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A Fixed Point Theorem for φ-Contraction Mappings in Metric and D-metric Spaces

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Abstract: In this study we state a type of contraction and then establish a fixed point theorem, then adapted results in D-metric spaces.

Key words: Complete metric space, D-metric space, fixed point, ϕ-contraction

INTRODUCTION

Fixed point theorems give the conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis, topology and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena.

In particular fixed point techniques have been applied in such diverse fields as biol-ogy, chemistry, economics, engineering, game theory, and physics (Beg, 2006; Mai and Liu, 2007). Historically fixed point started in 1922, the Polish mathematician, Banach proved a theorem which ensures under appropriate conditions, the existence and uniqueness of a fixed point, his result is called Banach's contraction principle or Banach's fixed point theorem. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways and different spaces such as D-metric spaces (Sedghi *et al*., 2007). Jungck (1976) introduced more generalized commuting mappings called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorem.

A mapping $T: X \to X$, where (X,d) is a metric space, is said to be contraction if there exists $k \in (0,1)$ such that for all $x, y \in X$:

$$d(Tx, Ty) \leq kd(x,y) \quad (1)$$

Rhoades (2001) assumed a weakly contractive mapping $T: X \to X$ which satisfies the condition:

$$d(Tx, Ty) \leq d(x,y) - \phi(d(x,y)) \quad (2)$$

where, $x, y \in X$ and $\phi: [0, \infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0 \Leftrightarrow t = 0(t) = 0$. If one takes $\phi(t) = kt$ where $0<k<1$, then (2) reduces to (1).

Generalization of the above contraction mapping has been a very active field of research during recent years (Chidume, 2002; Berinde, 2003; Zhang, 2009).

In this study we introduce a type of ϕ contraction and a theorem state and prove it, then the results adapted in D-metric spaces.

MATERIALS AND METHODS

Basic definitions:

Definition: If $T: X \to X$ be a mapping on the metric space (X, d) and for all $x, y \in X$, T satisfying:

$$\Phi(d(Tx, Ty)) \leq \phi(d(x,y)) - t$$

where, $t>0$ is a constant and $\phi[0,\infty) \to [0,\infty)$ is a mapping, then we say T is a ϕ- Contraction map.

Definition: Let X be a nonempty set. A D-metric on X is a function, $D: X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$:

- $D(x, y, z) \geq 0$
- $D(x, y, z) = 0 \Leftrightarrow x = y = z$
- $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation on $\{x, y, z\}$
- $D(x, y, z) \leq D(x, y, a) + D(a, a, z)$ in this case we say the pair (X, D) is D-metric space

Immediate example of such a function is:

$$D(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

Lemma: Let (X, D) be a D-metric space, $x, y \in X$, then

$$D(x, x, y) = D(x, y, y)$$

Proof: It's immediate from the following inequalities:

$$D(x, x, y) \leq D(x, x, x) + D(x, y, y)$$

$$D(x, y, y) \le D(y, y, y) + D(x, x, y)$$

Definition: Let (X, D) be a D-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X:

- $\{x_n\}$ converges to *x*, if and only if $D(x_n, x_n, x) \to 0$
- $\{x_n\}$ is a Cauchy sequence if for all $\epsilon > 0$, there exists positive integer K such that for all $m, n \ge K$, $D(x_n, x_n, x_m) < \epsilon$

Lemma: Let (X, D) be a D-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X such that converges to *x*, than *x* is unique.

Proof: Let $\{x_n\}$ converges also to $y \ne x$. For each $\epsilon > 0$ there exists K_1, K_2 such that for all $n \ge K_1$, $D(x, x, x_n) < \epsilon/2$ and for all $n \ge K_2$, $D(y, y, x_n) < \epsilon/2$

Now if $n \ge \max\{K_1, K_2\}$, we have:

$$D(x, x, y) \le D(x, x, x_n) + D(y, y, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence $D(x, x, y) = 0$ which is a contradiction. So the limit is unique.

Lemma: Let (X, D) be a D-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X such that converges to *x*, then $\{x_n\}$ is a Cauchy sequence.

Proof: Since $\{x_n\}$ converges to x for each $\epsilon > 0$, there exists positive integer K such that for all $n \ge K$:

$$D(x_n, x, x) < \epsilon/2$$

Then for each $m, n \ge K$ we have:

$$D(x_n, x_n, x_m) \le D(x_n, x_n, x) + D(x_m, x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon$$

hence $\{x_n\}$ is a Cauchy sequence.

Definition: Let (X, D) be a D-metric space, then we say D is a continuous on X^3 if:

$$\lim_{n\to\infty} D(x_n, y_n, z_n) = D(x, y, x)$$

Whenever the sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to $(x, y, z) \in X^3$, that is:

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} z_n = z$$

Lemma: Let (X, D) be a D-metric space, then D is a continuous function on X^3.

Proof: Let $\epsilon > 0$ be arbitrary and $\{(x_n, y_n, z_n)\}$ in X^3 converges to $(x, y, z) \in X^3$, so there exists K_1, K_2 and K_3 such that for all $n \ge K_1$, $D(x, x, x_n) < \epsilon/2$, $n \ge K_2$, $D(y, y, y_n) < \epsilon/3$ and $n \ge K_3$ $D(z, z, z_n) < \epsilon/3$. Now if $n \ge \max\{K_1, K_2, K_3\}$, we have:

$$\begin{aligned} D(x_n, y_n, z_n) &\le D(x_n, y_n, z) + D(z, z, z_n) \\ &\le D(x_n, z, y) + D(y, y_n, y_n) + D(z, z_n, z_n) \\ &\le D(z, y, x) + D(x, x_n, x_n) + D(y, y_n, y_n) \\ &\quad + D(z, z_n, z_n) \\ &< D(x, y, z) + \epsilon/3 + \epsilon/3 + \epsilon/3 = D(x, y, z) + \epsilon \end{aligned}$$

hence $D(x_n, y_n, z_n) - D(x, y, z) < \epsilon$.
Similarly we can prove that:

$$D(x, y, z) - D(x_n, y_n, z_n) < \epsilon$$

Hence,

$$|D(x, y, z) - D(x_n, y_n, z_n)| < \epsilon$$

That is, $\lim_{n\to\infty} D(x_n, y_n, z_n) = D(x, y, z)$

Definition: Let (X, D) be a D-metric space. We say that X is a complete D-metric space if every Cauchy sequence in X^3 converges in X^3.

RESULTS

Theorem: If $T: X \to X$ be a mapping on complete metric space X, and for all x and y, T satisfying:

$$\phi(d(Tx, Ty)) \le \phi(d(x,y)) - t \qquad (1)$$

where $t \in (0, \infty)$ is a constant and the function $\phi: [0, \infty) \to [0, \infty)$ be a monotone nondecreasing and continuous on $(0, \infty)$, then T has an unique fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point, $x_1 = T(x_0)$ and $x_{n+1} = T(x_n)$ for all natural numbers. First we show that $\{x_n\}$ is a Cauchy sequence.

Put $x = x_n$ and $y = x_{n-1}$ in (1), so we have:

$$\phi(d(Tx_n, Tx_{n-1})) \le \phi(d(x_n, x_{n-1})) - t$$

But from definition of the sequence:

$$\phi(d(x_{n+1}, x_n)) \le \phi(d(x_n, x_{n-1})) - t \le \phi(d(x_n, x_{n-1}))$$

Since ϕ is monotone nondecreasing:

$$d(x_{n+1}, x_n) \le d(x_n, x_{n-1})$$

So the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and bounded in R, and this show that exists $r \ge 0$ such that:

$$d(x_{n+1}, x_n) \to r$$

as $n \to \infty$ now we show that $r = 0$. If $r > 0$, put $x = x_n$ and $y = x_{n-1}$ in (1), so:

$$\phi(d(x_{n+1}, x_n)) \leq \phi(d(x_n, x_{n-1})) - t$$

From continuity of ϕ at $(0, \infty)$, if $n \to \infty$
The above relation yield,

$$\phi(r) \leq \phi(r) - t$$

and this is a contradiction, so $r = 0$.

If $\{x_n\}$ is not a Cauchy sequence, there exists a $\epsilon > 0$ and a subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is smallest index for which $n(k) > m(k) > k$ and, $d(x_{n(k)}, x_{m(k)}) \geq \epsilon$. This implies that for all $k > 1$, $d(x_{n(k)-1}, x_{m(k)}) \geq \epsilon$.

Using the triangle inequality we have:

$$\epsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{n(k)}, x_{n(k)-1})$$

If $k \to \infty$ we obtain $\lim_{n\to\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon$

Put $x = x_{n(k)-1}$ and $y_{m(k)-1}$ in (1), we obtain:

$$\phi(d(x_{n(k)}, x_{m(k)})) \leq \phi(d(x_{n(k)-1}, x_{m(k)-1})) - t \qquad (2)$$

But,

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1})$$

So if $k \to \infty$, $\lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) \leq \epsilon$.

Hence from (2), continuity and monotone nondecreasing condition of ϕ we obtain:

$$\phi(\epsilon) \leq \phi(\epsilon) - t$$

That is a contradiction. So $\{x_n\}$ is a Cauchy sequence and since X is a complete metric space there exists $u \in X$ such that $x_n \to u$

If $Tu = u$ then conclusion holds. If $Tu \neq u$ put $x = u$ and $y = x_{n-1}$ in (1):

$$\phi(d(Tu, x_n)) \leq \phi(d(u, x_{n-1})) - t$$

But the above relation yield that:

$$d(Tu, x_n) \leq d(u, x_{n-1})$$

If $n \to \infty$, $d(Tu, u) \leq d(u, u)$, that is a contradiction, so $Tu = u$.

If $Tv = v$ and $u \neq v$, put $x = u$ and $y = v$ in (1), we have:

$$\phi(d(Tu, Tv)) \leq \phi(d(u, v)) - t$$

So, $\phi(d(u, v)) \leq \phi(d(u, v)) - t$, that is a contradiction. So u is a unique fixed point of T.

Theorem: If $T: X \to X$ be a mapping on complete D-metric space (X, D) and for all x and y, T satisfying:

$$\phi(D(Tx, Tx, Ty)) \leq \phi(D(x, x, y)) - t \qquad (2)$$

where $t \in (0, \infty)$ is a constant and the function $\phi: [0, \infty) \to [0,\infty)$ be a monotone nondecreasing and continuous on $(0, \infty)$. The T has a unique fixed point.

Proof: The proof is similar to the proof of before theorem, just use axiom of D-metric spaces and lemmas that prove in basic definition section.

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