Properties of $T_\delta$-spaces and pairwise $T_\delta$-spaces

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Abstract: The separation axioms play an important role in the application of topological spaces. The concepts of $T_\delta$-spaces and pairwise $T_\delta$-spaces are introduced and studied by Chandrasekhar Rao and Narasimhan (2006). The aim of this study is to continue the study of some characterizations of $T_\delta$-spaces and pairwise $T_\delta$-spaces in topological and bitopological spaces.

Keywords: $T_\delta$-space, Pairwise $T_\delta$-space, pairwise $gT_\delta$-space, pairwise $\delta g_{\alpha \delta}$-space, pairwise $\alpha g_{\gamma}$-space, $T_\delta$-space, $gT_\delta$-space.

INTRODUCTION

Levine (1963, 1970) introduced semi open sets and g-closed sets. Njastad (1965) introduced $\alpha$-open sets. Maki et al. (1993a,b) introduced $\alpha$ g-closed sets. Bhattacharyya and Lahiri (1987), Arya and Nour (1988), Dontchev (1995), Dontchev and Ganster (1996), Gnanambal (1997) and Chandrasekhar Rao and Joseph (2000) investigated $g_{\alpha}$-closed sets, $g_{\delta}$-closed sets, $g_{\delta}$-closed sets, $g_{\delta}$-closed sets, $g_{\delta}$-closed sets, $g_{\delta}$-closed sets, $g_{\delta}$-closed sets, $g_{\delta}$-closed sets, respectively. Maki et al. (1993) introduced $T_{\alpha}, T_{\delta}$ and $T_{\alpha \delta}$ spaces respectively. Chandrasekhar Rao and Thangavelu (2003) studied complemented spaces. Veera Kumar (2000, 2002, 2006a,b) introduced $T_{\alpha \delta}, T_{\delta}, T_{\alpha \delta}$ spaces. Chandrasekhar Rao and Narasimhan (2007, 2009) introduced $T_\delta$-spaces.

Meanwhile, Kelly (1963) introduced the concept of bitopological spaces by using quasi metric space as a natural structure. Further work in this area were done by Fletcher (1965), Lane (1967), who introduced pairwise regularity independently. The concept of pairwise $T_\gamma$ (pairwise semi Hausdorff) was introduced by Kim (1968).

The concept of pairwise $T_\gamma$-space was initiated by Sunder Lal and Gupta (1999) and they classified some of pairwise $T_\gamma$-axioms by affixing strong, weak, minimally and almost. The $\alpha, \beta, \delta, \gamma$ closed sets were introduced by Sheik and Sundaram (2004). Rajamani and Vishwanthan (2005) introduced $\alpha$ $g_{\delta}$-closed sets and defined new spaces known as $T_\gamma, T_\delta, T_\alpha, T_{\alpha \delta}$-spaces and investigated some of their properties.

The concept of pairwise complemented spaces (2006) and $T_\delta$-spaces are introduced and studied by Chandrasekhar Rao and Narasimhan (2008). The aim of this paper is to continue the study of some characterizations of $T_\delta$-spaces and pairwise $T_\delta$-spaces in topological and bitopological spaces.

PRELIMINARIES

Let $(X, \tau)$ or simply $X$ denote a topological space. For any subset $A \subseteq X$, the closure [resp. $\delta$-closure, $\alpha$-closure] of a subset $A$ of a space $(X, \tau)$ is the intersection of all closed [resp. $\delta$-closed, $\alpha$-closed] sets that contain $A$ and is denoted by $cl(A)$ [resp. $cl_\delta(A), cl_\alpha(A)$]. We shall require the following known definitions.

Definition: A set $A$ of a topological space $(X, \tau)$ is called

- **Semi open** if there exists an open set $U$ such that $U \subseteq A \subseteq cl(U)$
- **Semi closed** if $X-A$ is semi open
- **Generalized closed** (g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$
• Generalized semi open (gs-open) if \( F \subseteq \text{ints}(A) \) whenever \( F \subseteq A \) and \( F \) is closed in \( X \)

• Generalized semi closed (gs-closed) if \( X - A \) is gs-open,

• Semi star generalized closed (s'g-closed) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi open in \( X \)

• \( \alpha \)-open if \( A \subseteq \text{int} \{ \text{cl}(A) \} \}

• \( \alpha \)-closed if \( \text{cl} \{ \text{int}(A) \} \subseteq A \)

• \( \alpha \) gs-closed if \( \alpha \text{ cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \)

• \( \alpha \) gs-open if \( \alpha \text{ cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \)

PROPERTIES OF \( T_\varsigma \)-SPACES

A topological space \((X, \tau)\) is called \( T_\varsigma \)-space if every \( s'g \)-closed set is closed in \( X \). Let \( X = \{a, b, c\}, \tau = \{\varnothing, X, \{a\}\} \). Then \( s'g \)-closed sets in \( X \) are \( \varnothing, X, \{b, c\} \), which are closed in \( X \). Hence \((X, \tau)\) is \( T_\varsigma \)-space.

Theorem: If \((X, \tau)\) is a \( T_\varsigma \)-space and \( T_1^\Omega \)-space then \( X \) is a \( T_\varsigma \)-space.

Proof: Let \( A \) be \( s'g \)-closed in \( X \). Then \( A \) is \( s'g \)-closed. Since \( X \) is \( T_\varsigma \)-space, \( A \) is \( s'g \)-closed in \( X \). Since \( X \) is a \( T_1^\Omega \)-space, we have \( A \) is closed in \( X \). Hence \( X \) is a \( T_\varsigma \)-space.

Theorem: If \((X, \tau)\) is both \( \Omega \)-space and \( T_1^\Omega \)-space then \( X \) is a \( T_\varsigma \)-space.

Proof: Let \( A \) be \( s'g \)-closed in \( X \). Then \( A \) is \( s'g \)-closed. Since \( X \) is \( \Omega \)-space, \( A \) is \( s'g \)-closed in \( X \). Since \( X \) is a \( T_1^\Omega \)-space, we have \( A \) is closed in \( X \). Hence \( X \) is a \( T_\varsigma \)-space.

Theorem: Suppose \( X \) is a \( s'T_\varsigma \)-space and \( T_1^s \)-space then \( X \) is a \( T_\varsigma \)-space.

Proof: Let \( A \) be \( s'g \)-closed set. Then \( A \) is \( s'g \)-closed. Since \( X \) is a \( s' \)-space, \( A \) is \( s'g \)-closed. Since \( X \) is a \( T_\varsigma \)-space, we have \( A \) is closed in \( X \). Hence \( X \) is a \( T_\varsigma \)-space.

Theorem: Suppose \( X \) is a \( s'T_\varsigma \)-space and \( T_1^s \)-space then \( X \) is a \( T_\varsigma \)-space.

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Theorem: Suppose \( X \) is a \( s' \)-space and \( T_1^s \)-space then \( X \) is a \( T_\varsigma \)-space.

Proof: Let \( A \) be \( s'g \)-closed set. Then \( A \) is \( s'g \)-closed. Since \( X \) is a \( s' \)-space, \( A \) is \( s'g \)-closed. Since \( X \) is a \( T_\varsigma \)-space, we have \( A \) is closed in \( X \). Hence \( X \) is a \( T_\varsigma \)-space.
Theorem: Suppose $X$ is a $\tau_T$-space and $\tau_T^g$ -space then $X$ is a $T_s$-space.

Proof: Let $A$ be a $g$-closed set. Then $A$ is $g$-closed. Since $X$ is a $T_{1\frac{1}{2}}$-space, we have $A$ is $g$-closed. Since $X$ is $\tau_T^g$-space, we have $A$ is closed. Hence, $X$ is a $T_s$-space.

Theorem: Suppose $X$ is a complimented space. If a subset $A$ of $X$ is $\delta g$-closed, then $A$ is $\delta g^*$ -closed.

Proof: Suppose $X$ is a complimented space. Let $A$ be a $\delta g$-closed set. Let $A \subseteq U$, $U$ is semiopen in $X$. Since $X$ is a complimented space, $U$ is open in $X$. Since $A$ is $\delta g$-closed, $cl_h (A) = U$. Hence $A$ is $\delta g^*$ -closed.

PROPERTIES OF PAIRWISE $T_s$-SPACES

First we recall some known definitions.

Definition:
A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called:

1. $\tau_1 \tau_2$-generalized closed if $\tau_2-ccl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-open in $X$.
2. $\tau_1 \tau_2$-semi generalized closed if $\tau_2-scl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-semi open in $X$.
3. $\tau_1 \tau_2$-generalized semi closed if $\tau_2-gscl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-open in $X$.
4. $\tau_1 \tau_2$-open if $A \subseteq \tau_1-int \{ \tau_2-cl \{ \tau_1-int (A) \} \}$.
5. $\tau_1 \tau_2$-$\alpha$ closed if $\tau_2-ccl \{ \tau_1-int \{ \tau_2-cl (A) \} \} = U$.
6. $\tau_1 \tau_2$-$\alpha$ $gs$ closed if $\tau_2-ccl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-open.
7. $\tau_1 \tau_2$-$\delta g$ closed if $\tau_2-ccl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-semi open in $X$.
8. $\tau_1 \tau_2$-$\delta g^*$ closed if $\tau_2-ccl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-$\delta g$ open.
9. $\tau_1 \tau_2$-$\Omega$ closed if $\tau_2-ccl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-$\Omega$ open.
10. $\tau_1 \tau_2$-$\alpha$ $gs$ closed if $\tau_2-scl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-$\alpha$ $g$ open.
11. $\tau_1 \tau_2$-$\delta g$ closed if $\tau_2-scl (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-$\delta g$ open.
12. $\tau_1 \tau_2$-$\Omega$ closed if $\tau_2-scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-$\Omega$ open.

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{sa}$-space if every $\tau_1 \tau_2$-$s g$ closed set is $\tau_s$-closed and every $\tau_1 \tau_2$-$g$ closed set is $\tau_{s'}$-closed.

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{ga}$-space if every $\tau_1 \tau_2$-$g s$ closed set is $\tau_{g'}$-closed and every $\tau_1 \tau_2$-$s$ closed set is $\tau_{g''}$-closed.

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{g a}$-space if every $\tau_1 \tau_2$-$s g$ closed set is $\tau_{g'}$-closed and every $\tau_1 \tau_2$-$g$ closed set is $\tau_{g''}$-closed.

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{gs}$-space if every $\tau_1 \tau_2$-$g s$ closed set is $\tau_{g'}$-closed and every $\tau_1 \tau_2$-$s$ closed set is $\tau_{g''}$-closed.

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{s g}$-space if every $\tau_1 \tau_2$-$s g$ closed set is $\tau_{s'}$-closed and every $\tau_1 \tau_2$-$g$ closed set is $\tau_{s''}$-closed.

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{ss}$-space if every $\tau_1 \tau_2$-$s g$ closed set is $\tau_{ss'}$-closed and every $\tau_1 \tau_2$-$g$ closed set is $\tau_{ss''}$-closed.

MAIN RESULTS

Definition: A bitopological space $(X, \tau_1, \tau_2)$ is called a pairwise $T_{sa}$-space if every $\tau_1 \tau_2$-$s g$ closed set is $\tau_{s'}$-closed and every $\tau_1 \tau_2$-$g$ closed set is $\tau_{s''}$-closed.

Example: Let $X = \{ a, b, c \}$, $\tau_1 = \{ \emptyset, X, \{ a \} \}$, $\tau_2 = \{ \emptyset, X, \{ a, c \} \}$. Then $(X, \tau_1, \tau_2)$ is a pairwise $T_{sa}$-space.

The necessary and sufficient condition for a bitopological space to be a pairwise $T_{sa}$-space is obtained in the following theorem.

Theorem: A bitopological space $(X, \tau_1, \tau_2)$ is a pairwise $T_{sa}$-space if and only if the singleton $\{ x \}$ is either $\tau_1$-open or $\tau_2$-$\tau_1$-$s$-semi closed, $i, j = 1, 2$ and $i \neq j$.

Proof: Let $X$ be a pairwise $T_{sa}$-space and suppose that $\{ x \}$ is not $\tau_{s'}$-semi closed. Then $X - \{ x \}$ is not $\tau_{s'}$-open. Consequently $X$ is the only $\tau_{s'}$-open set containing the set $X - \{ x \}$. Therefore, $X - \{ x \}$ is $\tau_{s'}$-$s g$ closed in $X$. Since $X$ is a pairwise $T_{sa}$-space, we have $X - \{ x \}$ is $\tau_{s'}$-closed in $X$. Consequently, $\{ x \}$ is $\tau_{s'}$-open in $X$. 

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Conversely, suppose that \( \{x\} \) is either \( \tau_i \)-open or \( \tau_j \)-semi closed, \( i, j = 1, 2 \) and \( i \neq j \). Let \( A \) be a \( \tau_i \)-\( \tau_j \)-\( \delta \)-\( g \) closed set in \( X \). Obviously \( A \subset \tau_{1\delta} \text{cl} (A) \). Let \( x \in \tau_{1\delta} \text{cl} (A) \).

**Case i:** Suppose that \( \{x\} \) is \( \tau_i \)-open. Since \( x \in \tau_{1\delta} \text{cl} (A) \), we have \( x \in A \). Thus, \( \tau_{1\delta} \text{cl} (A) \subset A \).

**Case ii:** Suppose that \( \{x\} \) is \( \tau_i \)-semi closed and \( x \notin A \). Then \( \tau_{1\delta} \text{cl} (A) \setminus A \) contains the \( \tau_i \)-semi closed set \( \{x\} \). This is a contradiction to the fact that \( A \) is \( \tau_i \)-\( \tau_j \)-\( \delta \)-\( g \) closed in \( X \). Hence, \( x \in A \), implies that \( \tau_{1\delta} \text{cl} (A) \subset A \). Therefore, \( \tau_{1\delta} \text{cl} (A) = A \).

Similarly, we can prove every \( \tau_i \)-\( \tau_j \)-\( \delta \)-\( g \) closed set is \( \tau_i \)-closed. Hence \( X \) is a pairwise \( T_{1\delta} \)-space.

**Theorem:** If a bitopological space \((X, \tau_i, \tau_j)\) is pairwise \( T_{1\delta} \)-space and \((X, \tau_i, \tau_j)\) is \( \tau_i^{*} \), \( i = 1, 2 \), then \( X \) is a pairwise \( T_{1\delta} \)-space.

**Proof:** Let \( A \) be a \( \tau_i \)-\( \tau_j \)-\( \delta \)-\( g \) closed set in \( X \), \( i, j = 1, 2 \) and \( i \neq j \). Then \( A \) is a \( \tau_i \)-\( \tau_j \)-\( \delta \)-\( g \) closed set in \( X \). Since \( X \) is a pairwise \( T_{1\delta} \)-space, we have \( A \) is \( \tau_i \)-\( \tau_j \)-\( \delta \)-\( g \) closed in \( X \).

Thus, we have studied some more characterizations of \( T_{1\delta} \)-spaces in both unital and bitopological spaces. In addition, the necessary and sufficient condition for a bitopological space to be a pairwise \( T_{1\delta} \)-space is obtained.

**CONCLUSION**

Thus, we have studied some more characterizations of \( T_{1\delta} \)-spaces in both unital and bitopological spaces. In addition, the necessary and sufficient condition for a bitopological space to be a pairwise \( T_{1\delta} \)-space is obtained.

**REFERENCES**


