

## A Reduction Factor for Buckling Load of Spherical Cap Shells

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**Abstract:** The classical buckling theory usually overestimates the buckling load of shells. In this study, a reduction factor is determined using geometrical parameters so as to reduce the classical buckling load to a more realistic value based on the post-buckling load. It is observed that the buckling load is directly proportional to the thickness and rise and inversely proportional to the span of the spherical cap. Finite element modeling and simulation using ABAQUS was conducted to determine the buckling behavior of a spherical cap shell subjected to different initial geometrical imperfections. The load-deflection curves drawn from the simulation formed a plateau at the post-buckling load. It is observed that as the initial geometrical imperfection is increased, the value of the initial buckling load is almost the same as the value of the post-buckling load on the plateau. The results obtained from different shells were used to derive a formula for the reduction factor.

**Key words:** Finite element, geometrical imperfections, geometrical parameters, initial buckling, plateau, post-buckling

### INTRODUCTION

Spherical cap shells are mostly used in roofs which span large unobstructed space. Their buckling load is calculated using the equations of a complete shell because their stress distribution is similar. The outstanding characteristics of shells are continuity and curvature (Calladine, 1983). These make them efficient load carrying capacity structures that carry transverse loading primarily by in-plane compression but their failure is mainly due to buckling. Therefore, the concern of many researchers has been to predict a more realistic buckling load because the classical buckling theory tends to overestimate this load (Prevost *et al.*, 1984; Zhu *et al.*, 2005). According to Timoshenko (1963), the classical buckling theory gives the buckling load as:

$$q_{cr} = \frac{2E}{\sqrt{3(1-\nu^2)}} \left( \frac{t}{R} \right)^2 \quad (1)$$

where,  $q_{cr}$  is the critical buckling load,  $R$  the radius of curvature,  $t$  the thickness of the shell wall,  $E$  the Young's modulus and  $\nu$  the Poisson's ratio of the material.

Many reasons have been given to explain the overestimation of the buckling load by the classical buckling theory. According to Teng (1996) and Forasassi and Frano (2007), geometrical parameters and initial imperfections are the major factors which contribute to this overestimation. The geometrical parameters for a

spherical cap include span ( $L$ ), rise ( $f$ ) and thickness ( $t$ ) which may be combined to form the  $f/L$  and  $t/L$  ratios. The classical buckling theory seem to have based on a complete shell which only has a radius of Curvature ( $R$ ) but a spherical cap has a span which is used to calculate  $R$ . There is an assumption that there is no need to consider  $L$  and  $f$  in the buckling load equation because they are included in the calculation of  $R$ , but it is proved in this study that  $R$  can be constant but have different  $f/L$  ratios which give different buckling loads.

In design of shells, safety factors are always introduced to check the buckling; for example a factor of safety of 2 was introduced over the working loads of Swindon dome (Makowski, 1984). The focus in this study is to determine a reduction factor that will be used to adjust the classical buckling load to a more realistic value which is the same as the values found during experiments on the buckling load of shells. Since the post-buckling load from load-deflection curves tends to be close to experimental values, this load will be used. The reduction factor will be expressed as a function of geometrical parameters because these parameters can be easily identified and in they affect the buckling load in their combined state.

### FINITE ELEMENT (FE) MODELING OF A SPHERICAL CAP SHELL

The structural behavior of shell structures during the whole loading process can be revealed by the load-

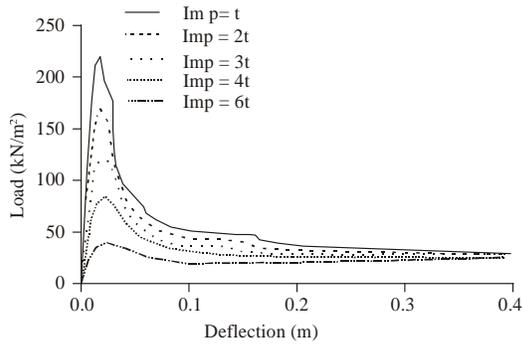


Fig. 1: Load-deflection curves for different initial geometrical imperfections;  $R = 21.25$  m,  $t = 0.02$  m,  $L = 20$  m,  $f = 2.5$  m,  $E = 206 \times 10^9$  N/m<sup>2</sup> and  $\nu = 0.3$

deflection curves and the buckling load can be predicted with sufficient accuracy using FE analysis (Fan *et al.* 2010). Using FE analysis also produces the optimum product; a product that is the least costly to produce, performs as intended and meets all of the specified requirements (Champion, 1992). Therefore, FE modeling was conducted for a steel spherical cap shell;  $L = 20$  m,  $t = 0.02$  m,  $f = 2.5$  m, Modulus of elasticity ( $E$ ) =  $206 \times 10^9$  N/m<sup>2</sup> and Poisson’s ratio ( $\nu$ ) = 0.3. Using ABAQUS, the shell was meshed with triangular shell elements-S3R (Hibbit, 2000a) and the Riks method (Hibbit, 2000b) with different magnitude of the initial geometrical imperfections (imp) was used to trace the load-deflection curves. The curves are shown in Fig. 1.

It is observed that when the initial geometrical imperfections are small, the initial buckling load is almost the same as the classical buckling load but as the imperfections are increased, the load drops significantly. This confirms the observation by Zhou *et al.* (2010) that a shell is imperfection sensitive. All the curves tend to meet at the plateau irrespective of their initial geometrical imperfections. This load is also closer to what is observed in experiments. Therefore this load gives a more realistic prediction of the buckling load. It is also observed that as the initial geometrical imperfection is increased, the initial buckling load tends to be the same as the plateau load. It will be assumed that the realistic initial buckling load is the one which is the same as the ‘plateau load’.

### RELATIONSHIP BETWEEN THE GEOMETRICAL PARAMETERS AND THE BUCKLING LOAD

**Test for individual parameters; L, f and t:** Using Eq. (1), the relationship of L, f and t and the buckling load was tested by varying the parameter while keeping all other parameters constant. To test for L, t and f were kept at 0.02 and 5 m, respectively, while changing L from

Table 1: Relationship between the geometrical parameters and the classical buckling load

L(m)	t(m)	f(m)	R(m)	Classical theory $q_{cr}$ (kN /m <sup>2</sup> )
20	0.02	2.5	21.25	221
20	0.03	2.5	21.25	497
20	0.02	5.0	12.50	638
30	0.02	5.0	25.00	160
30	0.02	3.0	39.00	66
30	0.02	3.75	31.875	98

20 to 30 m. To test for f, L and t were kept at 20 and 0.02 m, respectively, while changing f from 3 to 3.75 m. To test for t, L and f were kept constant at 20 and 2.5 m respectively, while changing t from 0.02 to 0.03 m. The results are shown in Table 1.

It is observed that the buckling load is directly proportional to t and f but inversely proportional to L. The relationship is not linear; therefore it can be expressed as:

$$q_{cr} \propto \frac{t^x f^x}{L^z} \tag{2}$$

**Test for t/L and f/L ratios:** From Table 1, it is observed that the initial buckling load (classical buckling load) is directly proportional to t/L and f/L ratios. To further test the effect of f/L on the initial buckling load and the post-buckling load, R and t were kept constant at 12.5 m and 0.02 m respectively and FE modeling conducted with varying f/L. The load-deflection curves from the modeling are shown in Fig. 2. An interesting observation is that f/L is directly proportional to the initial buckling load but inversely proportional to post-buckling load because the curves with lower f/L ratio are seen to change to the top at the ‘plateau’ of the post-buckling load.

**Derivation of the reduction factor ( $\alpha$ ) using FE modeling:** Since it is easy to identify the geometrical parameters,  $\alpha$  will be derived to be a function of these parameters in form of Eq. (2) as:

$$\alpha = k \frac{t^x f^x}{L^z} \tag{3}$$

So that the realistic buckling load is given by:

$$q_{cr} = \alpha \frac{2E}{\sqrt{3(1-\nu^2)}} \left(\frac{t}{R}\right)^2 \tag{4}$$

To determine the values of k, x, y and z, FE modeling for shells with different geometrical parameters was done and the load-deflection from each modeling shown in Fig. 3. A summary of the results are shown in Table 2.

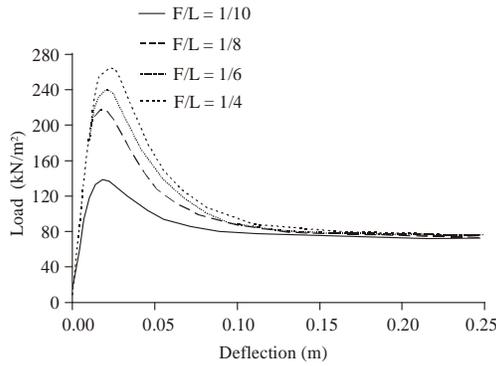


Fig. 2: Load-deflection curves for different f/L; R = 12.5m, t = 0.02m, E = 206 × 10<sup>9</sup> N/m<sup>2</sup>

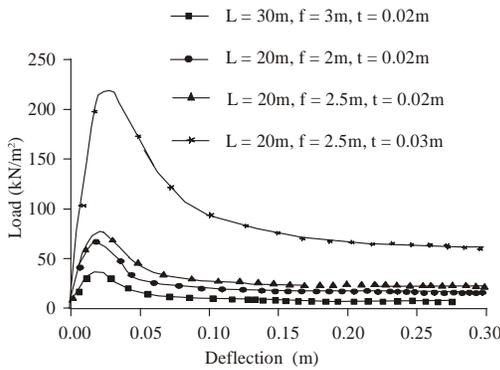


Fig. 3: Load-deflection curves for different shells

Substituting the parameters in Eq. (4) using the shell where, L = 20 m, t = 0.02 m and f = 2.5 m:

$$k \frac{0.02^x \times 2.5^y}{L^z} \times 220881 = 25000 \quad (5)$$

Substituting the parameters in Eq. (4) using the shell where L = 20 m, t = 0.03 m and f = 2.5 m:

$$k \frac{0.03^x \times 2.5^y}{L^z} \times 496982 = 69000 \quad (6)$$

Dividing Eq. (6) by Eq. (5):

$$\frac{0.03^x}{0.02^x} \times 2.25 = 2.76 \quad (7)$$

Solving for x using logarithms:

$$x = -0.5$$

Table 2: Buckling load of different shells from the FE modeling

L(m)	t(m)	f(m)	Observed q cr (kN /m <sup>2</sup> )	Classical theory q cr (kN /m <sup>2</sup> )
30	0.02	3.0	9700	65576
20	0.02	2.0	18000	147547
20	0.02	2.5	25000	220881
20	0.03	2.5	69000	496982

Using the same procedure as shown in Appendix to solve for the other unknowns:

$$k = 1.2, y = 0.7, z = 0.35$$

Therefore,

$$\alpha = \frac{1.2t^{0.5} f^{0.7}}{L^{0.35}} \quad (8)$$

**Validation of the reduction factor (α):** The buckling load from the experiments done by Carlson *et al.* (1966) on spherical shells ranged from 10 to 19% of the classical buckling theory load. Chia (1971) observed that theoretical results of the buckling load by many researchers range approximately from 7 to 67% of the classical buckling. As mentioned by Narayanan (2006), a deflection-to-thickness ratio of three which may be caused by some imperfections in the shells has been shown both experimentally and theoretical to reduce the buckling load of a thin shell to about 5% of the classical buckling load.

To test the validity of the findings in this study, the buckling load of several different shells was calculated using α and the value expressed as a percentage of the classical buckling load. The results are shown in Table 3. It is observed that α gives a buckling load which ranges between 11-18% of the classical buckling load which is within the range by other researchers. Therefore α is valid.

### CONCLUSION

The initial geometrical imperfections in real construction may be difficult to be quantified because they are introduced during construction. Therefore, it is safer to use a more realistic buckling load during the design stages. The post-buckling load on the plateau is independent from any imperfection so reducing the classical buckling load to that of the plateau load ensures that there is no overestimation. This has been done by using a reduction factor (α) which is a function of L, t and f. The relationship between these geometrical parameters has been established so as to be used to derive α. The reduced classical buckling load has been found to be in the range of loads obtained by other researchers. It has been observed that the way the individual parameters relate to the buckling load may not be the same when they are combined to form ratios. The f/L ratio is directly

Table 3: Percentages of loads given by the reduction factor as compared to classical buckling load

f/L	L(m)	t(m)	f(m)	Realistic q cr (kN /m <sup>2</sup> )	Classical theory q cr (kN /m <sup>2</sup> )	Percentage (%)
1/4	20	0.02	5.00	117	638	18
1/6	15	0.02	2.50	80	638	13
1/8	30	0.03	3.75	29	221	13
1/10	30	0.03	3.00	17	148	11

proportional to the classical buckling load but inversely proportional to the post-buckling load at the plateau. Since this study has mainly focused on the classical buckling load, there is need to determine how the geometrical parameters relate to post-buckling load so that the findings can be extended to determine the buckling load of reticulated shells using continuum analogy method.

**Appendix:** Solving for k, y and z  
Using the shell where L = 20 m, t = 0.02 m, f = 2.5 m:

$$k \frac{0.02^{0.5} \times 2.5^y}{L^z} \times 220881 = 25000$$

$$\frac{k \times 2.5^y}{L^z} = 0.8003254 \tag{1}$$

Using the shell where L = 20 m, t = 0.02, f = 2 m:

$$k \frac{0.02^{0.5} \times 2^y}{20^z} \times 147547 = 18000$$

$$\frac{k \times 2^y}{20^z} = 0.8626374 \tag{2}$$

Dividing Eq. (2) by Eq. (1):

$$1.25^y = 0.9277656$$

$$y = 0.7$$

Using the shell where L = 20 m, t = 0.02, f = 2.5 m:

$$k \frac{0.02^{0.5} \times 2.5^{0.7}}{L^z} \times 220881 = 25000$$

$$\frac{k}{20^z} = 0.4214136 \tag{3}$$

Using the shell where L = 30 m, t = 0.02, f = 3 m:

$$k \frac{0.02^{0.5} \times 3^{0.7}}{30^z} \times 65576 = 9700$$

$$\frac{k}{30^z} = 0.484760 \tag{4}$$

Dividing Eq. (4) by Eq. (3):

$$30^z/20^z = 1.15$$

Introducing logs:

$$z \log 30 - z \log 20 = z \log 1.15$$

$$0.17609z = 0.0608183$$

$$z = 0.35$$

Substituting the value of z in Eq. (3) and solving:

$$k = 1.2$$

### REFERENCES

Calladine, C.R., 1983. Theory of Shell Structures. Cambridge University Press, Cambridge.

Carlson, R.L., R.L. Sendelbeck and N.J. Hoff, 1966. Experimental studies of the buckling of complete spherical shells. Exp. Mech., 7: 281-288.

Champion, E.R., 1992. Finite Element Analysis in Manufacturing Engineering. APC- Based Approach, McGraw-Hill, Inc., New York.

Chia, C.Y., 1971. Buckling of thin spherical shells. Ingenieur-Archiv, 40: 227-237.

Fan, F., Z. Cao and S. Shen, 2010. Elasto-plastic stability of single-layer reticulated shells. Thin walled Struct., 48: 827-836.

Forasassi, G. and R.L. Frano, 2007. Curved thin shell buckling behavior. J. Achiev. Mater. Manuf. Eng., 23(2): 55-58.

Hibbit, Karlsson & Sorensen Inc., 2000a. ABAQUS Theory Manual Version 6.1.

Hibbit, Karlsson & Sorensen Inc., 2000b. ABAQUS user's Manual Version 6.1.

Makowski, Z.S., 1984. Analysis, Design and Construction of Braced Domes, Granada, London.

Narayanan, S., 2006. Space Structures: Principles and Practice. Multi-Science Publishing Co. Ltd., Brentwood.

Prevost, J.H., D. P. Billington, R. Rowland and C.C. LIM, 1984. Buckling of Spherical dome in a centrifuge. Exp. Mech., pp: 203-207.

Teng, G.J., 1996. Buckling of thin shells: Recent advances and trends. Appl. Mech., 49: 263-274.

Timoshenko, S. P. and J.M. Gere, 1963. Theory of Elastic Stability, McGraw-Hill Book Company, London.

Zhou, H.Z., F. Fan and E.C. Zhu, 2010. Buckling of reticulated laminated veneer lumber shells in consideration of the creep. Eng. Struct., 32: 2912-2918.

Zhu, E.C., Z.W. Guan, P.D. Rodd and D.J. Pope, 2005. Buckling of oriented strand board webbed wood I-joists. J. Struct. Eng., 131(10): 1629-1636.