodes (2003). However, Suzuki (2007) has shown that Meir-Keeler contraction of Integral type is still Meir-Keeler contraction and Alionche and Fisher (2007) extends function for three complete and compact metric spaces. In this study, we prove the following fixed-point theorem for complete metric spaces.

Now, Behzad and Moradi (2010), extends the common fixed point of Multivalued Generalized φ-Weak Contractive Mappings.

If (X, d) be a metric space. We denote the family of all nonempties closed and bounded subsets of X by $CB(X)$. A mapping $T: X \to X$ is said to be ϕ -weak contractive if there exists a map φ: $[0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$, such that

$$d(T_x, S_y) \le M(x,y) - \varphi(M(x,y)) \tag{1}$$

For all $x, y \in X$.

Also two mappings $T, S: X \to X$ are called generalized ϕ-weak contractions if there exists a map ϕ: $[0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$, such that:

$$d(T_x, S_y) \le M(x,y) - \varphi(M(x,y)) \tag{2}$$

For all $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y), d(x,T_x), d(y,T_y), \frac{1}{2}\left[d(x,S_y) + d(y,T_x)\right]\right\} \tag{3}$$

A mapping $T: X \to CB(X)$ is said to be a weak contraction if there exists $0 \le \alpha < 1$ such that:

$$H(T_x, T_y) \le \alpha N(x,y) \tag{4}$$

for all $x, y \in X$, where H denotes the Hausdorff metric on $CB(X)$ induced by d, that is:

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \tag{5}$$

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for all $A, B \in CB(X)$, and where,

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,T_y), \tfrac{1}{2}\left[d(x,T_y) + d(y,Tx)\right]\right\}$$

A mapping $T : X \to CB(X)$ is said to be -weak contractive if there exists a map, $\varphi: [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that:

$$d(T_x, T_y) \le d(x,y) - \varphi(d(x,y)) \tag{6}$$

for all $x, y \in X$.

The concepts of weak and φ -weak contractive mappings were defined by Daffer and Kaneko (1995). Many authors have studied fixed points for multivalued mappings. Interested reader may consult (2010).

MATERIALS AND METHODS

In this study, (X, d) denotes a complete metric space and H denotes the Hausdorff metric on $CB(X)$.

Definition 1: Two mappings $T, S : X \to CB(X)$ are called generalized weak contractions if there exists $0 \le \alpha < 1$ such that:

$$\int_0^{H(T_x,T_y)} \varphi(t)dt \le \alpha \int_0^{M(x,y)} \varphi(t)dt \text{ all } x, y \in X. \tag{7}$$

Definition 2: Two mappings $T, S : X \to CB(X)$ are called generalized ϕ-weak contractive if there exists a map ϕ: $[0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$, such that:

$$\int_0^{H(T_x,S_y)} \varphi(t)dt \le \int_0^{M(x,y)} \varphi(t)dt - \varphi\int_0^{M(x,y)} \varphi(t)dt$$

for all $x, y \in X$. (8)

In the proof of our main results, we will use the following well-known lemma, and refer to Nadler (1969) or Assad and Kirk (1972) for its proof.

RESULTS AND DISCUSSION

Theorem 2: Let (X, d) be a complete metric space. Suppose $T : X \to CB(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,

$$\int_0^{H(T_x,T_y)} \varphi(t)dt \le \alpha \int_0^{d(x,y)} \varphi(t)dt \tag{9}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$ (i.e., x is a fixed point of T).

Daffer and Kaneko (1995) proved the existence of a fixed point for a multivalued weak contraction mapping of a complete metric space X into $CB(X)$.

Theorem 3: Let (X, d) be a complete metric space and $T : X \to CB(X)$ be such that:

$$\int_0^{H(T_x,T_y)} \varphi(t)dt \le \alpha \int_0^{N(x,y)} \varphi(t)dt \tag{10}$$

for some $0 \le \alpha < 1$ and for all $x, y \in X$ (i.e., weak contraction). If $x \to d(x, Tx)$ is lower semicontinous , then there exits $x_0 \in X$ such that $x_0 \in Tx_0$.

Now extend Nadler and Daffer-Kaneko's theorems (2010) to multivalued generalized weak contraction mappings in (Definition 1). Rhoades (2001).Theorem 1 proved the following fixed point theorem for ϕ-weak contractive single valued mappings, giving another generalization of the Banach Contraction Principle.

Theorem 4: Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a mapping such that

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \int_0^{d(x,y)} \varphi(t)dt - \varphi\int_0^{d(x,y)} \varphi(t)dt \tag{11}$$

for every $x, y \in X$ (i.e., weak contraction). where ϕ: $[0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

By choosing $\psi(t) = t - \phi(t)$ ϕ-weak contractions become mappings of Boyd and Wong (1969) type and by defining $k(t) = (t - \phi(t))/t$, for $t > 0$ and $k(0) = 0$, then ϕ-weak contractions become mappings of Reich type (1974). Recently Zhang and Song (2009) proved the following theorem on the existence of a common fixed point for two single valued generalized ϕ-weak contraction mappings.

Theorem 5: Let (X, d) be a complete metric space and T: $X \to CB(X)$ be a two mapping such that for all $x, y \in X$.

$$\int_0^{d(T_x,S_y)} \varphi(t)dt \le \int_0^{M(x,y)} \varphi(t)dt - \varphi\int_0^{M(x,y)} \varphi(t)dt \tag{12}$$

(i.e., generalized ϕ-weak contractions), where $\phi: [0, +\infty) \to [0, +\infty)$ is an lower semi continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then there exists a unique point $x \in X$ such that $x = Tx = Sx$.

we extend Theorem 4 by assuming ϕ to be only lower semi continuous, and extend Theorem 5 to multivalued mappings.

Lemma: If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that:

$$d(a,b) \leq \alpha H(A,B) + \varepsilon \tag{13}$$

Theorem 6: Let (X, d) be a complete metric space and let $T : X \to CB(X)$ and $S : X \to CB(X)$ be two mappings such that for all $x, y \in X$:

$$\int_0^{H((Tx),Sy)} \varphi(t)dt \leq \int_0^{M(x,y)} \varphi(t)dt - \varphi \int_0^{M(x,y)} \varphi(t)dt \tag{14}$$

(i.e., generalized φ-weak contractive), where $\varphi: [0, +\infty) \to [0, +\infty)$ is an lower semi continuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then there exists a unique point $x \in X$ such that $x = Tx = Sx$.

Proof: Unicity of the common fixed point follows from Eq. (14) and (16). Obviously $M(x, y) = 0$ if and only if $x = y$ is a common fixed point of T and S. Let $x_0 \in X$ and $x_1 \in Sx_0$. Let $x_2 = Sx_1$. By Lemma, there exists $x_3 \in Sx_2$ such that:

$$\int_0^{d(x_3,x_2)} \varphi(t)dt \leq \int_0^{H(Sx_2,(Tx_1))} \varphi(t)dt + \frac{1}{2}\varphi \int_0^{M(x_2,x_1)} \varphi(t)dt \tag{15}$$

We let $x_4 = Tx_3$ inductively and we let $x_{2n} = T_{x_{2n-1}}$ and by Lemma we choose $x_{2n+1} \in Sx_{2n}$ such that:

$$\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt \leq \int_0^{H(Sx_{2n},(Tx_{2n-1}))} \varphi(t)dt + \frac{1}{2}\varphi \int_0^{M(x_{2n},x_{2n-1})} \varphi(t)dt \tag{16}$$

We break the argument into four steps.

Step 1: $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$

Proof: Using (14) and (16),

$$\begin{aligned} &\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt \leq \int_0^{H((T_{x_{2n+1}}),S_{x_{2n}})} \varphi(t)dt \\ &+ \frac{1}{2}\phi \int_0^{M(x_{2n-1},x_{2n})} \varphi(t)dt \leq \int_0^{M(x_{2n-1},x_{2n})} \varphi(t)dt - \frac{1}{2}\phi \int_0^{M(x_{2n-1},x_{2n})} \varphi(t)dt \end{aligned} \tag{17}$$

where $\int_0^{d[x_{2n-1},x_{2n})} \varphi(t)dt \leq \int_0^{M[x_{2n-1},x_{2n})} \varphi(t)dt$

$$= \int_0^{\max\left\{d(x_{2n-1},x_{2n}), d(x_{2n-1},Tx_{2n-1}), d(x_{2n},Sx_{2n}), \frac{d(x_{2n-1},Sx_{2n})+d(x_{2n},Tx_{2n-1})}{2}\right\}} \varphi(t)dt$$

$$\leq \int_0^{\max\left\{d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1}),\frac{d(x_{2n-1},x_{2n+1})+0}{2}\right\}} \varphi(t)dt$$

$$= \int_0^{\max\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\}} \varphi(t)dt = \int_0^{\{d(x_{2n-1},x_{2n})\}} \varphi(t)dt \qquad \text{(by 17)} \qquad (18)$$

So $\int_0^{M(x_{2n-1},x_{2n})} \varphi(t)dt = \int_0^{d(x_{2n-1},x_{2n})} \varphi(t)dt$

Hence by (17)

$$\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt \leq \int_0^{d(x_{2n},x_{2n-1})} \varphi(t)dt \qquad (19)$$

also

$$\int_0^{d(x_{2n+2},x_{2n+1})} \varphi(t)dt = \int_0^{d(x_{2n+1},x_{2n+1})} \varphi(t)dt \leq \int_0^{d(Tx_{2n+1},Sx_{2n})} \varphi(t)dt$$

$$\int_0^{M(x_{2n+1},x_{2n})} \varphi(t)dt = \phi\int_0^{M(x_{2n+1},x_{2n})} \varphi(t)dt \qquad (20)$$

where, $\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt \leq \phi\int_0^{M(x_{2n+1},x_{2n})} \varphi(t)dt$

$$= \int_0^{\max\left\{d(x_{2n+1},x_{2n}),d(x_{2n+1},Tx_{2n+1}),d(x_{2n},Sx_{2n}),\frac{d(x_{2n+1},Sx_{2n})+d(x_{2n},Tx_{2n+1})}{2}\right\}} \varphi(t)dt$$

$$\leq \int_0^{\max\left\{d(x_{2n+1},x_{2n}),d(x_{2n+1},Tx_{2n+2}),d(x_{2n},Sx_{2n+1}),\frac{0+d(x_{2n},Tx_{2n+2})}{2}\right\}} \varphi(t)dt \qquad (21)$$

$$= \int_0^{\max\{d(x_{2n+1},x_{2n}),d(x_{2n+1},x_{2n+2})\}} \varphi(t)dt = \int_0^{\{d(x_{2n+1},x_{2n})\}} \varphi(t)dt \qquad \text{(by (20)).}$$

So $\int_0^{M(x_{2n+1},x_{2n})} \varphi(t)dt = \int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt$. Hence by (20),

$$\int_0^{d(x_{2n+2},x_{2n+1})} \varphi(t)dt \leq \int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt \qquad (22)$$

Therefore, by (14) and (16), we conclude that:

$$\int_0^{d(x_{2k+1},x_{2k})} \varphi(t)dt \leq \int_0^{d(x_k,x_{k-1})} \varphi(t)dt \quad \text{for all } k \in \mathrm{N}. \qquad (23)$$

Therefore, the sequence $\{d(x_{k+1}, x_k)\}$ is monotone non-increasing and bounded below.
So there exists $r \geq 0$ such that

$$\lim_{n\to\infty}(x_{n+1}, x_n) = \lim_{n\to\infty} M(x_{n+1}, x_n) = r \tag{24}$$

Since ϕ *is l.s.c.,*

$$\phi(r) \leq \lim_{n\to\infty}\inf \phi(M(x_n, x_{n-1})) \leq \lim_{n\to\infty}\inf \phi(M(x_{2n-1}, x_{2n})) \tag{25}$$

By 17, we conclude that $r \leq r - \frac{1}{2}\phi(r)$ and so ϕ (r) =0. Hence $r = 0$.

Step 2: $\{x_n\}$ is a bounded sequence.

Proof. If $\{x_n\}$ were unbounded, then by Step 1, $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are unbounded. We choose the sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$,n(2) > $n(1)$ is even and minimal in the sense that $\int_0^{d(x_n(2), x_n(1))}\varphi(t)dt > 1$, $\int_0^{d(x_n(2)-2, x_n(1))}\varphi(t)dt \leq 1$ and similarly $n(3)>n(2)$ is odd and minimal in the sense that, $\int_0^{d(x_n(3), x_n(2))}\varphi(t)dt > 1$ and $\int_0^{d(x_n(3), x_n(2))}\varphi(t)dt \leq 1, \ldots, n(2k) > n(2k-1)$ is even and minimal in the sense that and $\int_0^{d(x_n(2k), x_n(2k-1))}\varphi(t)dt > 1$, and $\int_0^{d(x_n(2k)-2, x_n(2k-1))}\varphi(t)dt \leq 1, n(2k+1) > n(2k)$ is odd and minimal in the sense that, $\int_0^{d(x_n(2k+1), x_n(2k))}\varphi(t)dt > 1$ and $\int_0^{d(x_n(2k+1), x_n(2k))}\varphi(t)dt \leq 1$.

Obviously $n(k) \geq k$ for every $k \in N$. By Step 1, there exists N_0 (----) N such that for all $k \geq N_0$ we have $\int_0^{d(x_{k+1}, x_k)}\varphi(t)dt < 1/4$. So for every $k \geq N_0$, we have $n(k+1) \geq n(k) \geq 2$ and $1 < \int_0^{d(x_n(k+1), x_n(k))}\varphi(t)dt$:

$$\leq \int_0^{d(x_n(k+1), x_n(k+1)-1)}\varphi(t)dt + \int_0^{d(x_n(k+1), x_n(k+1)-2)}\varphi(t)dt + \int_0^{d(x_n(k+1)-2, x_n(k))}\varphi(t)dt$$

$$\leq \int_0^{d(x_n(k+1), x_n(k+1)-1)}\varphi(t)dt + \int_0^{d(x_n(k+1)-1, x_n(k+1)-2)}\varphi(t)dt + 1 \tag{26}$$

Hence $\lim_{k\to\infty} d(x_{n(k+1)}, x_{n(k)}) = 1$ also $1 < \int_0^{d(x_n(k+1), x_n(k))}\varphi(t)dt$

$$\leq \int_0^{d(x_n(k+1), x_n(k+1)+1)}\varphi(t)dt + \int_0^{d(x_n(k+1)+1, x_n(k)+1)}\varphi(t)dt + \int_0^{d(x_n(k)+1, x_n(k))}\varphi(t)dt$$

$$\leq \int_0^{d\left(x_{n(k+1)},x_{n(k+1)+1}\right)} \varphi(t)dt + \int_0^{d\left(x_{n(k+1)+1},x_{n(k+1)}\right)} \varphi(t)dt + \int_0^{d\left(x_{n(k+1)},x_{n(k)}\right)} \varphi(t)dt$$

$$+ \int_0^{d\left(x_{n(k)},x_{n(k+1)}\right)} \varphi(t)dt + \int_0^{d\left(x_{n(k+1)},x_{n(k)}\right)} \varphi(t)dt \leq 2\int_0^{d\left(x_{n(k+1)},x_{n(k+1)+1}\right)} \varphi(t)dt$$

$$+ \int_0^{d\left(x_{n(k+1)},x_{n(k)}\right)} \varphi(t)dt + 2\int_0^{d\left(x_{n(k)+1},x_{n(k)}\right)} \varphi(t)dt \tag{27}$$

and this shows that $\lim_{k\to\infty} d\left(x_{n(k+1)},x_{n(k)+1}\right) = 1$. So if n $(k+1)$ is odd, then:

$$+ \int_0^{d\left(x_{n(k+1)+1},x_{n(k)+1}\right)} \varphi(t)dt \leq \int_0^{M\left(x_{n(k+1)},x_{n(k)+1}\right)} \varphi(t)dt - \phi\int_0^{M\left(x_{n(k+1)},x_{n(k)}\right)} \varphi(t)dt \tag{28}$$

where , $1 < \int_0^{d\left(x_{n(k+1)},x_{n(k)}\right)} \varphi(t)dt \leq \int_0^{M\left(x_{n(k+1)},x_{n(k)}\right)} \varphi(t)dt$

$$= \int_0^{\max\left\{d\left(x_{n(k+1)},x_{n(k)}\right),d\left(x_{n(k+1)},Tx_{n(k+1)}\right),d\left(x_{n(k)},Sx_{n(k)}\right),\frac{d\left(x_{n(k+1)},Sx_{n(k)}\right)+d\left(x_{n(k)},x_{n(k+1)}\right)}{2}\right\}} \varphi(t)dt$$

$$\leq \int_0^{\max\left\{d\left(x_{n(k+1)},x_{n(k)}\right),d\left(x_{n(k+1)},x_{n(k+1)+1}\right),d\left(x_{n(k)},x_{n(k)+1}\right),\frac{d\left(x_{n(k+1)},x_{n(k)+1}\right)+d\left(x_{n(k)},x_{n(k+1)+1}\right)}{2}\right\}} \varphi(t)dt$$

$$\leq \int_0^{\max\left\{d\left(x_{n(k+1)},x_{n(k)}\right),d\left(x_{n(k+1)},x_{n(k+1)+1}\right),d\left(x_{n(k)},x_{n(k)+1}\right),\frac{2d\left(x_{n(k+1)},x_{n(k)}\right)+d\left(x_{n(k)+1},x_{n(k)}\right)+d\left(x_{n(k+1)+1},x_{n(k+1)}\right)}{2}\right\}} \varphi(t)dt \tag{29}$$

and this shows that $\lim_{k\to\infty} M\left(x_{n(k+1)},x_{n(k)}\right) = 1$. Since ϕ is l.s.c. and (28) holds, we have $1 \leq 1 - \phi(1)$. So $\phi(1) = 0$ and this is a contradiction.

Step 3: $\{x_n\}$ is Cauchy.

Proof: Let $C_n = \sup\left\{d\left(x_i,x_j\right) \mid i,j \geq n\right\}$. Since $\{x_n\}$ is bounded, $C_n < +\infty$ for all $n \in N$ Obviously $\{C_n\}$ is decreasing. So there exists $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. We need to show that $C = 0$.

For every $k \in$ N, there exists $n(k)$, $m(k) \in$ N such that $m(k) > n(k) \geq k$ and:

$$C_n - \frac{1}{k} \leq \int_0^{d\left(x_{m(k)},x_{n(k)}\right)} \varphi(t)dt \leq C_k \tag{30}$$

By 30, we conclude that:

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = C \tag{31}$$

From Step 1 and Eq. (31), we have:

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right)$$

$$= \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) = C \tag{32}$$

So we may assume that for every $k \in$ N, $m(k)$ is odd and $n(k)$ is even. Hence:

$$\int_0^{d\left(x_{m(k)+1}, x_{n(k)+1}\right)} \varphi(t)dt \le \int_0^{M\left(x_{m(k)}, x_{n(k)+1}\right)} \varphi(t)dt \tag{33}$$

where, $\int_0^{d\left(x_{m(k)}, x_{n(k)}\right)} \varphi(t)dt \le \int_0^{M\left(x_{m(k)}, x_{n(k)}\right)} \varphi(t)dt$

$$\le \int_0^{H\left(\left(Tx_{m(k)}\right), Sx_{n(k)}\right)} \varphi(t)dt \le \int_0^{M\left(x_{m(k)}, x_{n(k)}\right)} \varphi(t)dt - \phi\int_0^{M\left(x_{m(k)}, x_{n(k)}\right)} \varphi(t)dt$$

$$= \int_0^{\max\left\{d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, Tx_{m(k)}\right), d\left(x_{n(k)}, Sx_{n(k)}\right), \frac{d\left(x_{m(k)}, Sx_{n(k)}\right)+d\left(x_{n(k)}, Tx_{m(k)}\right)}{2}\right\}} \varphi(t)dt \tag{34}$$

$$\le \int_0^{\max\left\{d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, Tx_{m(k)+1}\right), d\left(x_{n(k)}, Sx_{n(k)+1}\right), \frac{d\left(x_{m(k)}, x_{n(k)+1}\right)+d\left(x_{n(k)}, x_{m(k)+1}\right)}{2}\right\}} \varphi(t)dt$$

This inequality shows that, $\lim_{k\to\infty} M\left(x_{n(k)}, x_{n(k)}\right) = C$. Since φ is lower semi continuous and (33) holds, we have $C \le C - \phi(C)$. Hence $\phi(C) = 0$ and so $C = 0$ Therefore, $\{x_n\}$ is a Cauchy sequence.

Step 4: T and S have a common fixed point.

Proof: Since (x, d) is complete and $\{x_n\}$ is Cauchy, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$

For every $n \in$ N:

$$\int_0^{d\left(x_{2n+2}, Sx\right)} \varphi(t)dt \le \int_0^{d\left(Tx_{2n+1}, Sx\right)} \varphi(t)dt$$

$$\leq H\int_0^{H(\{Tx_{2n+1}\},Sx)}\varphi(t)dt \leq \int_0^{M(x_{2n+1},x)}\varphi(t)dt - \phi\int_0^{M(x_{2n+1},x)}\varphi(t)dt \tag{35}$$

where.

$$\int_0^{M(x_{2n+1},x)}\varphi(t)dt = \int_0^{\max\left\{d(x_{2n+1},x),d(x_{2n+1},Tx_{2n+1}),d(x,Sx),\frac{d(x_{2n+1},Sx)+d(x,Tx_{2n+1})}{2}\right\}}\varphi(t)dt$$

$$= \int_0^{\max\left\{d(x_{2n+1},x),d(x_{2n+1},Tx_{2n+2}),d(x,Sx),\frac{d(x_{2n+1},Sx)+d(x,x_{2n+2})}{2}\right\}}\varphi(t)dt \tag{36}$$

and this shows that, $\lim_{k\to\infty} M(x_{2n+1},x) = d(x,Sx)$. Since ϕ is *l.s.c* and 35 holds, letting $n \to \infty$ in (35) we get:

$$\int_0^{d(x,Sx)}\varphi(t)dt \leq \int_0^{d(x,Sx)}\varphi(t)dt - \phi\int_0^{d(x,Sx)}\varphi(t)dt \tag{37}$$

So $\phi\int_0^{d(x,Sx)}\varphi(t)dt = 0$ and hence $\int_0^{d(x,Sx)}\varphi(t)dt = 0$. Since $Sx \in CB(X)$, Also

$$\int_0^{d(x,Sx)}\varphi(t)dt \leq \int_0^{H(\{Tx\},Sx)}\varphi(t)dt \leq \phi\int_0^{M(x,x)}\varphi(t)dt - \phi\int_0^{M(x,x)}\varphi(t)dt \tag{38}$$

$$\int_0^{M(x,x)}\varphi(t)dt = \int_0^{\max\left\{d(x,x),d(x,Tx)d(x,Sx),\frac{d(x,Sx)+d(x,Tx)}{2}\right\}}\varphi(t)dt = \int_0^{d(x,Tx)}\varphi(t)dt \tag{39}$$

So from (38), we have:

$$\int_0^{d(Tx,x)}\varphi(t)dt \leq \int_0^{d(Tx,x)}\varphi(t)dt - \phi\int_0^{d(Tx,x)}\varphi(t)dt \tag{40}$$

Thus $\phi\int_0^{d(Tx,x)}\varphi(t)dt = 0$ and hence $\int_0^{d(Tx,x)}\varphi(t)dt = 0$

Therefore, $x = Tx$.

Remark: In the proof of Zhang and Song (2009), the boundedness of the sequence $\{C_n\}$ is used, but not proved. Also, for the proof that $\{x_n\}$ is a Cauchy sequence, the monotonicity of ϕ is used, without being explicitly mentioned.

In our proof of Theorem 6, which is different from Zhang and Song (2009) ϕ is not assumed to be non decreasing The following theorem extends Rhoades' theorem by assuming to be only *l.s.c*.

Theorem 7: Let (X, *d*) be a complete metric space and let $T : X \to X$ be a mappings such that:

$$\int_0^{d(Tx,Sx)}\varphi(t)dt \leq \int_0^{d(x,y)}\varphi(t)dt - \phi\int_0^{d(x,y)}\varphi(t)dt \tag{41}$$

for every $x, y \in X$. (i.e., ϕ weak contractive), where $\phi: [0,+\infty) \to [0,+\infty)$ is an lower semi continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

Proof. The proof is similar to the proof of Theorem 6, by taking $S = T$, and replacing $M(x, y)$ with (x, y),

Remark: With a similar proof as in Theorem 2 in Theorem 6 we can replace the inequality (41) by the following inequality (42) for two single valued mappings $T, S : X \to X$.

$$\int_0^{d(Tx,Sx)} \varphi(t)dt \le \int_0^{M(x,y)} \varphi(t)dt - \phi\int_0^{d(x,y)} \varphi(t)dt \tag{42}$$

CONCLUSION

We establish a unique Common Fixed Point of Multivalued Generalized φ-Weak Contractive Mappings in Integral type inequality.

REFERENCES

Abbas, M. and B.E. Rhodes, 2007. Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of Integral type. Fixed Point Theor. A., pp: 4 -9, Article Id: 274793, doi: 10.1155/2008/274973.

Alionche, A., 2006. A Common fixed point theorems for weakly compatible mappings in symmetric spaces satisfying contractive condition of Integral type. J. Math. Anal. Appl., 322(2): 796-802.

Alionche, A. and B. Fisher, 2007. Fixed point theorem on three complete and compact matric spaces. Unir. Din. Bacen. Stu. Cer. Stiin., pp: 13-20.

Assad, N.A. and W.A. Kirk, 1972. Fixed point theorems for set-valued mappings of contractive type. Paci. C. J. Math., 43: 553-562.

Behzad, D.R., and S. Moradi, 2010. Common fixed point of multivalued generalized φ-weak contractive mappings. Fixed Point Theor. A., Article ID 708984, pp: 1-13.

Boyd, D.W. and J.S.W. Wong, 1969. On nonlinear contractions. P. Am. Math. Soc., 20: 458-464.

Branciari, A., 2002. A Fixed point theorem for mappings satisfying a general contractive condition of Integral type. Int. J. Math. Sci., 29(9): 531-536.

Daffer, P.Z. and H. Kaneko, 1995. Fixed points of generalized contractive multi-valued mappings. J. Math. Anal. Appl., 192(2): 655-666.

Gairola, U.C. and A.S. Rawat, 2008. A fixed point theorem for Integral type Inequality. Int. J. Math. Anal., 2(15): 709-712.

Kumar, S., R. Chug and R. Kumar, 2007. Fixed point theorem for compatible mappings satisfying a contractive condition of Integral type. Sochow. J. Math., 33(2): 181-185.

Nadler, S.B., 1969. Multi-valued contraction mappings. Pacific J. Math., 30: 475-488.

Reich, S., 1974. Some fixed point problems. Atti della accademia nazionale dei lincei. Rendiconti. Classe diScienze Fisiche, Matematiche e Naturali, 57(3-4): 194-198.

Rhoades, B.E., 2001. Some theorems on weakly contractive maps. Nonlinear Anal-Theor., 47(4): 2683-2693.

Rhodes, B.E., 2003. Two fixed point theorems for mapping satisfying general contractive condition of Integral type. Int. J. Math. Sci., 63: 4007-4013.

Suzuki, T., 2007. Meri-keeler Contractions of integral type are still Meir-keeler contraction, Int. J. Math. Sci., Article ID: 39281, doi: 10.1155/2007/39281.

Zhang, Q. and Y. Song, 2009. Fixed point theory for generalized φ-weak contractions. Appl. Math. Lett., 22(1): 75-78.