Research Journal of Applied Sciences, Engineering and Technology 4(17): 2893-2897, 2012
ISSN: 2040-7467

Submitted: December 02, 2011 Accepted: January 13, 2012 Published: September 01, 2012

Approximate Explicit Solution of Falkner-Skan Equation by Homotopy Perturbation Method

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Abstract: In this study, by mean`s of He`s Homotopy Perturbation Method (HPM) an approximate solution of Falkner-Skan equation obtained. In boundary layer theory, we have seen how similarity methods combine two independent variables into one, and therefore our problems our simplified to ODE Equations. If we use HPM we can deforms a difficult ordinary differential equation into a simple problem which can be easily solved. Comparison is made between the solution of Falkner Skan equation for 4 cases and those in open literature to verify accuracy of this work. Results show that the method is very effective and simple.

Keywords: Boundary layer theory, Falkner-Skan equation, Homotopy Perturbation Method (HPM)

INTRODUCTION

Recent years, there has appeared an ever increasing interest of scientists and engineers in analytical techniques for studying nonlinear problems. Such techniques have been dominated by the perturbation methods and have found many applications in science, engineering (He, 2006). However, like other analytical techniques, perturbation methods have their own limitations. For example, all perturbation methods require the presence of a small parameter in the non linear equation and the approximate solutions of equation containing this parameter are expressed as series expansions in the small parameter. Selection of small parameter requires a special skill. A proper choice of small parameter gives acceptable results, while an improper choice may result in incorrect solutions. Therefore, an analytical method is welcome which does not require a small parameter in the equation modeling the phenomena (He, 2003a).

He (2006) proposed such a technique which is a coupling of the traditional perturbation method and homotopy in topology. In this paper, by mean`s of He`s homotopy perturbation method (HPM) an approximate solution of boundary layer equation for two-dimensional laminar viscous flow is obtained. (He, 2006) investigate a simple perturbation approach to Blasius equation. Most recently (Sajid *et al*., 2006) discussed application of He's homotopy perturbation method to boundary layer equation flow and convection heat transferover a flat plate.

Governed equations give rise to highly nonlinear differential equations. If we use (HPM) we can deforms a difficult differential equation into a simple problem which can be easily solved.

METHODOLOGY

Fundamental of the homotopy perturbation method: To illustrate the Homotopy Perturbation Method (HPM) for solving nonlinear differential equations, He considered the following nonlinear differential equation:

$$A(U) = f(r),\ r \in \Omega \tag{1}$$

subject to the boundary condition:

$$B(u, \partial u / \partial n) = 0,\ r \in \Gamma \tag{2}$$

where, A is a general differential operator, B is a boundary operator, f(r) is a known analytic function, Γ is the boundary of the domain Ω and, $\partial/\partial n$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts M and *N*. Therefore, (1) can be rewritten as follows:

$$M(u) + N(u) = f(r)\ r \in \Omega \tag{3}$$

(He, 2000) constructed a homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ which satisfies:

$$H(v, p) = (1-p)[M(v) - M(u_o)] + p\,[A(v) - f(r)] = 0 \tag{4}$$

which is equivalent to:

$$H(v, p) = M(v) - M(u_o) + pM(u_o) + [A(v) - f(r)] = 0, \tag{5}$$

where, $p \in [0,1]$ is an embedding parameter, and u_o is the first approximation that satisfies the boundary condition. Obviously, we have:

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$$H(v,0) = M(v) - M(u_o) = 0 \quad (6)$$

$$H(v,1) = A(v) - f(r) = 0 \quad (7)$$

The changing process of p from zero to unity is just that of H(v, p) from M (v) – M(u_o) to A (v) – f(r). In topology, this is called deformation and M (v) – M(u_o) and A (v) – f(r) are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of Eq. (4) can be expressed as a series in p in the form:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \quad (8)$$

when p-1, Eq. (4) corresponds to the original one, Eq. (3) and (8) becomes the approximate solution of Eq. (3), i.e.,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \quad (9)$$

the convergence of the series in Eq. (9) is discussed by He (2000).

Problem formulation: We will consider two-dimensional laminar viscous flow governed by Eq. (10) which is named as Falkner Skan equation:

$$f(\eta) + \alpha f(\eta) f(\eta) + \beta\left(1 - \left(f(\eta)\right)^2\right) = 0 \quad \text{for } \eta \in [0, +\infty] \quad (10)$$

with boundary conditions:

$$f(0) = f(0) = 0,\ f(+\infty) = 1 \quad (11)$$

where the prime denotes the derivative with respect to h which is defined as:

$$\eta = y\sqrt{\frac{U}{vx}} \quad (12)$$

and f (h) is relative to stream function y by:

$$f(\eta) = \frac{\psi}{\sqrt{vUx}} \quad (13)$$

Here U is the velocity at infinity, v is the kinematic viscosity coefficient, x and y are the two independent coordinates (He, 2003b).

Analysis Falkner Skan equation using homotopy perturbation method: To obtain the solution of Eq. (10) we use HPM. First we consider operators M and N as follows:

$$M = d^3/d\eta^3(\bullet) \quad (14)$$

and

$$N = \alpha\left(\frac{d^2}{d\eta^2}(\bullet)\right)(\bullet) + \beta\left[1 - \left(\frac{d}{d\eta}(\bullet)\right)^2\right] \quad (15)$$

then, construct homotopy v (r, p) :Ω×[0, 1] →ℜ witch satisfies:

$$H(f, p) = (1-p)\, M(f) - M(g_o)) + p[M(f) + N(f)] = 0 \quad (16)$$

$$(1-p)\left(\frac{d^3 f}{d\eta^3} - \frac{d^3 g_o}{d\eta^3}\right) + p\left[\begin{array}{l}\frac{d^3 f}{d\eta^3} + \alpha\left(\frac{d^2 f}{d\eta^2}\right) \\ (f) + \beta\left[1 - \left(\frac{d}{d\eta}(f)\right)^2\right]\end{array}\right] = 0 \quad (17)$$

where, p∈[0,1] is the embedding parameter and u_o is the initial guess, accordingly to HPM and with respect to boundary conditions Eq. (11), we assume that Eq. (17) has the solution of the form:

$$f(\eta, p) = f_o(\eta) + p\, f_1(\eta) + p^2 f_2(\eta) + \ldots \quad (18)$$

by substituting Eq. (18) into (17) and (16), then equating the same powers of p and choosing powers of p from zero to three with respect to our approximation in this study.

It should be noted that at about $\eta = \eta_\delta$ we have u/U = 0.99 approximately. In this study, polynomial functions are used to obtain solution for our problem, so we use $\eta = \eta_\delta$ instead of $\eta = \infty$. In the other hand third condition is replaced as follow $df/d\eta\ (\eta_\delta) = 1$.

Zero-order: The differential equation of the zero order with the boundary conditions is obtained as follows:

$$N(f) - N(g_o) = 0 \quad (19)$$

$$f(0) = d/d\eta\ f(0) = 0,\ d/d\eta\ f(\eta_\delta) = 1 \quad (20)$$

note that u_0 is initial guess and with respect to our problem, it is considered parabola:

$$g_0 = 1/2\ \eta^2/\eta_\delta \quad (21)$$

it should be noted that initial guess assumption satisfies the boundary conditions. Since N is linear operator, therefore, we conclude that:

$$f_0 = go = 1/2\ \eta^2 / \eta_\delta \quad (22)$$

First-order: The first order equation is:

$$\frac{d^3 f_1}{d\eta^3} + \alpha f_0 \frac{d^2 f_0}{d\eta^2} + \beta\left(1 - \left(\frac{d^2 f_0}{d\eta^2}\right)^2\right) = 0 \tag{23}$$

subject to boundary conditions:

$$f(0) = d/d\eta\ f(0) = 0,\ d/d\eta\ f(\eta_\delta) = 0 \tag{24}$$

for first order solution, we substitute the zero-order solution f_0 in to Eq. (23) and some simplification along with boundary conditions, so we obtain the first order solution in the form:

$$f_1 = \frac{1}{120}\frac{\eta^5}{\eta_\delta^2}(2\beta - \alpha) - \frac{1}{6}\beta\eta^3 + \frac{1}{2}\left(\frac{5}{12}\beta\eta_\delta + \frac{1}{24}\alpha\eta_\delta\right)\eta^2 \tag{25}$$

Second-order: The second order equation along with boundary condition is:

$$\frac{d^3 f_2}{d\eta^3} + \alpha\left(f_0 \frac{d^2 f_1}{d\eta^2} + f_1 \frac{d^2 f_0}{d\eta^2}\right) - 2\beta\left(\frac{df_1}{d\eta}\frac{df_0}{dn}\right) = 0 \tag{26}$$

$$f(0) = d/d\eta\ f(0) = 0,\ d/d\eta\ f(\eta_\delta) = 0 \tag{27}$$

the resulting differential equation subjected to the boundary condition, f_2 is obtained as follows:

$$\begin{aligned} f_2 = {} & \frac{1}{120}\frac{1}{\eta_\delta^3}\Big(\frac{1}{336}(-32\alpha\beta + 11\alpha^2 + 20\beta^2)\eta^8 + \frac{1}{120}(80\alpha\beta_\delta^2 - 120\beta^3\eta_\delta^2)\eta^6 \\ & + \frac{1}{60}(-5\alpha^2\eta_\delta^3 - 40\alpha\beta\eta_\delta^3 + 100\beta^2\eta_\delta^3)\eta^5 \\ & + \frac{1}{2}\Big(\frac{1}{1260}\alpha\beta\eta_\delta^3 + \frac{13}{10080}\alpha^2\eta_\delta^3 - \frac{59}{2520}\beta^2\eta_\delta^3\Big)\eta^2 \end{aligned} \tag{28}$$

Third-order: And the third order equation is:

$$\frac{d^3 f_1}{d\eta^3} + \alpha f_0 \frac{d^2 f_0}{d\eta^2} + \beta\left(1 - \left(\frac{d^2 f_0}{d\eta^2}\right)\right) = 0 \tag{29}$$

subject to boundary conditions:

$$f(0) = d/d\eta\ f(0) = 0,\ d/d\eta\ f(\eta_\delta) = 0 \tag{30}$$

with some simplifications along with boundary conditions, we obtain the third order solution in the form:

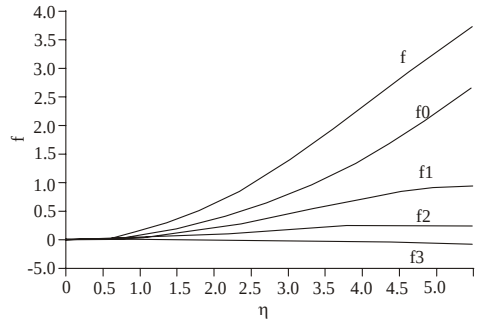

Fig. 1: Effect of each iteration for Blasius equation

$$\begin{aligned} f_3 = {} & \frac{1}{40320}\frac{1}{\eta_\delta^4}\Big(\frac{1}{990}(1398\beta\alpha^2 - 375\alpha^3 + 600\beta^3 - 1596\beta^2\alpha)\eta^{11} + \frac{1}{504}(12656\beta^2\alpha\eta_\delta^2 - 7392\beta^3\eta_\delta^2 \\ & - 5040\alpha^2\beta\eta_\delta^2)\eta^9 + \frac{1}{336}(-12600\beta^2\alpha\eta_\delta^3 + 8400\beta^3\eta_\delta^2 + 3276\beta\alpha^2\eta_\delta^2 + 462\alpha^3\eta_\delta^3)\eta^8 + \frac{1}{210}(-6720\alpha\beta^2\eta_\delta^4 \\ & - 10080\beta^3\eta_\delta^4)\eta^7 + \frac{1}{120}(9520\beta^2\eta_\delta^5\alpha - 16800\beta^3\eta_\delta^5 + 1120\alpha^2\beta\eta_\delta^5)\eta^6 + \frac{1}{60}(-87\alpha^3\eta_\delta^6 + 5112\beta^3\eta_\delta^6 \\ & - 1092\beta^2\eta_\delta^6\alpha - 558\beta\eta_\delta^6\alpha^2)\eta^5\Big) + \frac{1}{2}\Big(-\frac{391}{1209600}\alpha^2\beta\eta_\delta^5 + \frac{1}{96768}\alpha^3\eta_\delta^5 + \frac{1}{12096}\beta^3\eta_\delta^5 - \frac{1039}{604800}\beta^2\eta_\delta^5\alpha\Big)\eta^2 \end{aligned} \tag{31}$$

final solution of Eq. (17) by using HPM up to third order is obtained as follow.

$$f(\eta) = \lim_{p\to 1} f(\eta, p) = f_0(\eta) + f_1(\eta) + f_2(\eta) + f_3(\eta) \tag{32}$$

Figure (1) shows iterations and effect of them for case I.

INVESTIGATION RESULTS

To verify accuracy and validate the obtained solution for Falkner Skan equation, result investigated in four cases. As we know for various coefficients a,L98\f"Symbol"\s12 we can govern many famous problems of boundary layer from Falkner Skan equation (Bansal, 2004)

Case 1: Blasius equation (Schlichting and Gerstan, 1999):

$$\begin{cases} \alpha = \dfrac{1}{2} \\ \beta = 0 \end{cases}$$

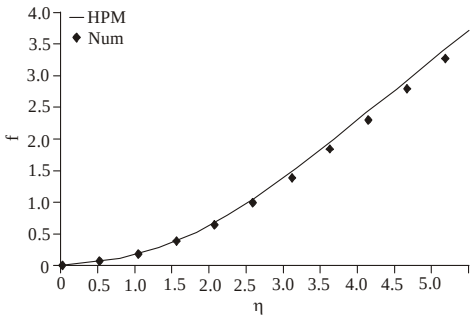

Fig. 2: Comparison between HPM and numerical solutions for blasius

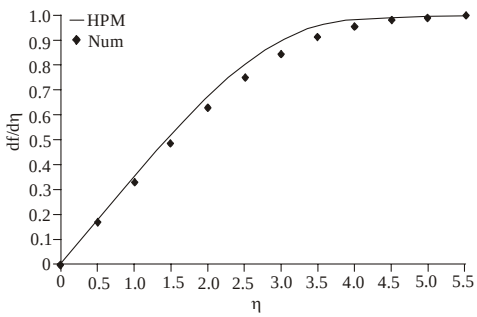

Fig. 3: Comparison between HPM and numerical solutions for blasius

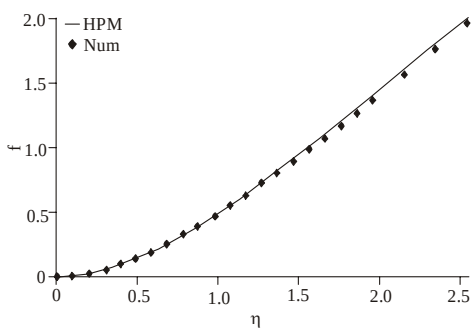

Fig. 4: Comparison between HPM and numerical solutions for plane stagnation-point flow

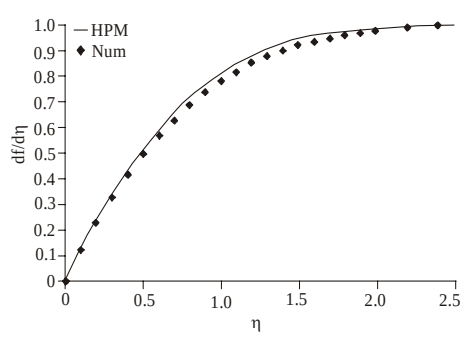

Fig. 5: Comparison between HPM and numerical solutions for plane stagnation-point flow

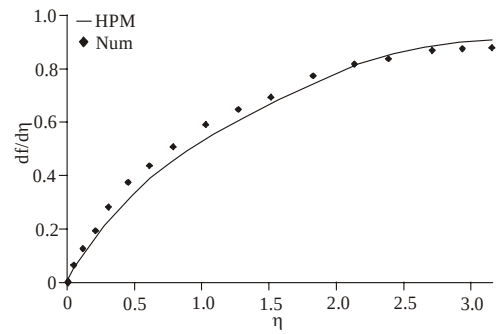

Fig. 6: Comparison between HPM and numerical solutions for convergent channel

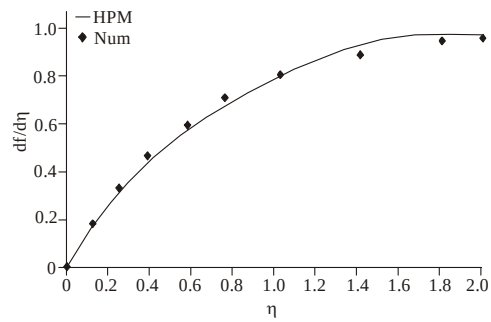

Fig. 7: Comparison between HPM and numerical solutions for sink at leading edge

Case 2: plane stagnation-point flow (Bansal 2004).

$$\begin{cases} \alpha = 1 \\ \beta = -1 \end{cases}$$

Case 3: convergent channel (Bansal 2004).

$$\begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}$$

Case 4: sink at leading edge (Kundu and Cohen, 2002).

$$\begin{cases} \alpha = -\dfrac{1}{2} \\ \beta = 2 \end{cases}$$

All four cases are graphed from tables in (Bansal 2004), 9, (Kundu and Cohen, 2002) and compared to HPM solution in Fig. (2-7).

CONCLUSION

Since, Eq. (10) cannot be easily solved by the analytical method; Eq. (10) is, therefore solved with

HPM. As we can see obtained solution for Falkner Skan equation by HPM has high accuracy and simple.

Although in this study we use four iterations but we can conclude from Fig. (1) that, solutions of three iterations are enough and they converge to proper solution.

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