

Second-Order Duality for Nondifferentiable Multiobjective Programming Involving (ϕ, ρ) -Univexity

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Abstract: The concepts of (ϕ, ρ) -invexity have been introduced by Carsiti, Ferrara and Stefanescu. We consider a second-order dual model associated to a multiobjective programming problem involving support functions and a weak duality result is established under appropriate second-order (ϕ, ρ) -univexity conditions.

Key words: Duality theorem, multiobjective programming, second-order (ϕ, ρ) -(pseudo/quasi)-convexity, second-order (ϕ, ρ) -(pseudo/quasi)-univexity

INTRODUCTION

For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual (Dorn, 1960; Hanson and Mond, 1982) are well known. While studying duality under generalized convexity, Mond and Weir (1981) proposed a number of deferent duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used by Gulati *et al.* (1997), Suneja *et al.* (2003) and Yang *et al.* (2003). Mangasarian (1975) considered a nonlinear programming problem and discussed second order duality under inclusion condition. Mond (1974) was the first who present second order convexity. He also gave simpler conditions than Mangasarian using a generalized form of convexity, which was later called bonvexity by Bector and Chandra (1987). Further, Jeyakumar (1985, 1986) and Yang *et al.* (2003) discussed also second order Mangasarian type dual formulation under ρ -convexity and generalized representation conditions respectively. Zhang and Mond (1996) established some duality theorems for second-order duality in nonlinear programming under generalized second-order B-invexity, defined in their study. Mond (1974) has shown that second order duality can be useful from computational point of view, since one may obtain better lower bounds for the primal problem than otherwise. The case of some optimization problems that involve n-set functions was studied by Preda (1998). Recently, Yang *et al.* (2003) proposed four second-order dual models for nonlinear programming problems and

established some duality results under generalized second-order F-convexity assumptions.

For $\phi(x, a, (y, r)) = F(x, a; y) + rd^2(x, a)$, where $F(x, a)$ is sublinear on R^n , the definition of (ϕ, ρ) -invexity reduces to the definition of (F, ρ) -convexity introduced by Preda (1992), which in turn Jeyakumar (1985) generalizes the concepts of F-convexity and ρ -invexity (Vial, 1983).

The more recent literature, Xu (1996), Ojha (2005) and Ojha and Mukherjee (2006) for duality under generalized (F, ρ) -convexity, Mishra (2000) and Yang *et al.* (2003) for duality under second order F-convexity. Liang *et al.* (2003) and Hachimi (2004) for optimality criteria and duality involving (F, α, ρ, d) -convexity or generalized (F, α, ρ, d) -type functions. The (F, ρ) -convexity was recently generalized to (ϕ, ρ) -invexity by Caristi *et al.* (2006), and here we will use this concept to extend some theoretical results of multiobjective programming.

Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity, too. This is not the case in multiobjective programming; Ferrara and Stefanescu (2008) showed that sufficiency Kuhn-Tucker condition can be proved under (ϕ, ρ) -invexity, even if Hanson's invexity is not satisfied, (Puglisi, 2009). The interested reader may consult Ahmad and Husain (2005), Chandra *et al.* (1985, 1998), Chandra and Husain (1981), Chandra and Prasad (1993), Craven (1977), Dorn (1960), Kim *et al.* (1997), Mishra (2000), Mond *et al.* (1991), Unger and Hunter (1974), Weir and Mond (1988) and Yang and Hou (2001).

Therefore, the results of this study are real extensions of the similar results known in the literature.

In this study, we define the second-order (ϕ, ρ) -invexity, we also consider a class of Multiobjective programming problems and for the dual model we prove a weak duality result.

MATERIALS AND METHODS

we denote by R^n the n -dimensional Euclidean space, and by R^+_n its nonnegative orthant. Further, $R^n_+ = \{x \in R^n | x > 0\}$. For any vector $x \in R^n, y \in R^n$ we denote:

$$x^T y = \sum_{i=1}^n x_i y_i$$

Let $C \subset R^n$ be a compact convex set. The support function of C is defined by $s(x|C) = \max \{x^T y | y \in C\}$. Being convex and every where finite, it has a subdifferential, that is, there exist $z \in R^n$ such that $s(y|C) \geq s(x|C) + z^T (y - x)$ for all $y \in C$.

The subdifferentials of $s(x|C)$ is given by $\partial s(x|C) = \{z \in C | z^T = s(x|C)\}$.

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by:

$$N_D(x) = \left\{ y \in R^n \mid (z - x) \leq 0 \right\} \text{ for all } z \in D$$

For a compact convex set C . We obviously have $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, if $x \in \partial s(y|C)$.

We consider $f: R^n \rightarrow R^p, g: R^n \rightarrow R^q$, are differential functions and $x \subset R^n$ is an open set. We define the following multiobjective programming problem:

(P) minimize $f(x) = (f_1(x), \dots, f_p(x))$ subject to $g(x) \geq 0, x \in X$

Let X_0 be the set of all feasible solutions of (P) that is, $X_0 = \{x \in X | g(x) \geq 0\}$. We quote some definitions and also give some new ones.

Definition: A vector $a \in X_0$ is said to be an efficient solution of problem (P) if there exit no $x \in X_0$ such that $f(a) - f(x) \in R^p_+ \setminus \{0\}$ i.e., $f_i(x) \leq f_i(a)$ for all $i \in \{1, \dots, p\}$ and for at least one $j \in \{1, \dots, p\}$ we have $f_j(x) < f_j(a)$.

Definition: A $a \in X_0$ point is said to be a weak efficient solution of problem (VP) if there is no $x \in X$ such that $f(x) < f(a)$.

Definition: A point $a \in X_0$ is said to be a properly efficient solution of (VP) if it is efficient and there exist a positive constant K such that for each $x \in X_0$ and for each $i \in \{1, 2, \dots, p\}$ satisfying $f_i(x) < f_i(a)$, there exist at least one $j \in \{1, 2, \dots, p\}$ such that $f_j(x) < f_j(a)$ and $f_j(a) - f_j(x) \leq K (f_i(x) - f_i(a))$.

Denoting by WE(P), E(P) and PE(P) the sets of all weakly efficient, efficient and properly efficient solutions of (VP), we have $PE(P) \subseteq E(P) \subseteq WE(P)$.

We denote by $\nabla f(a)$ the gradient vector at a of a differentiable function $f: R^p \rightarrow R$, and by $\nabla^2 f(a)$ the Hessian matrix of f at a . For a real valued twice differentiable function $\Psi(x, y)$ defined on an open set in $R^p \times R^q$, we denote by $\nabla_x \Psi(a, b)$ the gradient vector of Ψ with respect to x at (a, b) , and by $\nabla_{xx} \Psi(a, b)$ the Hessian matrix with respect to x at (a, b) . Similarly, we may define:

$$\nabla_y \Psi(a, b), \nabla_{xy} \Psi(a, b) \text{ and } \nabla_{yy} \Psi(a, b)$$

For convenience, let us write the definitions of (ϕ, ρ) -univexity from (1), Let $\varphi: X_0 \rightarrow R$ be a differentiable function ($X_0 \subseteq R^n, X \subseteq X_0$, and $a \in X_0$). An element of all $(n+1)$ - dimensional Euclidean Space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ and $r \in R$, ρ is a real number and ϕ is real valued function defined on $X_0 \times X_0 \times R^{n+1}$, such that $\phi(x, a)$ is convex on R^{n+1} and $\phi(x, a(0, r)) \geq 0$, for every $(x, a) \in X_0 \times X_0$ and $r \in R_+$. $b_0, b_1: X \times X \times [0, 1] \rightarrow R_+$ $b(x, a) = \lim_{\lambda \rightarrow 0} b(x, a, \lambda) \geq 0$, and b does not depend upon λ if the corresponding functions are differentiable. $\psi_0, \psi_1: R \rightarrow R$ is an n -dimensional vector-valued function and $h: X \times R^n \rightarrow R$ be differentiable function.

We assume that $\psi_0, \psi_1: R \rightarrow R$ satisfying $u \leq 0 \Rightarrow \psi_0(u) \leq 0$ and $u \leq 0 \Rightarrow \psi_1(u) \leq 0$, and $b_0(x, a) > 0$ and $b_1(x, a) \geq 0$ and $\psi_0(\alpha) = -\psi_0(\alpha)$ and $\psi_1(-\alpha) = -\psi_1(\alpha)$.

Example:

$$\begin{aligned} \min f(x) &= x-1 \\ g(x) &= -x-1 \leq 0, x \in X_0 \in [1, \infty) \\ \phi(x, a; (y, r)) &= 2(2^r-1) |x-a| \langle y, x-a \rangle \end{aligned}$$

for $\psi_0(x) = x, \psi_1(x) = -x, \rho = \frac{1}{2}$ (for f) and $\rho = 1$ (for g), then this is (ϕ, ρ) -univex but it is not (ϕ, ρ) -invex.

Definition: A real-valued twice differentiable function $f(., y): X \times X \rightarrow \mathbb{R}$ is said to be second-order (Φ, ρ) -univex at $u \in X$ with respect to $p \in R^n$, if for all $b: X \times X \rightarrow \mathbb{R}_+$, $\Phi: X \times X \times R^{n+1} \rightarrow \mathbb{R}$, ρ is a real number, we have:

$$b(x, u) \left[\psi \left(f_i(x, y) - f_i(u, y) + \frac{1}{2} p^T \nabla^2 f_i(u, y) p \right) \right] \geq \Phi \left(x, u, \left(\nabla f_i(u, y) + \nabla^2 f_i(u, y) p, \rho_i \right) \right) \quad (1)$$

Definition: A real-valued twice differentiable function $f(., y): X \times X \rightarrow \mathbb{R}$ is said to be second-order (Φ, ρ) -pseudounivex at $a \in X$ with respect to $p \in R^n$, if for all $b: X \times X \rightarrow \mathbb{R}_+$, $\Phi: X \times X \times R^{n+1} \rightarrow \mathbb{R}$, ρ is a real number, we have:

$$\Phi \left(x, u, \left(\nabla f_i(u, y) + \nabla^2 f_i(u, y) p, \rho_i \right) \right) \geq 0 \Rightarrow b(x, u) \left[\psi \left(f_i(x, y) - f_i(u, y) + \frac{1}{2} p^T \nabla^2 f_i(u, y) p \right) \right] \geq 0 \quad (2)$$

Definition: A real-valued twice differentiable function $f(., y): X \times X \rightarrow \mathbb{R}$ is said to be second-order (Φ, ρ) -quasiunivex at $a \in X$ with respect to $p \in R^n$, if for all $b: X \times X \rightarrow \mathbb{R}_+$, $\Phi: X \times X \times R^{n+1} \rightarrow \mathbb{R}$, ρ is a real number, we have:

$$b(x, u) \left[\psi \left(f_i(x, y) + \frac{1}{2} p^T \nabla^2 f_i(u, y) p \right) \right] \leq 0 \Rightarrow \Phi \left(x, u, \left(\nabla f_i(u, y) + \nabla^2 f_i(u, y) p, \rho_i \right) \right) \leq 0 \quad (3)$$

Remark:

- If we consider the case $b = 1$, $\Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ (with F is sublinear in third argument, then the above definition reduce to Definition 4 of Chen [4].
- When $h(u, y) = y^T \nabla_{yy} f(u) y / 2$ and $\Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u)) = \eta(x, u)^T \nabla f(u)$ where $\eta: X \times X \rightarrow R^n$, the above definition reduce to η - (pseudo/quasi)-bonvexity.

Example:

We present here a function which is second-order (Φ, ρ) -univex for $b = 1$. Let us consider $X = (0, \infty)$ and

$$f: X \rightarrow \mathbb{R}, f(x) = x \log x, h: X \times \mathbb{R} \rightarrow \mathbb{R}, h(u, y) = -y \log u$$

We have:

$$\begin{aligned} \nabla_u f(u) &= 1 + \log u, \nabla_{uu} f(u) \\ &= \frac{1}{u}, \nabla_y h(u, y) = -\log u \end{aligned}$$

$\Phi: X \times X \times R^{n+1} \rightarrow \mathbb{R}$, taking:

$$\rho = 0 \quad \Phi(x, y; b) = |b| + |b|^2$$

It is obvious our mapping is more generalized rather than previous ones.

Hence $f(x) = x \log x$ is second-order (Φ, ρ) -univex at $u \in X$, with respect to $h(u, y) = -y \log u$.

A real valued twice differentiable function g is second order F-pseudoconcave if $-g$ is second order F-pseudoconvex.

We shall make use of the following generalized Schwartz inequality:

$$x^T A y \leq (x^T A x)^{\frac{1}{2}} (y^T A y)^{\frac{1}{2}}$$

where $x, y \in R^n$ and $A \in R^n \times R^n$ is a positive semidefinite matrix. Equality holds if for some $\lambda \geq 0$, $Ax = \lambda Ay$.

RESULTS AND DISCUSSION

Mond-Weir type second order symmetric duality: We consider here the following pair of second order nondifferentiable multiobjective with r -objectives and establish weak, strong and converse duality theorems.

(MP) Minimize:

$$H(x, y, w, p) = \{H_1(x, y, w, p), H_2(x, y, w, p), \dots, H_r(x, y, w, p)\}$$

subject to:

$$\sum_{i=1}^r \lambda_i \left[\nabla_y f_i(x, y) - C_i w_i + \nabla_{yy} f_i(x, y) p_i \right] \leq 0 \quad (4)$$

$$y^T \sum_{i=1}^r \lambda_i \left[\nabla_y f_i(x, y) - C_i w_i + \nabla_{yy} f_i(x, y) p_i \right] \geq 0 \quad (5)$$

$$w_i^T C_i w_i \leq 1, i = 1, 2, \dots, r \tag{6}$$

$$\lambda > 0 \tag{7}$$

$$x \geq 0 \tag{8}$$

(MD) Maximize:

$$J(u, v, a, q) = \{J_1(u, v, a, q), J_2(u, v, a, q), \dots, J_r(u, v, a, q)\}$$

subject to:

$$\sum_{i=1}^r \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_{xx} f_i(u, v) q_i] \geq 0 \tag{9}$$

$$u^T \sum_{i=1}^r \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_{xx} f_i(u, v) q_i] \leq 0 \tag{10}$$

$$a_i^T E_i a_i \leq 1, i = 1, 2, \dots, r \tag{11}$$

$$\lambda > 0 \tag{12}$$

$$v \geq 0 \tag{13}$$

where;

$$H_i(x, y, w, p) = f_i(x, y) + (x^T E_i x)^{\frac{1}{2}} - y^T C_i w_i - \frac{1}{2} p_i^T \nabla_{xy} f_i(x, y) p_i$$

$$J_i(u, v, a, q) = f_i(u, v) - (v^T C_i v)^{\frac{1}{2}} + u^T E_i a_i - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i$$

$\lambda_i \in R, p_i \in R^n, q_i \in R^n, i = 1, 2, \dots, r$ and $f_i, i = 1, 2, \dots, r$ are thrice differentiable function from $R^n \times R^n \rightarrow R, E_i$ and $C_i, i = 1, 2, \dots, r$ are positive semi definite matrices. Also, we mean here:

$$b_i = R^n \times R^m \times R^n \times R^m \rightarrow R_+ p = (p_1, p_2, \dots, p_r),$$

$$q = (q_1, q_2, \dots, q_r), w = (w_1, w_2, \dots, w_r), a = (a_1, a_2, \dots, a_r)$$

Remark:

Since the objective functions of (MP) and (MD) contain the support functions $s(x | C_i)$ and $s(v | D_i), i = 1, 2, \dots, p$, these problems are nondifferentiable multiobjective programming problems.

Theorem (Weak duality):

Let (x, y, λ, w, p) be a feasible solution of (MP) and (u, v, λ, a, q) a feasible solution of (MD). Then the inequalities can not hold simultaneously:

- $\sum_{i=1}^r \lambda_i [f_i(\cdot, v) + (\cdot)^T E_i a_i]$ is second order (ϕ, ρ) -pseudounivex at u ,
- $\sum_{i=1}^r \lambda_i [f_i(x, \cdot) - (\cdot)^T C_i w_i]$ is second order (ϕ, ρ) -pseudounicave at y
- $\phi(x, u; (\xi, \rho)) + u^T \xi \geq 0$, for $\xi \in R^n$, and
- $\phi(v, y; (\zeta, \rho)) + y^T \zeta \geq 0$, for $\zeta \in R^n$, then

$$H(x, y, w, p) \leq J(u, v, a, q).$$

Proof:

With the help of:

$$\sum_{i=1}^r \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_{xx} f_i(u, v) q_i]$$

we have:

$$\phi \left(x, u; \left(\sum_{i=1}^r \lambda_i [\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v) q_i], \rho_i \right) \right) + u^T \sum_{i=1}^p \lambda_i \left(\nabla_u f_i(u, v) + \nabla_u g_i(u, v, u_i) \right) \geq 0$$

(By hypothesis (iii) and (10), also by the second order (ϕ, ρ) -pseudounivexity of:

$$\sum_{i=1}^r \lambda_i [f_i(\cdot, v) + (\cdot)^T E_i a_i]$$

at u , with property of b and Ψ , provides:

$$\sum_{i=1}^r \lambda_i [f_i(x, v) + (x)^T E_i a_i] \geq \sum_{i=1}^r \lambda_i \left(f_i(u, v) + u^T E_i a_i - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i \right) \tag{14}$$

Now,

$$\zeta = -\sum_{i=1}^r \lambda_i \left[\nabla_y f_i(x, y) - C_i w_i + \nabla_{yy} f_i(x, y) p_i \right]$$

we have:

$$\phi(v, y; (\zeta, \rho)) + y^T \zeta \leq 0$$

(by hypothesis (iv),(5) and by the second order (ϕ, ρ) -pseudoncavity $\sum_{i=1}^r \lambda_i \left[f_i(x, \cdot) - (\cdot)^T C_i w_i \right]$ at y , with property of b and ψ , gives:

$$\begin{aligned} & \sum_{i=1}^r \lambda_i \left[f_i(x, v) - (v)^T C_i w_i \right] \leq \\ & \sum_{i=1}^r \lambda_i \left[f_i(x, y) - y^T C_i w_i - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i \right] \end{aligned} \quad (15)$$

Combining (14) and (15), we get:

$$\begin{aligned} & \sum_{i=1}^r \lambda_i \left[(x)^T E_i a_i + v^T C_i w_i \right] \geq \sum_{i=1}^r \lambda_i \\ & \left[\left\{ f_i(u, v) + u^T E_i a_i - \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i \right\} \right. \\ & \left. - \left\{ f_i(x, y) + y^T C_i w_i + \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i \right\} \right] \end{aligned}$$

Applying Schwartz inequality, (6) and (11), we get:

$$\begin{aligned} & \sum_{i=1}^r \lambda_i \left\{ f_i(x, y) + (x^T E_i x)^{\frac{1}{2}} \right. \\ & \left. - \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i \right\} \end{aligned}$$

$$\begin{aligned} & \geq \sum_{i=1}^r \lambda_i \left\{ \left(f_i(u, v) - (v^T C_i v)^{\frac{1}{2}} \right. \right. \\ & \left. \left. + \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i \right) \right\} \end{aligned}$$

Hence, $H(x, y, w, p) \leq J(u, v, a, q)$.

Theorem (Strong duality): Let f be thrice differentiable on $R^n \times R^n$ and $(x', y', \lambda', w', p')$ be a weak efficient solution for (MP), and $\lambda = \lambda'$, assume that:

- $\nabla_{yy} f_i$ is nonsingular for all $i = 1, 2, \dots, r$;
- the matrix $\sum_{i=1}^r \lambda'_i (\nabla_{yy} f_i p'_i)$ is positive or negative definite
- the set $\left[\begin{array}{l} \nabla_y f_1 - C_1 w'_1 + \nabla_{yy} f_1 p'_1, \\ \nabla_y f_2 - C_2 w'_2 + \nabla_{yy} f_2 p'_2, \dots, \\ \nabla_y f_r - C_r w'_r + \nabla_{yy} f_r p'_r \end{array} \right]$ are linearly independent

where; $f_i = f_i(x', y')$, $i = 1, 2, \dots, r$. Then $(x', y', \lambda', a', q' = 0)$ is a feasible solution of (MD), $b_i(x', y', u', v') > 0, i = 1, 2, \dots, r$ and the two objectives have the same values. Also, if the hypothesis of previous theorem are satisfied for all feasible solutions of (MP) and (MD), then $(x', y', \lambda', w', p' = 0)$ is an efficient solution for (MD).

Proof: Since $(x', y', \lambda', w', p')$ is a weak efficient solution of (MP), by Fritz-John condition, there exist $\alpha \in R^r, \beta \in R^n, \gamma \in R, v \in R^r$ and $\xi \in R^n$ such that:

$$\begin{aligned} & \sum_{i=1}^r \alpha_i \left[\nabla_x f_i + E_i a'_i - \frac{1}{2} (\nabla_{yy} f_i p'_i) x p'_i \right] + \\ & \sum_{i=1}^r \lambda'_i \left[\nabla_{yx} f_i + (\nabla_{yy} f_i p'_i) x \right] (\beta - \gamma y') - \xi = 0 \end{aligned} \quad (16)$$

$$\sum_{i=1}^r \alpha_i \left[\nabla_y f_i - C_i w_i' + \frac{1}{2} (\nabla_{yy} f_i p_i')_y p_i' \right] - \gamma \sum_{i=1}^r \lambda_i' \left[\nabla_y f_i - C_i w_i' + (\nabla_{yy} f_i p_i')_y \right] (\beta - \gamma y')$$

$$- \gamma \sum_{i=1}^r \lambda_i' \left[\nabla_y f_i - C_i w_i' + (\nabla_{yy} f_i p_i')_y \right] = 0$$

$$(\beta - \gamma y')^T \left[\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i' \right] - \delta_i = 0 \quad (17)$$

$$i = 1, 2, \dots, r$$

$$\alpha_i C_i y' + (\beta - \gamma y')^T \lambda_i' C_i = 2v_i C_i w_i', \quad (18)$$

$$i = 1, 2, \dots, r$$

$$\left[(\beta - \gamma y') \lambda_i' - \alpha_i p_i' \right]^T \nabla_{yy} f_i = 0, \quad (19)$$

$$i = 1, 2, \dots, r$$

$$x'^T E_i a_i' = \left(x'^T E_i \lambda_i' \right)^{\frac{1}{2}}, i = 1, 2, \dots, r \quad (20)$$

$$\beta^T \sum_{i=1}^r \lambda_i' \left[\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i' \right] = 0 \quad (21)$$

$$\gamma y'^T \sum_{i=1}^r \lambda_i' \left[\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i' \right] = 0 \quad (22)$$

$$v_i \left(w_i'^T C_i w_i' - 1 \right) = 0, i = 1, 2, \dots, r \quad (23)$$

$$\delta^T \lambda' = 0 \quad (24)$$

$$x'^T \xi = 0 \quad (25)$$

$$a_i'^T E_i a_i' \leq 1, i = 1, 2, \dots, r \quad (26)$$

$$(\alpha, \beta, \gamma, v, \delta, \xi) \geq 0 \quad (27)$$

$$(\alpha, \beta, \gamma, v, \delta, \xi) \neq 0 \quad (28)$$

Since $\lambda' > 0$ and $\delta \geq 0$, (25) implies $\delta = 0$.

Consequently, (18) gives:

$$(\beta - \gamma y')^T \left[\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i' \right] = 0 \quad (29)$$

Since $\nabla_{yy} f_i$ is nonsingular for $i = 1, 2, \dots, r$, from (20), it follows that:

$$(\beta - \gamma y') \lambda_i' = \alpha_i p_i', i = 1, 2, \dots, r$$

from (17), we get:

$$\sum_{i=1}^r (\alpha_i - \gamma \lambda_i') (\nabla_y f_i - C_i w_i') + \sum_{i=1}^r \lambda_i' \nabla_{yy} f_i (\beta - \gamma \lambda_i' - \gamma p_i') + \sum_{i=1}^r (\nabla_{yy} f_i p_i')_y \left[(\beta - \gamma y') \lambda_i' - \frac{1}{2} \alpha_i p_i' \right] = 0$$

using (31), we get:

$$\sum_{i=1}^r (\alpha_i - \gamma \lambda_i') (\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i') + \frac{1}{2} \sum_{i=1}^r \lambda_i' (\nabla_{yy} f_i p_i') (\beta - \gamma y') = 0 \quad (30)$$

Premultiplying (32) by $(\beta - \gamma y')^T$ and using (30), we get:

$$(\beta - \gamma y')^T \sum_{i=1}^r \lambda_i' (\nabla_{yy} f_i p_i')_y (\beta - \gamma y') = 0$$

by hypothesis (ii) implies:

$$\beta = \gamma y' \quad (31)$$

Therefore, from (32), we get:

$$\sum_{i=1}^r (\alpha_i - \gamma \lambda_i') (\nabla_y f_i - C_i w_i' + \nabla_{yy} f_i p_i') = 0$$

which by hypothesis (iii) gives:

$$\alpha_i = \gamma \lambda'_i, i = 1, 2, \dots, r \tag{33}$$

If $\gamma = 0$, then $\alpha_i = 0, i = 1, 2, \dots, r$ and from (33), $\beta = 0$. Also from (16) and (19), we get, $\xi_i = 0, v_i = 0, i = 1, 2, \dots, r$. Thus $(\alpha, \beta, \gamma, v, \delta, \xi) = 0$, a contradiction to (29). Hence $\gamma > 0$, since $\lambda'_i > 0, i = 1, 2, \dots, r$, (34) implies $\alpha_i > 0, i = 1, 2, \dots, r$. Using (33) in (31), $\alpha_i p'_i = 0, i = 1, 2, \dots, r$, hence $p'_i = 0, i = 1, 2, \dots, r$. Using (33) and $p'_i = 0$

$i = 1, 2, \dots, r$ in (16), it gives $\sum_{i=1}^r \alpha_i [\nabla_x f_i + E_i a'_i] = \xi$,

which by (34) gives:

$$\sum_{i=1}^r \lambda_i [\nabla_x f_i + E_i a'_i] = \frac{\xi}{\gamma} \geq 0 \tag{34}$$

$$x'^T \sum_{i=1}^r \lambda_i [\nabla_x f_i + E_i a'_i] = x'^T \frac{\xi}{\gamma} = 0 \tag{35}$$

Also, from (33), we get:

$$y' = \frac{\beta}{\gamma} \geq 0 \tag{36}$$

Hence from (27) and (35-37), $(x', y', \lambda', a', q' = 0)$ is feasible for (MD).

Let $2 \frac{v_i}{\alpha_i} = t$. Then $t \geq 0$ and from (19) and (33):

$$C_i y' = t C_i w'_i \tag{37}$$

which is the condition in the Schwartz inequality. Therefore:

$$y'^T C_i w'_i = (v'^T C_i y')^{\frac{1}{2}} (w'_i{}^T C_i w'_i)^{\frac{1}{2}}$$

In case, $v_i > 0$, (24) gives $w'_i{}^T C_i w'_i = 1$ and so

$$y'^T C_i w'_i = (y'^T C_i y')^{\frac{1}{2}}. \text{ In case } v_i = 0,$$

gives $C_i y' = 0$ and so:

$$y'^T C_i w'_i = (y'^T C_i y')^{\frac{1}{2}} = 0 \tag{38}$$

Thus in either case:

$$y'^T C_i w'_i = (y'^T C_i y')^{\frac{1}{2}} \tag{39}$$

Hence:

$$\begin{aligned} H_i(x', y', w', p' = 0) &= f_i(x', y') + (x'^T E_i x')^{\frac{1}{2}} - y'^T C_i w'_i \\ &= f_i(x', y') - (y'^T C_i y')^{\frac{1}{2}} + x'^T E_i a'_i \\ &= J_i(x', y', a', q' = 0) \end{aligned}$$

(using (21) and (39)).

Now follows from Previous Theorem that $(x', y', \lambda', a', q' = 0)$ is an efficient solution for (MD). A converse duality theorem may be merely stated as its proof would run analogously to this Theorem.

Theorem (Converse duality): Let f be thrice differentiable on $R^n \times R^n$ and $(u', v', \lambda', a', q')$ be a weak efficient solution for (MD), and $\lambda = \lambda'$ fixed in (MP). Assume that:

- $\nabla_{xx} f_i$ is nonsingular for all $i = 1, 2, \dots, r$
- the matrix $\sum_{i=1}^r \lambda'_i (\nabla_{xx} f_i q'_i)_x$ is positive or negative definite
- the set $[\nabla_x f_1 + E_1 a'_1 + \nabla_{xx} f_1 q'_1, \nabla_x f_2 + E_2 a'_2 + \nabla_{xx} f_2 q'_2, \dots, \nabla_x f_r + E_r a'_r + \nabla_{xx} f_r q'_r]$ are linearly independent

where, $f_i = f_i(u', v'), i = 1, 2, \dots, r$. Then $(u', v', \lambda', w', p' = 0)$ is a feasible solution of (MP), $b_i(x', y', u', v') > 0, i = 1, 2, \dots, r$, and the two objectives have the same values. Also, if the hypothesis of

Theorem (Weak duality) are satisfied for all feasible solutions of (MP) and (MD), then $(u', v', \lambda', w', p' = 0)$ is an efficient solution for (MP).

CONCLUSION

- If $b = 1, \psi \equiv 1, E_i = C_i = 0, i = 1, 2, \dots, r$, and $\phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ for $\rho = 0$ then (MP) and (MD) reduce to the second order multiobjective symmetric dual program studied by Suneja *et al.* (2003) with omission of non-negativity constraints from (MP) and (MD). If in addition $p = q = 0$, and $r = 1$, then we get the first order symmetric dual programs of Chandra *et al.* (1998).
- If $b = 1, \psi \equiv 1$, we set $p = q = 0$, and $\phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ for $\rho = 0$ in (MP) and (MD), then we obtain a pair of first order symmetric dual nondifferentiable multiobjective programs considered by Mond *et al.* (1991).
- If we set, $b = 1, \psi \equiv 1, \phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))$ for $\rho = 0$ in (MP) and (MD), then we obtain a pair of second order symmetric dual nondifferentiable multiobjective programs considered by Ahmad and Husain (2005).

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