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Lyapunov Equations Approach for Robust Nonlinear Optimal Control Problems

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Abstract: In this study, nonlinear constrained optimal control problems with uncertain parameters which can be addressed by a robust worst-case formulation are considered. The robust worst-case formulation leads to a bi-level min-max optimization problem. We propose a method to solve this min-max optimization problem based on Lyapunov differential equations. Employing first order derivatives of both the reference states and uncertainty parameters, the linear approximation of the dynamic system and inequality constraints can be obtained. After that, the Lyapunov differential equations can be formed based on the linear approximation and the upper bound for the worst case of the inequality state constraints accepted by the uncertain parameters can be obtained. Then the bi-level min-max optimization problem is transformed into a normal single-level optimization control problems which can be solved easily. To show the effectiveness of the proposed method, the simulation results of two robust constrained nonlinear optimal control problems are presented.

Key words: Optimal control, lyapunov equations, robust optimization, uncertainty

INTRODUCTION

It is a common problem solving constrained nonlinear optimal control problems with uncertain parameters. In the past decades, a number of articles have addressed to find solutions of such nonlinear optimal control problems.

Two general min-max formulations for solving the robust control problem is summarized well (Lee and Yu, 1997). One is open-loop formulation where the uncertainty and feedback in the future time step are ignored. The second is min-max formulation from the viewpoint of closed-loop control where a dynamic program is solved. Based on the two general formulations, many different approaches have been introduced. A scenario tree formulation (Scokaert and Mayne, 1998) for linear systems with additive disturbances is suggested to treat a single optimization problem for one initial state only. Linear matrix inequality techniques (Wan and Kothare, 2003) are employed to efficiently compute the worst-case performance. A computationally approximate dynamic programming approach is presented (Nosair *et al.*, 2010) by using piecewise parametric quadratic approximation. In the method of feasibility and flexibility measures (Halemane and Grossmann, 1983; Swaney and Grossmann, 1985), the optimal variables are partitioned into design and control variables. The design variables are specified by outer optimal problem. Flexibility and feasibility in the presence of parametric uncertainty is addressed by formulating a nested max-min-max constraint for the feasibility constraints and the control variables are obtained by the minimization problem and the uncertain parameters by the outer optimal problem. A numerical method is presented (Diehl *et al.*, 2008) to solve the min-max optimal problem in the case that the underlying maximization problems always has its solution on the boundary of the uncertainty set. By formulating equilibrium constraint employing first order derivatives of uncertainty set and the user defined constraints, this approach adopt the local reduction approach to solve the generalized semi-infinite programs.

All the methods above must solve a bi-level min-max optimization problem and it is always difficult to solve. A constrained Lyapunov differential equation is introduced (Diehl *et al*., 2008) providing robustness interpretations with respect to L_2 -bounded disturbance in the context of inequality state constraints. Then the bi-level min-max optimization problem is transformed into a normal single-level optimal problem. But Houska and Diehl (2009). considered the linear system only (Houska and Diehl, 2009).

In this study, we present an approach for solving nonlinear constrained optimal control problems with uncertain parameters which can be addressed by a robust worst-case formulation are considered. By employing first order derivatives of both the reference states and uncertainty parameters, the linear approximation of the dynamic system and inequality constraints can be obtained. After that, the Lyapunov differential equations can be formed based on the linear approximation. And the upper bound for the worst case of the inequality state

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constraints affected by the uncertain parameters can be obtained. Then the bi-level min-max optimization problem is transformed into a normal single-level optimization control problems which can be solved easily.

PROBLEM STATEMENT

The problem we are considering is uncertain nonlinear problems (NLP) of the form:

$$\min_{x(t),u(t),p,t_f} J(x(t),u(t),p(t),t_f) \tag{1}$$

s.t.

$$\dot{x}(t) = f(x(t),u(t),p(t),t), x(0) = x_0 \tag{2}$$

$$h(x(t),u(t),p(t),t) \leq 0 \tag{3}$$

where, t_f is the final time, $x(t) \in X$ is the n_x vector of states, $u(t) \in U$ is the n_u set of all possible control trajectories, $p(t) \in P$ is the $n_p$vector of model parameters in the uncertainty set P, $J(x(t)u,(t),p(t),t_f)$ is the objective function $J:X \rightarrow R$, f is the vector function $f:X \rightarrow R^{nx}$which describes the dynamic equations of the system, $h:X \rightarrow R^{nx}$ is the vector of functions which describes the linear and nonlinear, time-varying or end-times algebraic constraints for the system and c is the number of these constraints.

We assume we have some knowledge about the parameters p. The uncertainty set P is a confidence ellipsoid for a Gaussian random variable pwith the expectation value $\hat{p}$, the $n_p \times n_p$ positive definite variance-covariance matrix $\sum$ and a scalar $\gamma>0$ depending on the desired confidence level

$$P = \{p \in R^{n_p} | (p - \hat{p})^T \sum^{-1} (p - \hat{p}) - \gamma \leq 0\} \tag{4}$$

Worst-case approximation by linearization: To solve the optimal control problem (1)-(3), we have to ensure the constraints are satisfied for all possible uncertainties $p(t) \in P$. The worst case ϕ_i (x,u), is chosen by:

$$\phi_i(x,u,t) := \max_{p \in P} h_i(x(t),u(t),p(t),t) \tag{5}$$

$$s.t.\ \dot{x}(t) = f(x(t),u(t),p(t),t), \quad x(0) = x_0 \tag{6}$$

Employing the functions $\phi_i(x(t),u(t),t)$, the following worst-case formulation as the robust counterpart of optimal control problem (1)-(3) is arrived:

$$\min_{u(t) \in U} J(x(t),u(t),t,t_f) \tag{7}$$

$$s.t. \phi_i(x(t),u(t),p(t),t) \leq 0, \quad i = 1,\ldots,n_c \tag{8}$$

The above problem has a bi-level/ min-max structure. It is difficult to solve with a normal approach. For the case the f(x(t), u(t), t) is nonlinear, the problem become even much harder.

In order to avoid the bi-level structure of the robust counterparts (5)-(7), approximations will be defined below to replace the functions Φ_i,i = 0,…,n_c.

By inspecting the expression (5) for the function ϕ_i, we linearize both state functions f(x(t), u(t), p(t), t) = 0and constraints h_i(x(t), u(t), p(t), t). Define $\hat{x}$ as the nominal state trajectory vector and Δx as the deviation about the nominal state vector, caused by uncertain parameters. Therefore, the state vector for the real system is:

$$x(t) = \hat{x}(t) + \Delta x(t) \tag{9}$$

For small uncertain parameters p(t), the dynamic equations of the system f(x(t),u(t),p(t),t) can be divided into two parts. One part includes the nominal dynamics functions of the system:

$$\dot{\hat{x}} = f(\hat{x}(t),u(t),\hat{p}(t),t) \tag{10}$$

$$\hat{x}(0) = x_0 \tag{11}$$

the other part includes the approximate linearization based dynamics functions affected by the uncertainty:

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta p(t) \tag{12}$$

$$\Delta x(0) = B(0)\Delta p(t) \tag{13}$$

where,

$$A(t) = \frac{\partial f(\hat{x}(t),u(t),\hat{p}(t),t)}{\partial x} \tag{14}$$

$$B(t) = \frac{\partial f(\hat{x}(t),u(t),\hat{p}(t),t)}{\partial p} \tag{15}$$

The constraint functions can be approximated as:

$$h_i(\hat{x}(t),u(t),t) + C(t)\Delta x < 0, i = 1,\ldots,n_c \tag{16}$$

where,

$$C_i = \frac{\partial h_i(\hat{x}(t),u(t),\hat{p}(t),t)}{\partial x} \tag{17}$$

Considering Eq. (5) and (16):

$$\phi_i(x(t), u(t), t) = \max_{p\in P} h_i(\hat{x}(t), u(t), t) + C(t)\Delta x, i = 1,\ldots,n_c$$

$$i = 1, \ldots, n_c$$

The value of $h_i(\hat{x}(t), u(t), t), i = 1,\ldots,n_c$ can be computing easily, therefor obtaining the value of $C(t)\Delta x$ in the worst-case is the key problem to find out the maximum of ϕ.

Lyapunov differential equations: Consider the linear system below:

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta p(t)\Delta(t), \Delta x(0) = B_0\Delta p_0 \quad (18)$$

$$y(t) = C(t)\Delta(t) \quad (19)$$

where, $t\in[0,t_f]$. $\Delta p(t)$ is a Gaussian white noise process with:

$$E\{\Delta p(t)\} = 0$$

$$E\{\Delta P(t_1)\Delta p(t_2)\} = \delta(t_1-t_2)$$

here, δ is the Dirac's function and $E\{.\}$denotes statistical expectation.

Based on system (18)-(19), $\Delta x(t)$ and $y(t)$ can been obtained as follows:

$$\Delta x = \Phi(t,0)B_0\Delta p_0 + \int_0^t \Phi(t,\tau)B(\tau)\Delta p(\tau)d\tau \quad (20)$$

$$y(t) = C(t)\Delta x(t) = H(t,0)\Delta p_0 + \int_0^t H(t,\tau)\Delta p(\tau)d\tau \quad (21)$$

where, $\Phi(t,\tau):R\times R\to Rn_x\times n_x$ is the state transition matrix of the system(18)-(20) and $H(t, \tau) = C(t)\Phi(t, \tau)B(\tau)$.

Lyapunov differential equations is introduced to describe the dependence of the $\Delta x(t)$ on the disturbance $\Delta p(t)$. Assuming $\Delta x(t)$ is asymptotically stable, there exists a unique and symmetric matrix function V: $R\to Rn_x \times n_x$ satisfying the functions below:

$$\dot{V}(t) = A(t)V(t) + V(t)A^T(t) + B(t)B^T(t) \quad (22)$$

$$V(0) = B_0B_0^T \quad (23)$$

Lemma 1: If Δx is asymptotically stable, the function V can uniquely be obtained as follow (Kemin and John, 1996):

$$V(t) = \Phi(t,0)B_0B_0^T\Phi(t,0)^T + \int_0^T \Phi(t,\tau)B(\tau)B(\tau)^T\Phi^T(t,\tau)d\tau \quad (24)$$

Lemma 2: The function V in Esq. (22) and (23) can be interpreted as the variance-covariance matrix of Δx.

Proof: Computing the variance-covariance matrix of Δx:

$$\begin{aligned}
&E\{\Delta x(t)\Delta x^T(t)\} \\
&= E\left\{\begin{matrix}\left(\Phi(t,0)B_0\Delta p_0 + \int_0^t \Phi(t,\tau)B(\tau)\Delta p(\tau)d\tau\right). \\ \left(\Phi(t,0)B_0\Delta p_0 + \int_0^t \Phi(t,\tilde{\tau})B(\tilde{\tau})\Delta p(\tilde{\tau})d\tilde{\tau}\right)^T\end{matrix}\right\} \\
&= E\left\{\Phi(t,0)B0\Delta p_0\Delta p_0^T B_0^T\Phi(t,0)^T\right\} \\
&+ E\left\{\int_0^{t_f}\int_0^{t_f} \Phi(t,\tau)B(\tau)\Delta p(\tau)\Delta p(\tilde{\tau})B(\tau)^T\Phi(t,\tilde{\tau})^T d\tau d\tilde{\tau}\right\} \\
&= \Phi(t,\tau)B_0B_0^T\Phi(t,0)^T + d\tau + \int_0^{t_f}\int_0^{t_f} \Phi(t,\tau)B(\tau)\delta(\tilde{\tau}-\tau) \\
&\quad B(\tau)^T\Phi(t,\tilde{\tau})^T d\tau d\tilde{\tau} \\
&= \Phi(t,0)B_0B_0^T\Phi(t,0)^T + \int_0^{t_f} \Phi(t,\tau)B(\tau)B(\tau)^T\Phi(t,\tau)^T d\tau \\
&= V(t)
\end{aligned}$$

The variance-covariance matrix of the Gaussian white noise Δp in the proof is assumed as an identity matrix. If the variance-covariance matrix of Δp is$\sum$, we can redefine B by $B(t)\to B(t)\sum^{0.5}(t)$. The result is same.

From the result above, the variance-covariance matrix of y can be obtained:

$$E\{y(t)y^T(t)\} = C(t)E\{x(t)x^T(t)\}C(t)^T=V(t)C(t)$$

Considering the worst case of the Lyapunov system based on the disturbance Δp, defining an inner product $<.|.>w:W\times W\to R$in the space W: = $R^{n\Delta p0}\times L_2$ and the corresponding W-norm $\|.\|_w:W\to R$ by:

$$< \Delta p_1, \Delta p_2 >_w := \Delta p_{0,1}^T\Delta p_{0,2} + \int_0^{tf} \Delta p_1(\tau)^T\Delta p_1(\tau)d\tau$$

$$\|\Delta p_1\|_w := \sqrt{< \Delta p_1, \Delta p_2 >_w},$$

for al $\Delta p_1,\Delta p_2\in W$. The Eq.(21) can be written as:

$$y_i(t) =< H_i^T|\Delta P >_w \quad (25)$$

and defining the γ ball $B:=\{\Delta p\in W\|_w \le \gamma\}$
The worst case of $y_i(t)$ is:

$$\begin{aligned}
&\max_{\Delta p \in B} y_i(t) \\
&= < H_i^T | \Delta p >_w \\
&\le \left\| H_i^T \right\|_w \left\| \Delta p \right\|_w \left(Cauchy - Schwarz \ \ inequality \right) \\
&\le \gamma \sqrt{C_i(t) \begin{pmatrix} \Phi(t,0) B_0 B_0^T \Phi(t,0)^T + \\ \int_0^t \Phi(t,\tau) B(\tau)^T \Phi^T(t,\tau) d\tau \end{pmatrix} C_i(t)^T} \\
&= \gamma \sqrt{C_i(t) V(t) C_i(t)^T}
\end{aligned} \tag{26}$$

Then the uncertain nonlinear problems (1)-(3) can be transformed as below:

$$\min_{x(t), u(t), p, t_f} J\left(x(t), u(t), p(t), t_f \right) \tag{27}$$

s.t.

$$\dot{\hat{x}} = A(t) x(t) + B(t) u(t), \hat{x}(0) = x_0 \tag{28}$$

$$\dot{V}(t) = A(t) V(t) + V(t) + A^T(t) + B(t) B^T(t), V(0) = B_0 B_0^T \tag{29}$$

$$h_i(\hat{x}(t), u(t), t) + \gamma \sqrt{C_i(t) V(t) C_i(t)^T} \le 0, i = 1, \ldots, n_c \tag{30}$$

Note that in this robust counterpart formulation, it is not the bi-level optimal problem as before, but a single-level optimal problem. And this formulation has no uncertain parameters. Hence, the former uncertain optimal problem is transformed into a normal optimal problem which can be easily solved by some numerical method (e.g. active-set method).

NUMERICAL SIMULATIONS

In this section, two nonlinear optimal control problems are solved with the proposed methods. All the computations were run on a desktop computer with dual-core processor 3.10 GHz and 2.0 GB of RAM and all the codes are written in MATLAB software.

Example 1: This example is adapted from (Kleinman *et al.*, 1968) and we modify it by mixing some uncertain parameters:

$$J = \int_0^1 \left(x_1^2(t) + x_2^2(t) + 0.005 u^2(t) \right) dt$$

subject to:

$$\begin{aligned}
&\dot{x}_1 = x_2(t), \ x_1(0) = 0 \\
&\dot{x}_2 = -x_2(t)(1 + p) + u(t), \quad x_2(0) = -1 \\
&x_2(t) \le 8(t - 0.5)^2 - 0.5
\end{aligned}$$

The variance-covariance matrix Σ of the Gaussian random variable p is [0.03] and expectation E{p}= 0. Based on the Eqs. (14), (15) and (17), the A, B, C_i can be obtained:

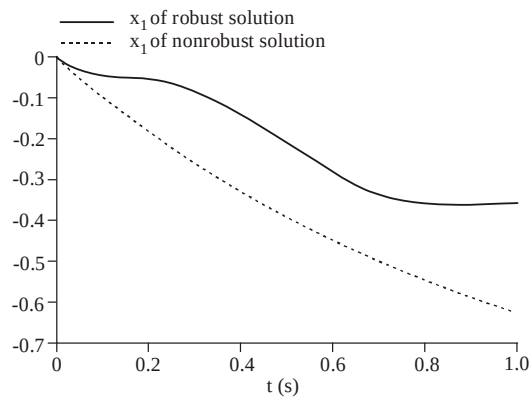

Fig. 1: The solutions of x_1 for example 1

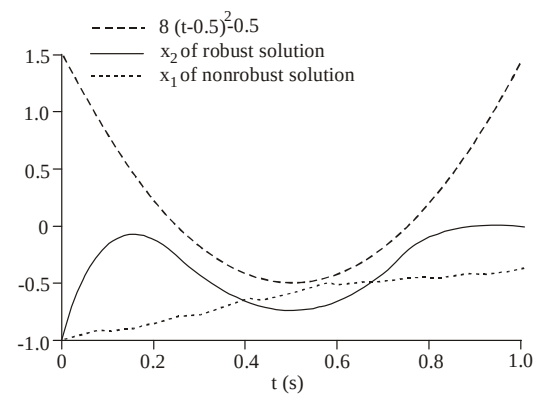

Fig. 2: The solutions of x_2 for example 1

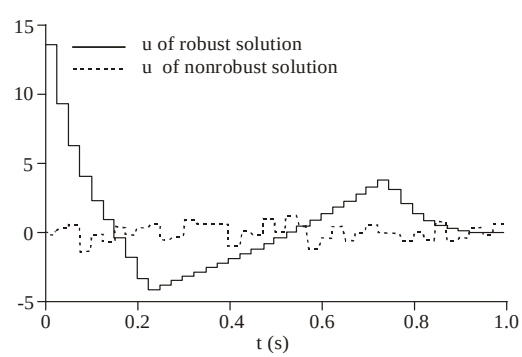

Fig. 3: The solutions of u for example 1

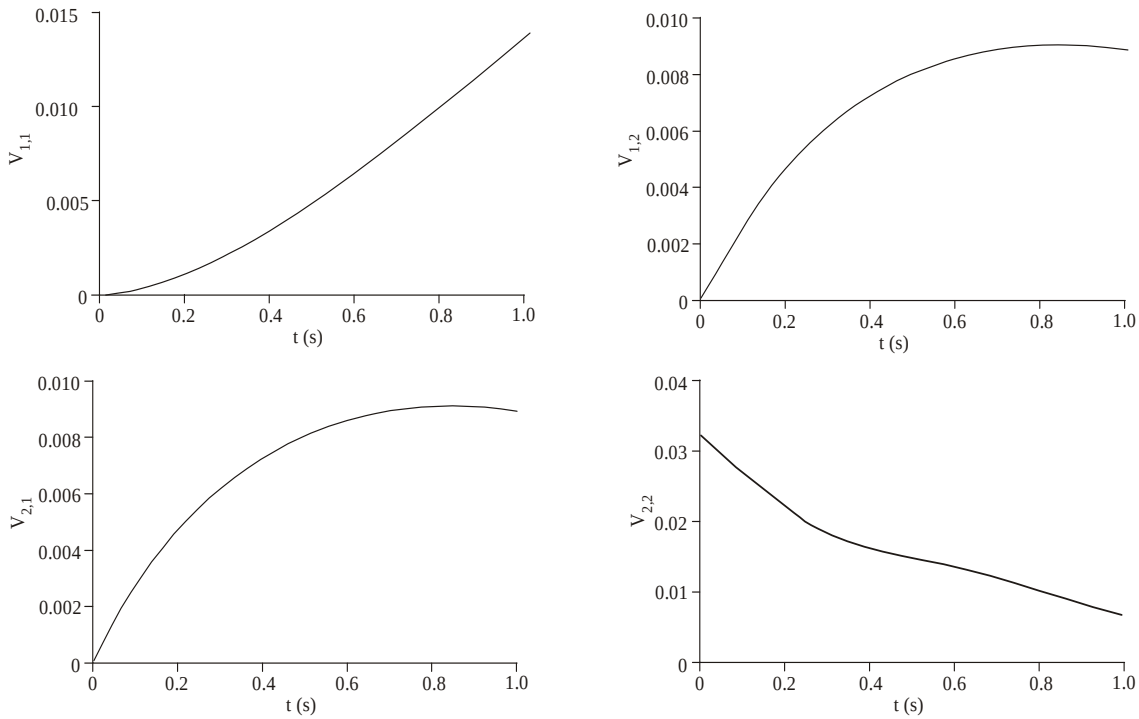

Fig. 4: The solutions of V for example 1

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_2^2(t) \end{pmatrix} \sum^{\frac{1}{2}}, C = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{31}$$

We transform the optimal problem into the form as bellows:

$$\min \quad J \tag{32}$$

subject to:

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t), \quad \hat{x}_1(0) = 0 \tag{33}$$

$$\dot{\hat{x}}_2(t) = -\hat{x}_2(t) + u(t), \quad \hat{x}_2(0) = -1 \tag{34}$$

$$\dot{V}(t) = AV(t) + V(t)A^T + BB^T, \quad V(0) = B_0 B_0^T \tag{35}$$

$$C\begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} + \sqrt{CVC^T} \le 8(t-0.5)^2 - 0.5 \tag{36}$$

where, V is a 4×4 matrix. We solve the above problem by a piecewise constant control parameterization with 40 pieces and active-set method. The solutions of optimal problems with uncertain parameters or not are compared in the Fig. 1, 2 and 3 for x_1, x_2 and u, respectively. The values of parameters in the matrixV are shown in the Fig. 4.

The performance index J in no robust condition is 0.1711 and the corresponding value of J in robust condition is 0.3221.

Example 2: This example is adapted from (Elnagar and Kazemi, 1998) and we modify it by mixing some uncertain parameters:

$$J = \int_{-5/4}^{5/4} \left(1 + u^2(t)\right)^{1/2} dt$$

s.t.

$$\dot{x} = u(t)(1+p)$$
$$x\left(-\frac{5}{4}\right) = x\left(\frac{5}{4}\right) = 0$$
$$1 - x(t) - t^2 \le 0$$

The variance-covariance matrix $\sum$ of the Gaussian random variable p is[0.03] and expectation E{p} = 0. Based on the Eq. (14), (15) and (17), the A, B, C_i can be obtained:

$$A = (0), B = (u) \sum^{\frac{1}{2}}, C = (-1) \tag{37}$$

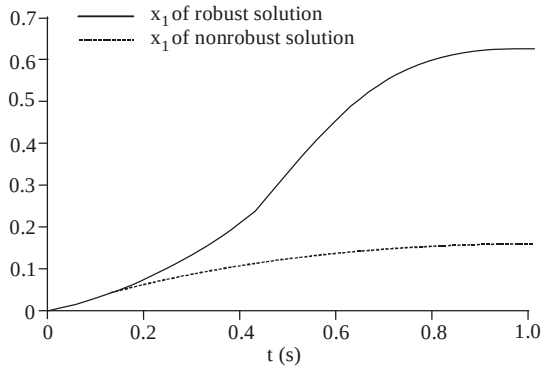

Fig. 5: The solutions of x for example 2

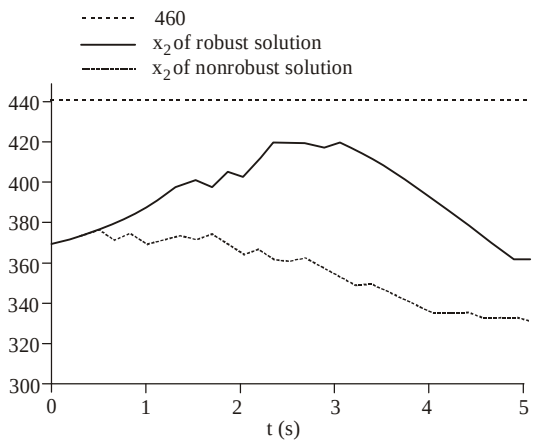

Fig. 6: The solutions of u for example 2

We transform the optimal problem into the form as bellows:

$$\min \quad J \tag{38}$$

s.t.

$$\dot{\hat{x}} = u(t)$$

$$\hat{x}\left(-\frac{5}{4}\right) = \hat{x}\left(\frac{5}{4}\right) = 0$$

$$\dot{V}(t) = AV(t) + V(t)A^T + BB^T$$

$$C\hat{x} + \sqrt{CVC^T} \le t^2 - 1$$

where, V is a 1 matrix. We solve the above problem by a piecewise constant control parameterization with 40 pieces and active-set method. The solutions of optimal problems with uncertain parameters or not are compared in the Fig. 5 and 6 for x and u respectively. The values of parameters in the matrix V are shown in the Fig. 7.

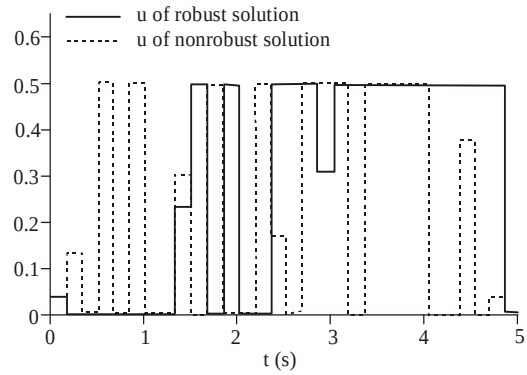

Fig. 7: The solutions of V

The performance index J in nonrobust condition is 3.2687 and the corresponding value of J in robust condition is 3.8682.

CONCLUSION

In this study, we develop a method for solving constrained nonlinear optimal control problems with uncertain parameters. The method considers the robust optimal control problems as worst-case bi-level optimal problems. To solve the bi-level optimal problems, we transform the optimal problems into single-level optimal problems by applying the Lyapunov differential equations. Then the single-level optimal problems can be easily solved by some normal numerical programming techniques. Illustrative examples are included to show the effectiveness of the proposed method.

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