

## Analysis of a New Joint Time-Frequency Distribution of Suppressing Cross-Term

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**Abstract:** In this study, we proposed a joint time-frequency distribution of suppressing cross-terms aiming at cross-terms interference of Wigner-Ville distribution. According to the analyzed signal, the Wigner-Ville distribution and Smooth Pseudo Wigner-Ville distribution can be obtained. Moreover, the prototype of time-frequency distribution can be obtained by non-linear filter for Smooth Pseudo Wigner-Ville distribution. In this way, the distributions the joint time-frequency distribution came to the result of the product of two. As a result, the joint time-frequency distribution can suppress cross-terms effectively and keep high time-frequency resolution. Finally, simulations and experiments prove the effectiveness of the proposed method.

**Key words:** Cross-terms, joint time-frequency distribution, non-linear filter, wigner-ville distribution

### INTRODUCTION

In signal processing, time-frequency analysis (Goswami and Chan, 2011; Beamish *et al.*, 1979) comprises those techniques that study a signal in both the time and frequency domains simultaneously, using various time-frequency representations. Rather than viewing a 1-dimensional signal (a function, real or complex-valued, whose domain is the real line) and some transform (another function whose domain is the real line, obtained from the original via some transform), time-frequency analysis studies a two-dimensional signal-a function whose domain is the two-dimensional real plane, obtained from the signal via a time-frequency transform. The basic idea of time frequency analysis is to design the joint function of time and frequency, which can describes the density or intensity of signal in different time and different frequency (Anderson *et al.*, 2012). Such function of Time and frequency is called time-frequency distribution. Using time-frequency distribution to analyze the signal not only can give the instantaneous frequency and amplitude, but also can carry out the time-frequency filtering and study the time varying signal.

There are several different ways to formulate a valid time-frequency distribution function, resulting in several well-known time-frequency distributions, such as (Boashash, 2003): Short-time Fourier transform (including the Gabor transform); Wavelet transform; Bilinear time-frequency distribution function (Wigner distribution function); Modified Wigner distribution function, Gabor-Wigner distribution function, and so on (see Gabor-Wigner transform).

The time-bandwidth product of Wigner-Ville Distribution (WVD) reaches the lower bound given by

Heisenberg uncertainty principle (Zhou and Chen, 2012), which have characteristics of high time-frequency resolution, energy concentration, satisfying frequency edge and so on. But at the same time there are cross-term interferences when the WVD is used as a quadratic time-frequency distribution with excellent features. The cross-term will produce false signal between every component of signal, which severely limits the application of WVD. Many scholars have studied the WVD, but most are trade-off process between overcoming the cross-term interference and maintaining the time-frequency resolution. Literature (Zou *et al.*, 2001) prove mathematically that there is no such time-frequency distribution not including cross-term interference and with WVD aggregation, also prove that there is generally no such time-frequency distribution under wide conditions, which is not including of cross-terms interference and the aggregation is sufficiently close to WVD. Based on the theory, this paper studies the method of cross-terms suppression, not from the view of mathematics, but combining WVD and smooth pseudo WVD (SPWVD) (Ning and Wang, 2011; Miroslaw *et al.*, 2010), an analysis method of joint time-frequency distribution is proposed. The smooth pseudo WVD is nonlinear filtered, to obtain time-frequency distribution prototype, then, product WVD by smooth pseudo WVD after filtered, we can have the time-frequency distribution results. The method takes full advantages of both, which can not only maintain the time-frequency aggregation of WVD, but also suppress the cross-term interference. At last, simulations and vibration data of a certain type of missile under water proved the effectiveness of this method.

This study analyzes a joint time-frequency distribution of suppressing cross-terms. Based on the

analyzed signal, the Wigner-Ville distribution and Smooth Pseudo Wigner-Ville distribution could be obtained, and the prototype of time-frequency distribution can be obtained by non-linear filter for Smooth Pseudo Wigner-Ville distribution. In this way, the distributions the joint time-frequency distribution came to the result of the product of two, so that the joint time-frequency distribution can suppress cross-terms effectively and keep high time-frequency resolution. Furthermore, simulations and experiments prove the effectiveness of our proposed method.

**METHODOLOGY**

**WVD and cross-terms:** WVD is the two-dimensional joint function of time and frequency, which can be seen as energy density function of time and frequency on 2-Dimensional plane. For any measurable and square integrable signal  $s \in L^2(\mathbb{R})$ , the definition of WVD is (Wigner, 1932):

$$W_s(t, f) = \int_{-\infty}^{\infty} s\left(t + \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \quad (1)$$

From Eq. (1), it can be seen that  $s(t)$  appear two times, and it does not contain any window function, thus avoid mutual constraint problem of time resolution and frequency resolution in the linear time-frequency analysis. We also know that WVD belongs to a class of quadratic form (or bilinear) time-frequency distributions, when the signal to be analyzed is composed by many component, it is necessary to produce cross-terms:

Let  $s \in L^2(\mathbb{R})$ ,  $s(t) = s_1(t) + s_2(t)$ , then

$$W_s(t, f) = W_{11}(t, f) + W_{22}(t, f) + W_{12}(t, f) + W_{21}(t, f) \quad (2)$$

where,

$$W_{12}(t, f) = \int_{-\infty}^{\infty} s_1^*\left(t - \frac{\tau}{2}\right) s_2\left(t + \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \quad (3)$$

we can prove:

$$W_{12}^*(t, f) = W_{21}(t, f) \quad (4)$$

Eq. (2) can be rewritten as:

$$W_s(t, f) = W_{11}(t, f) + W_{22}(t, f) + 2\text{Re}[W_{12}(t, f)] \quad (5)$$

where  $W_{11}(t, f)$  and  $W_{22}(t, f)$  is characterized by self-terms energy distribution of  $s_1(t)$  and  $s_2(t)$  respectively,  $2\text{Re}[W_{12}(t, f)]$  is cross-term of WVD.

From the physical point of view,  $W_s(t, f)$  is the signal energy distribution in the time  $t$  and at the frequency  $f$ , so it is expected that  $W_s(t, f)$  is sum of  $W_{11}(t, f)$  and  $W_{22}(t, f)$ , the cross-term  $2\text{Re}[W_{12}(t, f)]$  only provide false spectral

value distribution, and it directly influence the physical explanation of WVD. If the signal is mono-component, then the WVD can accurately depict 2-D time-frequency feature of the signal. But when the signal is composed by many components, the existence of cross-term will pollute WVD, which limits the engineering application of WVD to a large degree.

**Method of suppressing cross-terms:** From the above analysis, it can be known that the cross-term of signal is real, mixing with self-terms, and the amplitude is 2 times of self-terms, moreover, the cross-term is oscillatory type (Cohen, 1994; Cohen, 1999), each two components can produce a cross-term. If the signal contains  $N$  components, it will produce  $C_N^2$  cross-terms (Zou *et al.*, 2000). In order to suppress cross-terms of WVD, 6 types of time-frequency distributions have been proposed, they are Cohen class bilinear time-frequency distribution, linear time-frequency distribution, affine class of bilinear time-frequency distribution, rearrangement class of bilinear time-frequency distribution, adaptive kernel function class time-frequency distribution, as well as parametric time-frequency distribution. These distributions are quite different, but they basically are compromise between keeping self-term and suppressing cross-terms.

Here pseudo WVD and SPWVD used in the paper will be introduced. Essentially, WVD and SPWVD can be written in the unified form of Cohen class time-frequency:

$$C(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s\left(u + \frac{\tau}{2}\right) s^*\left(u - \frac{\tau}{2}\right) \phi(\tau, \nu) e^{-j2\pi(\nu + f\tau - u\nu)} d\nu d\tau d\xi$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod(\eta - t, \xi - f) W_s(\eta, \xi) d\eta d\xi \quad (6)$$

where,

$$\prod(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\xi, \tau) e^{-2j\pi(f\tau + \xi\tau)} d\tau d\xi \quad (7)$$

$\prod(\cdot)$  can be considered as a smooth function, thus Cohen class time-frequency distribution can be interpreted as WVD after smoothing. When kernel function  $\phi(\xi, \tau) = \delta(\xi, \tau)$  it is WVD.

For pseudo WVD, in fact, it is WVD added windows and smoothed in frequency, define:

$$PWVD_s(t, f) = \int_{-\infty}^{\infty} h(\tau) s\left(t + \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{\infty} H(f - \xi) W_s(t, f) d\xi \quad (8)$$

where  $H(f)$  is the Fourier transform of  $h(\tau)$ .

In order to control smooth windows in the time domain and frequency domain independently, combining Cohen class time-frequency distribution with that of pseudo WVD, smooth window function can be defined as:

$$\Pi(t,f) = g(t)H(-f) \tag{9}$$

Use the smooth window function to modify WVD, obtain:

$$SPW_s(t, f) = \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} g(\eta - t) s\left(\eta + \frac{\tau}{2}\right) s^*\left(\eta - \frac{\tau}{2}\right) d\eta e^{-j2\pi f \tau} d\tau \tag{10}$$

Equation (10) also be called as SPWVD.

The kernel function  $\phi(\xi, \tau)$  of Cohen class time-frequency distribution determines the characteristics of the distribution. By designing the kernel function, we can obtain the required distribution. Due to the oscillatory characteristics of the cross-terms, so the selected kernel function generally has the characteristic of 2-D low-pass filter. SPWVD is a special case of Cohen class time-frequency distribution and it has good performance in suppressing cross-terms. But smooth filter play a role in energy loss when surpassing the cross-terms, at the expense of time-frequency resolution of the whole time-frequency distribution. And it is no longer meet many important properties of WVD, such as edge characteristics.

**Improved joint time-frequency distribution analysis:**

Keeping the time-frequency resolution of WVD and suppressing the cross-term interferences is expected by time-frequency analysis. According to the characteristics SPWVD can suppress the cross-terms, combining WVD and SPWVD, the joint time-frequency distribution analysis is proposed.

Comparing WVD with SPWVD, WVD contains cross-terms, but with good time-frequency aggregation, SPWVD suppress the cross-terms at the expense of time-frequency resolution. Combing both advantages, the aim of ideal time-frequency distribution analysis can be achieved.

From the definition of WVD, it is known that the energy distribution is a joint function of time  $t$  and frequency  $f$ , and the self-term in time-frequency interval is included in that of SPWVD. That is, if SPWVD can completely suppress the cross-terms, and self-terms at the time-frequency plane is within  $\Delta t [t_1, t_2] \Delta f \in [f_1, f_2]$ , suppose the distribution interval of self-terms is  $\Delta t' \in [t'_1, t'_2]$ ,

$$\Delta f' \in [f'_1, f'_2]$$

If the smooth pseudo-WVD cross terms can be completely suppressed, and its since the key ingredient in the time-frequency plane distribution, within the set term component from the WVD of the time-frequency plane distribution interval, it is inevitable,  $\Delta t' < \Delta t, \Delta f' < \Delta f$ , and:

$$\begin{cases} t_1 < t'_1 < t'_2 < t_2 \\ f_1 < f'_1 < f'_2 < f_2 \end{cases} \tag{11}$$

So it can be considered that SPWVD certainly contains the prototype of time-frequency distribution at the interval  $[\Delta t, \Delta f]$ . Multiplying WVD by SPWVD directly can suppress the cross-term interferences to a certain degree, and can improve time-frequency resolution of SPWVD. But the improvement degree is limited. A more effective method is to make SPWVD pass a nonlinear filter. The nonlinear filter has the bellowed characteristic:

$$P_1(t, f) = \begin{cases} 1 & \text{if } |SPW_s(t, f)| \geq thr \\ 0 & \text{else} \end{cases} \tag{12}$$

Multiplying WVD by SPWVD after nonlinear filter can not only keep self-terms, but also suppress the cross-terms interference.

For the actual collected data of engineering, probably there is energy distribution between self-terms and cross-terms, because of the influence of equipments and environments, for example, noise. Then the same method (nonlinear filter for WVD) can be used, but the filter results should keep the energy spectral value of WVD:

$$P_2(t, f) = \begin{cases} W_s(t, f) & \text{if } |W_s(t, f)| \geq thr1 \\ 0 & \text{else} \end{cases} \tag{13}$$

The product of filter results can suppress the cross-terms, and at the same time, can keep joint time-frequency distribution with time-frequency aggregation of WVD:

$$P(t, f) = P_1(t, f)P_2(t, f) \tag{14}$$

**SIMULATIONS ANALYSIS AND EXPERIMENTS**

The computation contains four Gaussian time-frequency distribution. The threshold value is  $thr = 3, thr1 = 0.5$ , Figure 1 shows the simulation signals in time domain and in time-frequency domain, Fig. 2 shows the Wigner-Ville distribution of the simulation signals, Fig. 3 is the smooth pseudo Wigner-Ville Distribution (SPWVD) of the signal, and the Fig. 4 is the improved joint time-frequency distribution.

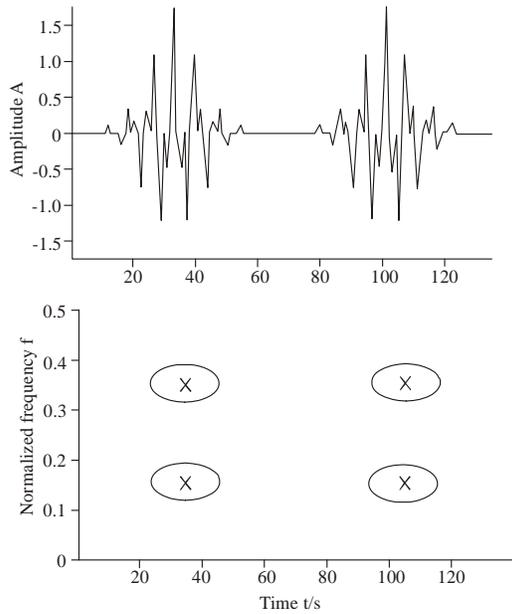


Fig. 1: Gaussian distribution in time domain and time-frequency distribution of 4 components

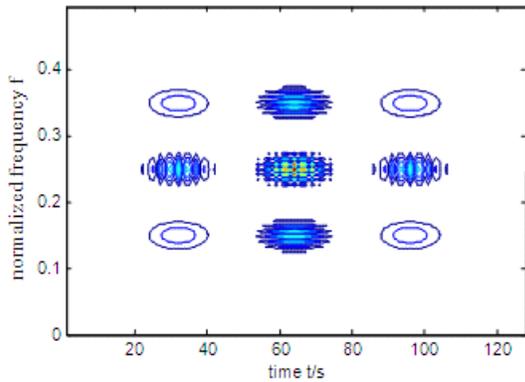


Fig. 2: WVD of the signal

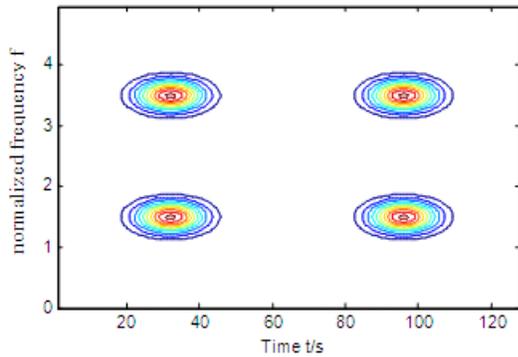


Fig. 3: SPWVD of the signal

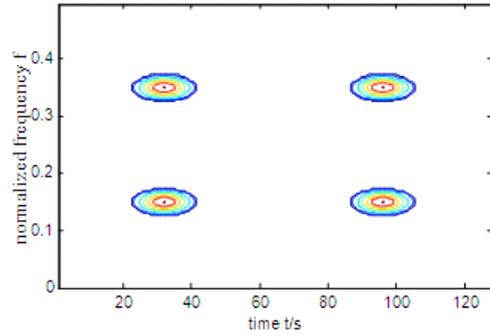


Fig. 4: The improved joint time-frequency distribution

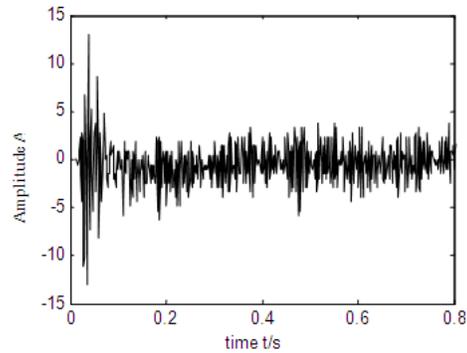


Fig. 5: The experimental data collected by some low frequency sensor

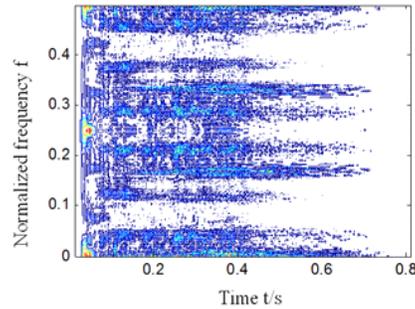


Fig. 6: WVD of signal

From the figures, we can see that there are cross-term interferences when the WVD is used, and false signal is produced. The SPW has good performance in suppressing cross-terms, but during the process the energy is lost, and it is no longer meet many important properties of WVD, such as edge characteristics. The analysis method of joint time-frequency distribution proposed in the paper effectively suppresses the cross-term interference and its time-frequency aggregation is close to WVD.

In the missile experiments, the underwater environment is extremely complicated, and the collected data such as vibration, shock, has non-stationary property and the time-frequency characteristics can not be extracted by Fourier analysis method, so the time-

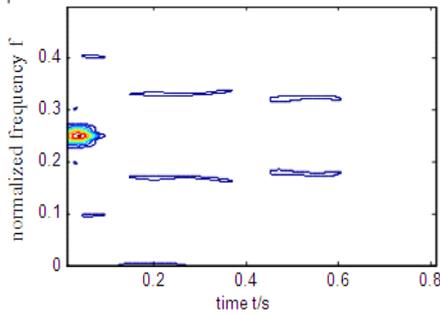


Fig. 7: SPWVD of signal

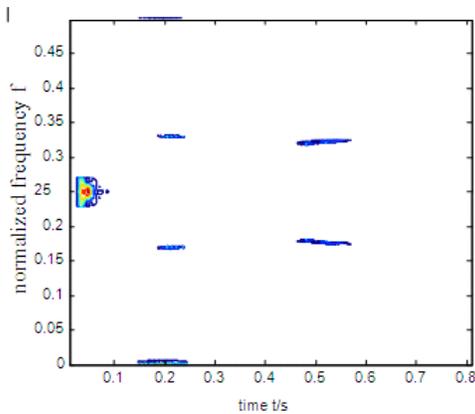


Fig. 8: The improved joint time-frequency distribution

frequency analysis method should be used for underwater environment data analysis. Take the underwater vibration data collected by sensors in some task as an example to verify the proposed method in the paper. The collected data, its WVD, SPWVD and the improved joint time-frequency distribution are shown in Fig. 5, 6, 7 and Fig. 8, respectively.

The selected threshold  $thr = 60$ ,  $thr1 = 12$ , in the processing. The processing results show that the WVD is influenced seriously by the cross-terms, the improved joint time-frequency distribution not only suppress the cross-terms, but also has a high time-frequency resolution.

### CONCLUSION

Simulations and experimental data processing results prove the effectiveness of the proposed method in the study. Use joint time-frequency distribution of smooth

pseudo WVD and WVD filtered nonlinear, which suppress the cross-terms interferences effectively and keep high time-frequency resolution. But in the actual application, the filter threshold should be selected according to the actual characteristics of the signal.

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