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On Levine's Generalized Closed Sets: A survey

K. Kannan
Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA University,
Kumbakonam-612 001, Tamil Nadu, India

Abstract: The aim of this short communication is to study the literature survey on Levine's (1961) generalized closed sets in topological spaces. Also, the important results are summarized in author's point of view.

Key words: g-closed sets, g-locally close sets, g-continuous functions, GO-compact, GO-connected, strongly g-closed sets, weakly g-closed sets

INTRODUCTION AND SURVEY

The concept of closed sets is a fundamental object in general topology. Levine (1970) initiated the study of generalized closed sets in a single topological space in order to extend many of the important properties of closed sets to a larger family. According to him, a set A of a topological space (X, τ) is called generalized closed set (g-closed) if cl (A) $\subseteq$U whenever A $\subseteq$U and U is open in X. He used the "closure operator" and "openness" of the superset in this definition.

Moreover, he characterized them as well as determines their behavior relative to unions, intersections and subspaces. Also it was shown that compactness, normality and completeness in a uniform space are inherited by g-closed subsets. He defined that the complement of g-closed set is a g-open set. Then the images and inverse images of g-closed and g-open sets under continuous closed transformations were explored by him.

Thus the important results are:

- A set A is g-closed if and only if cl (A) $-$A contains no non empty closed set.
- If A and B are g-closed, then A$\cup$B is g-closed.
- Suppose that B $\subseteq$A $\subseteq$X, B is a g-closed set relative to A and that A is a g-closed subset of X. Then B is g-closed relative to X.
- Let (X, τ) be a compact topological space and suppose that A is a g-closed subset of X. Then A is compact.
- Let (X, τ) be a normal space and suppose that Y is a g-closed subset of X. Then (Y, Y $\cap$ τ) is normal.
- If (X, μ) is a complete uniform space and if A is a g-closed subset of X, then (A, A$\times$A $\cap$ μ) is complete.
- If (X, τ) is regular and locally compact and if A is a g-closed subset of X, then A is locally compact in the relative topology.
- If A is a g-closed set in X and if f: X $\to$ Y is continuous and closed, then f [A] is g-closed in Y.
- If B is a g-closed (g-open) set in Y and if f: X $\to$ Y is continuous and closed, then f^{-1}[B] is g-closed (g-open) in X.
- Suppose that A is g-open in X and that B is g-open in Y. Then A$\times$B is g-open in X$\times$Y.
- A space (X, τ) will be termed symmetric if and only if for x and y in X, then x$\in$cl (y) implies that y$\in$cl (x). A space (X, τ) is symmetric if and only if {x} is g-closed for each x$\in$X.

Further he used g-closed sets to define the new separation axiom, called $T_{1/2}$ -spaces. He defines a $T_{1/2}$ -space to be one in which the closed sets and the g-closed sets coincide. As the notation suggests, $T_{1/2}$ is strictly between T_1 and T_0. Also the study of Cartesian products of g-closed sets, g-open sets and $T_{1/2}$ -spaces were undertaken by him.

Dunham (1977) continued the study of $T_{1/2}$ -spaces and characterized $T_{1/2}$ -spaces through singleton set by defining "every singleton of X is either open or closed if and only if every subset of X is the intersection of all open sets and all closed sets containing it". Further he gave some characterizations of $T_{1/2}$ -spaces which are independent of generalized closed sets. In particular, he investigated their behavior with respect to subspaces, transformations and products. For example:

- If X is a $T_{1/2}$ -space and Y $\subseteq$X, then Y is $T_{1/2}$ -space. i.e) $T_{1/2}$ -space is a hereditary property.
- If X is a $T_{1/2}$ -space and f: X$\to$Y is continuous, closed and onto, then Y is $T_{1/2}$.
- If X = Π X α, $\alpha\in\Delta$ is $T_{1/2}$, then X_α is $T_{1/2}$ for all $\alpha\in\Delta$.

Further results on generalized closed sets in topological spaces by Levine and Dunham (1980) were obtained. Some of the important results are:

- Every derived set is g-closed and hence derived sets in a compact space are compact.

- If A is a compact subset of a weakly Hausdorff space, then A is g-closed and hence a compact subset of a regular space is g-closed.
- If A is a retract of a weakly Hausdorff space, then A is g-closed and hence a retract of a regular space is g-closed.

Moreover, they proved that "closed" can be replaced by "g-closed" in the statement of Tietze Extension Theorem and hence established "Generalized Tietze Extension Theorem" which states that "A continuous, real valued function defined on a g-closed subset of a normal space has continuous extension to the entire space". Consequently, they obtained the following:

- A continuous, real valued function defined on a complete subspace of a pseudometric space has continuous extension to the entire space.
- But "pseudometric" can't be replaced by "uniform" in the above result.

Further Dunham (1982) defined the new closure operator c* with the help of g-closed sets in such a way that for any topological space (X, τ), $c^*(E) = \cap \{A: E \subseteq A \in D\}$, where $D = \{A: A \subseteq X, A \text{ is g-closed}\}$ and then he proved that c* is a Kuratowski closure operator on X. Further, he defined that $\tau^* = \{O^*: c^*(O^{*C}) = O^{*C}\}$ is the topology generated by c*. Then he proved (X, τ^*) is always a T_0-space and he improved that result by establishing the stronger result: (X, τ^*) is a $T_{1/2}$-space, for any topological space (X, τ). Moreover, he characterized the discreteness of (X, τ^*) in the following theorem:

Theorem: The following conditions are equivalent.
- (X, τ^*) is discrete
- For each $x \in X$, $\{x\}^C$ is g-closed in (X, τ).
- If $\{x\}$ is closed in (X, τ), then $\{x\}$ is open in (X, τ).

Balachandran *et al*. (1991a, b) introduced the notion of generalized continuous (g-continuous) functions by using g-closed sets and obtained some of their properties. Further they introduced the stronger form of connectedness, called GO-connectedness in 1991 by using g-closed sets. A topological space X is said to be GO-connected if X cannot be written as a disjoint union of two non-empty g-open sets. A subset of X is GO-connected if it is GO-connected as a subspace.

Moreover, they (1991) defined generalized homeomorphisms via generalized closed sets and gc-homeomorphisms in terms of preserving generalized closed sets. Then it was proved that every homeomorphism is a generalized homeomorphism but not vice versa. These two concepts coincide when both the domain and the range satisfy the weak separation axiom $T_{1/2}$. They showed that the class of gc-homeomorphisms is properly placed between the classes of homeomorphisms and g-homeomorphisms.

Caldas (1993) improved the Theorem 6.3 of Levine (1970). The following is the slight improvement of that theorem.

If B is a g-closed (g-open) set in Y and if $f: X \rightarrow Y$ is g-continuous and closed, then $f^{-1}[B]$ is g-closed (g-open) in X. Also it is proved if X is Hausdorff, $A \subseteq X$ and $r: X \rightarrow A$ is g-continuous retraction, then A is a g-closed set in X.

Baker (1996) introduced new forms of continuity and closure, called a-continuity and a-closure and used to strengthen several results concerning the preservation of g-closed sets. Two of them are given below:

- If B is a g-closed (g-open) set in Y and if $f: X \rightarrow Y$ is g-continuous and a-closed, then $f^{-1}[B]$ is g-closed (g-open) in X. This is the improvement of Caldas's result by replacing the closure requirement by "a-closure".
- Levine proved that a closed continuous image of a g-closed set is g-closed. This result was modified slightly by replacing continuity requirement with a-continuity. If B is a g-closed set in X and if $f: X \rightarrow Y$ is a-continuous and closed, then f(B) is g-closed in Y.

Meanwhile, Ganster and Reilly (1989) introduced locally closed sets in topological spaces and studied three different notions of generalized continuity, namely, LC-continuity, LC-irresoluteness and sub LC-continuity using the concept of locally closed set due to Bourbaki (1966). In this sequel, Balachandran *et al*. (1996) introduced generalized locally closed sets by using generalized closed sets and then continued the study of different notions of generalizations of continuous functions in topological spaces.

According to them, a subset A of a topological space (X, τ) is said to be:

- g-locally closed set if $A = G \cap F$ where G is g-open and F is g-closed in X,
- g-locally closed* if $A = G \cap F$ where G is g-open and F is closed in X,
- g-locally closed** if $A = G \cap F$ where G is open and F is g-closed in X.

Then they showed that:

- Every locally closed set is both g-locally closed* and g-locally closed**,
- Every g-locally closed* (resp. g-locally closed**) set is g-locally closed.

Also the important characterization of g-locally closed* set was done in the following theorem.

Theorem: For a subset A of a topological space (X, τ), the following assertions are equivalent:

- $A \in GLC^* (X, \tau)$
- $A = G \cap [cl(A)]$ for some g-open set G
- $A \cup \{X- [cl(A)]\}$ is g-open
- $[cl (A)] - A$ is g-closed

Concerning the submaximality, they introduced the new type of submaximal space, namely g-submaximal space in which every dense subset is g-open. Moreover, the following necessary and sufficient condition for a space X to be a g-submaximal space was discussed. That is, a topological space (X, τ) is a g-submaximal space if and only if $GLC^* (X, \tau) = P(X)$. Also it is proved that every submaximal space is g-submaximal space. Further they defined the following new types of maps:

- A function $f: (X, \tau) \to (Y, \sigma)$ is glc-continuous {resp. glc*-continuous, glc**-continuous} if $f^{-1}(U)$ is g-locally closed {resp. g-locally closed*, g-locally closed**} for each open set U in Y.
- A function $f: (X, \tau) \to (Y, \sigma)$ is glc-irresolute {resp. glc*-irresolute, glc**-irresolute} if $f^{-1}(U)$ is g-locally closed {resp. g-locally closed*, g-locally closed**} for each g-locally closed set {resp. g-locally closed* set, g-locally closed** set} U in Y.

Then the following relations were obtained:

- Every LC-continuous function is both GLC*-continuous and GLC**-continuous.
- Every GLC*-continuous {resp. GLC**-continuous} function is GLC-continuous.
- Every GLC-irresolute {resp. GLC*-irresolute, GLC**-irresolute} function is GLC-continuous {resp. GLC*-continuous, GLC**-continuous}.
- If f is both continuous and closed then f is GLC-irresolute, GLC*-irresolute and GLC**-irresolute.

Sundarm and Nagaveni (1998) introduced weakly generalized closed sets. According to them, a subset A of (X, τ) is called weakly generalized closed (briefly, weakly g-closed) if $cl [int(A)] \subseteq G$ whenever $A \subseteq G$ and G is open in X.

Sundaram and Pushpalatha (2001) introduced the concepts of strongly generalized closed sets and strongly generalized open sets, which are generalizations of closed sets and open sets in topological spaces. A subset A of (X, τ) is called strongly generalized closed (briefly, strongly g-closed) if $cl [int (A)] \subseteq G$ whenever $A \subseteq G$ and G is g-open in X. Further, they introduced u_1 and v_1-spaces and studied some of their properties.

Chawalit (2003a, b) continued the study of g-continuous mappings and proved some relationships between continuous functions and g-continuous functions and also studied g-continuous functions from any topological spaces into product spaces. In general, every continuous function is a g-continuous function but not conversely. Also he gave some more characterizations of this type of continuity. According to him, the main objectives are:

- If X is a $T_{1/2}$ -space and Y is a topological space, then $f: X \to Y$ is continuous if and only if f is g-continuous.
- If X and Z are topological spaces and Y is a $T_{1/2}$ -space, then if $f: X \to Y$ and $g: Y \to Z$ are g-continuous, then g o f is g-continuous.
- If Y is a $T_{1/2}$ -space and $\{X_\alpha: \alpha \in \Delta\}$ is a family of topological spaces, then a function $f: Y \to \prod_{\alpha \in \Delta} X_\alpha$ is g-continuous if and only if the composite function $\prod_\alpha$ o $f: Y \to X_\alpha$ is g-continuous for each $\alpha \in \Delta$.

Further he (2003) defined and studied the concepts g-Hausdorff spaces and investigated preservation theorems concerning them in the same year. A topological space X is said to be g-Hausdorff if whenever x and y are distinct points of X, there are disjoint g-open sets U and V with $x \in U$ and $y \in V$. It is obvious that every Hausdorff space is a g-Hausdorff space, but the converse is not in true in general. He obtained the following preservation theorems on some topological spaces under g-continuous, gc-irresolute functions:

- Let X be a topological space and Y be Hausdorff. If $f: X \to Y$ is injective and g-continuous, then X is g-Hausdorff.
- Let X be a topological space and Y be g-Hausdorff. If $f: X \to Y$ is injective and gc-irresolute, then X is g-Hausdorff.

Jin and Jin (2004) introduced and studied the class of mildly generalized closed sets, which is properly placed between the classes of strongly generalized closed sets due to Sundaram and Pushpalatha (2001) and weakly generalized closed sets due to Sundarm and Nagaveni (1998). The relations with other notions directly or indirectly connected with generalized closed were investigated. Moreover, they used it to obtain new characterizations and preservation theorems of almost normal spaces due to Singal and Shashi (1970) and mildly normal spaces due to Singal and Asha (1973) respectively.

Meanwhile, Popa (1979) introduced the notion of rare continuity as a generalization of weak continuity (1961) which had been further investigated by Long and Herrington (1982) and Jafari (1995, 1997). In this sequel, Caldas and Jafari (2005) introduced the concept of rare g-continuity in topological spaces as a generalization of rare continuity and weak continuity. A function $f: X \to Y$ is called rarely g-continuous if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap cl(R_G) =$

ϕ and $U \in GO(X, x)$ such that $f(U) \subseteq G \cup R_G$. Further they investigated several properties of rarely g-continuous functions. The notion of Ig-continuity was also introduced which is weaker than g-continuity and stronger than rare g-continuity. It was shown that when the codomain of a function is regular, then the notions of rare g-continuity and Ig-continuity are equivalent.

Caldas *et al*. (2008) studied some more properties of GO-compact spaces and to introduce and investigate some properties of g-continuous multifunctions. They also investigated GO-compact spaces in the context of multifunctions. They also introduced the notion of g-complete accumulation points by which they gave some characterizations of GO-compact spaces. A space X is GO-compact if and only if each infinite subset of X has a g-complete accumulation point. Also, the following two statements are equivalent for a space X:

- X is GO-compact.
- Every upper g-continuous multifunction from X into the subsets of a T_1-space attains a maximal value with respect to set inclusion relation.

The following result concerns the existence of a fixed point for multifunctions on GO-compact spaces.

Theorem: Suppose that $F: X \rightarrow Y$ is a multifunction from a GO-compact domain X into itself. Let F(S) be g-closed for S being a g-closed set in X. If $F(x) \neq \phi$ for every point $x \in X$, then there exists a nonempty, g-closed set C of X such that $F(C) = C$.

Finally, they arised the following open question: Give a nontrivial example of a GO-compact space.

CONCLUSION

This study provides the development of generalized closed sets over the period of time. Also it will be very useful for the researchers those who are doing their research on generalized closed sets.

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