

Numerical Approach to Construction of Lyapunov Function for Nonlinear Stability Analysis

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Abstract: This study proposes a numerical approach using Adaptive Tabu search (ATS) to construct Lyapunov function that maximizes region of attraction. The proposed methods is directly useful for stability analysis of nonlinear dynamical systems. Two illustrative examples including a synchronous generator possessing multiple equilibriums and a non-autonomous system are given to demonstrate the practicality and effectiveness of the approach.

Keywords: Adaptive tabu search, lyapunov's direct method, nonlinear stability analysis, region of attraction

INTRODUCTION

Difficulty in stability analysis of a nonlinear system using the direct method of Lyapunov's is to find a suitable Lyapunov function candidate. As the classic Lyapunov's theorems are explained in textbooks (Slotine and Li, 1991; Khalil, 1992), review of the theorems is thus omitted in this article. To find a Lyapunov function candidate, one may apply algebraic method (Golub *et al.*, 1979; Hachicho, 2007), numerical algorithm (Zhaolu, Chuanqing, 2008; Sorensen and Zhou, 2003) or genetic programming (Grosman and Lewin, 2008). This article presents an application of Adaptive Tabu Search (ATS) (Sujitjorn *et al.*, 2006) as a global optimization tool to search for a Lyapunov function in quadratic form. The next section presents the definitions of the quadratic Lyapunov function and the domain of attraction (also known as basin, attraction region and stability region). Then, we present the specific implementation of the ATS for stability analysis problems. Two illustrative examples are given in Results and Discussions section. Conclusion follows in the last section.

MATERIALS AND METHODS

The Lyapunov's direct method for stability analysis calls for a scalar function referred to as Lyapunov function. Asymptotic stability of a dynamical system can be concluded if and only if the function is positive definite and its derivative is negative definite. It is often difficult to find such a function for a nonlinear system. Nonetheless, it is well-known that a general quadratic

function can be used as a Lyapunov function candidate. Difficulty still exists in finding coefficients of the quadratic Lyapunov function. This section explains the proposed method to identify the coefficients. The approach is to use Adaptive Tabu Search (ATS), one type of metaheuristics, to search for possible parameters.

Definition 1: (Quadratic form), (Ogata, 1997). The quadratic Lyapunov function is defined by:

$$V(x) = x^T P x \tag{1}$$

where, x is the real n -column vector, P is a real symmetric matrix, i.e., $P = P^T$.

For a stable system, it is essential to ensure the positive definiteness of $V(x)$ and the negative definiteness

of $\dot{V}(x)$. In engineering practice, the stability region is an important issue. Assume that D is a region enclosing an equilibrium point and $\Omega_d \subset D$ is the set containing an estimated area formed from a negative definite function.

$\dot{V}(x)$, the domain of attraction is defined next.

Definition 2: (Domain of attraction), (Khalil, 1992).

Assume that $f : D \rightarrow R^N$ is locally Lipschitz and $D \subset R^n$ is the domain containing the origin, which is an asymptotically stable equilibrium point. Let $V(x)$ be a Lyapunov function for a system in the domain:

$$\Omega_d = \{x \in R^n \mid V(x) \leq d\}, d > 0 \tag{2}$$

where, d is a real value indicating an estimated area. If Ω_d encloses the equilibrium point and if $\dot{V}(x)$ is negative definite in the Ω_d domain, such equilibrium point is said to be asymptotically stable. As such, every solution inside the Ω_d domain converges to the equilibrium as $t \rightarrow \infty$. The domain Ω_d is said to be the domain of attraction.

To analyse stability of a nonlinear system, one has to compute the coefficients of the Lyapunov function, its first derivation with-respect-to time and the domain of attraction (or stability region). This article proposes a use of the ATS to search for the required solutions. The Adaptive Tabu Search (ATS) has several strong features compared with the conventional tabu search (Glover, 1989, 1990). As a result, the ATS converges faster to a global solution. Readers can find the generic algorithms of the ATS in the literature (Sujitjorn *et al.*, 2006). The procedural list of the ATS below contains specific modifications for nonlinear stability analysis. Notice that the mathematical model of the system of interest is embedded in the Lyapunov function candidate (refer to Steps 3-4).

Algorithm implementation:

Step 1: Initialize search parameters, search spaces, iteration max and Tabu List (TL).

Step 2: Assign random values to the elements of the matrix P.

Step 3: Define symbolic variables $V(x)$, $\dot{V}(x)$, x_1 , x_2 , x_1 and x_2 , respectively.

Step 4: Define the dynamic system $\dot{x} = f(x, u, t)$.

Step 5: Compute $V(x)$ and $\dot{V}(x)$.

Step 6: Extract (x, y) coordinates from the graph of $V(x)$.

Step 7: Evaluate the positive definiteness and the negative definiteness of $V(x)$ and $\dot{V}(x)$, respectively.

If ($V(x) > 0$ and $\dot{V}(x) < 0$) then

- Compute the area of $V(x)$
- As $\text{sign } J \leq \text{area}^{-1}$.

If ($V(x) \leq 0$ and $\dot{V}(x) \geq 0$) then

- Assign $J = 10^5$

Step 8: Create a set of solutions in the neighborhood of the initial solutions. Evaluate costs of all solutions (refer to Steps 5-7). Do minimum sorting. Assign the set of solutions with the minimum cost as “current-best”.

Step 9: If (the cost of current-best < the cost of initial solutions) then:

- Assign global-solution = current-best
- Put initial solution in the TL, otherwise
- Global-solution = initial solution
- Put current-best in the TL

Step 10: If termination criterion is satisfied, then exit with the global-solution.

Step 11: If solution cycling (solutions of similar quality revisited) occurs 5 times, then retrieve the 5th-backward solutions in the TL, assign them to be new initial solutions, go to Step 8.

Step 12: If the search comes close to an elite set of solutions; invoke Adaptive search Radius (AR) procedures.

Step 13: Go to Step 8.

Remarks 1: To keep the CPU time low, ones should not plot the graph according to Step 6. Extraction of the coordinates is possible using suitable MATLAB™ commands while graph plotting is inactive.

Remarks 2: The implementation of the AR procedures for each example is as follows:

Example 1 AR: {if best_error ≤ 0.6 then R = 0.8; if best_error ≤ 0.14 then R = 0.25; if best_error ≤ 0.13 then R = 0.2; if best_error ≤ 0.12 then R = 0.15; if best_error ≤ 0.11 then R = 0.1}.

Example 2 AR: {if best_error ≤ 0.5 then R = 0.3; if best_error ≤ 0.3 then R = 0.2; if best_error ≤ 0.1 then R = 0.1; if best_error ≤ 0.05 then R = 0.05}.

RESULTS AND DISCUSSION

The following examples used the ATS coded in MATLAB™ running on a Pentium™ Core 2 Duo, 2.0 GHz and 2.83 Gbytes of RAM.

Example 1: The dynamics of the synchronous generator (Genesio and Vicino, 1984) are given by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -Dx_2 - \sin x_1 + \sin \delta_0 \end{aligned} \tag{3}$$

where, x_1 is the power angle and x_2 is the corresponding speed deviation. Let $D = 0.5$ and, $\delta_0 = 0.412$, the equilibrium point of interest is $x_1 = 0.412$, $x_2 = 0$; the

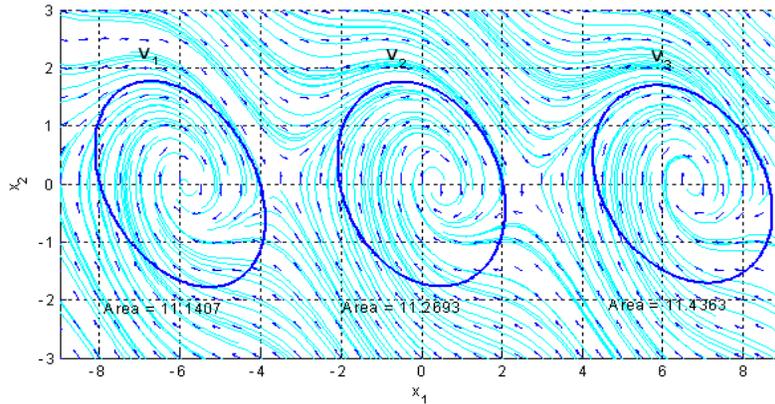


Fig. 1: Stability regions of example 1

Table 1: Obtained parameters V(x)

No	$V(x) = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 - d$				Areas
	p_{11}	p_{12}	p_{22}	d	
1	0.0624	0.0084	0.0174	0.7483	148.127
2	0.0698	0.0097	0.0352	0.7426	163.329
3	0.0511	0.0106	0.0237	0.6754	157.283
4	0.0511	0.0106	0.0237	0.6754	163.690

other critical points to be considered are $x_1 = 0.412 + \pi$, $x_2 = 0$ and $x_1 = -0.412 + \pi$, $x_2 = 0$. The termination criterion is either $J \leq 0.1$ or maximum iteration = 1,000.

The results of $V(x)$ s all terminated by the minimum J criterion are shown in (4)-(6) for three equilibriums:

$$V_1(x) = 0.2183x^2 + 2.6196x + 0.1606xy + 0.9633y + 0.3719y^2 + 6.8634 \quad (4)$$

$$V_2(x) = 0.2308x^2 + 0.1265xy + 0.322y^2 - 0.9514 \quad (5)$$

$$V_3(x) = 0.2354x^2 - 3.0602x + 0.1932xy - 1.256y + 0.3351y^2 + 9.01015 \quad (6)$$

The time derivative of the $V(x)$ s are:

$$\dot{V}_1(x) = 0.0643x + 2.4350y + 0.3562xy - 0.2113y^2 + 0.3858 - 0.9636\sin(x) - 0.1606x\sin(x) - 0.7438y\sin(x) \quad (7)$$

$$\dot{V}_2(x) = 0.0506x + 0.3983xy + 0.2578y - 0.1955y^2 - 0.644y\sin(x) - 0.1265x\sin(x) \quad (8)$$

$$\dot{V}_3(x) = 0.0773x - 2.1647y + 0.3743xy - 0.1418y^2 - 0.5028 + 1.2558\sin(x) - 0.1932x\sin(x) - 0.6701y\sin(x) \quad (9)$$

Figure 1 illustrates the phase portraits with the corresponding stability regions, which cover all the asymptotically stable equilibriums.

Example 2: Consider the following non-autonomous system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2 \end{aligned} \quad (10)$$

The time varying term is $f(t) = \sin(t)$. The termination criterion is either $J \leq 0.0072$ or maximum iteration = 1,000. The ATS returns the parameters of $V(x)$ s as tabulated in Table 1. The time derivative of $V(x)$ s are shown in (12)-(15):

$$\dot{V}_1(x) = -0.0869x_1^2 + 0.0057x_1x_2 - 0.0163x_2^2 + 0.0173x_1\sin(t) + 0.0168x_2\sin(t) \quad (12)$$

$$\dot{V}_2(x) = -0.1762x_1^2 - 0.0283x_1x_2 - 0.0037x_2^2 + 0.0352x_1\sin(t) + 0.0194x_2\sin(t) \quad (13)$$

$$\dot{V}_3(x) = -0.1186x_1^2 - 0.0507x_1x_2 - 0.0185x_2^2 + 0.0237x_1\sin(t) + 0.0211x_2\sin(t) \quad (14)$$

$$\dot{V}_4(x) = -0.0678x_1^2 - 0.0112x_1x_2 - 0.0198x_2^2 + 0.0135x_1\sin(t) - 0.0166x_2\sin(t) \quad (15)$$

Several Lyapunov functions can be used to form a large stability region by doing a geometrical union as shown in Fig. 2. The phase portraits with four ellipses represent the Lyapunov functions. The solid line represents the union of the sub-regions. Figure 3 illustrates the substitutions of extracted coordinates (x, y)

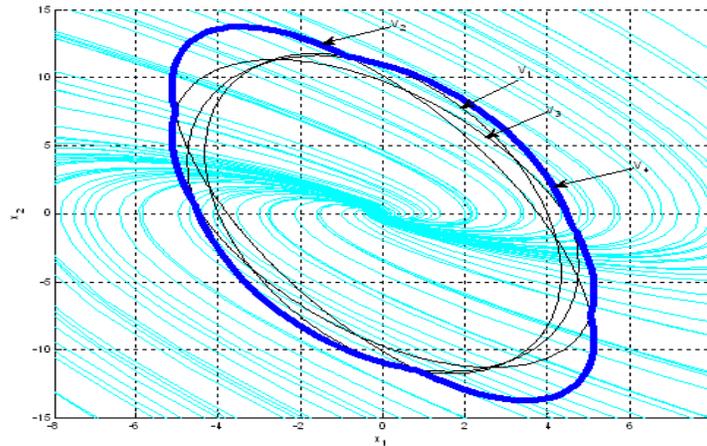


Fig. 2: Stability regions of example 2

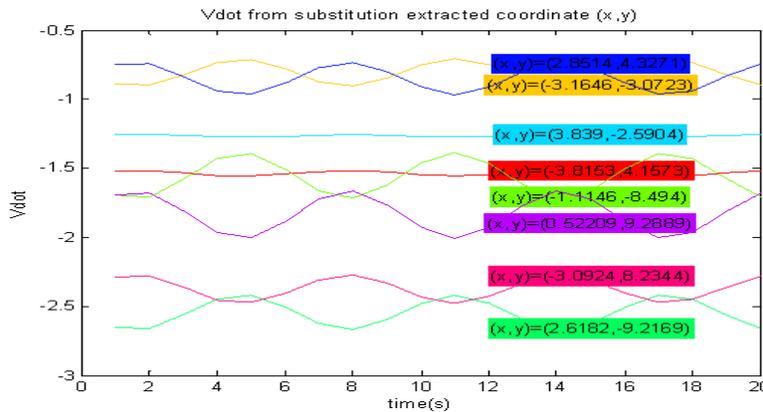


Fig. 3: Some of the substitution of coordinates (x, y) into $V(x)$

into $\dot{V}(x)$ which are all negative. According to the results, the system is asymptotically stable at the origin.

CONCLUSION

This article has presented the ATS applied to nonlinear stability analysis problems. Regarding this, an optimization problem can be formulated and solved by the ATS as the algorithms are elaborated by the article. Results of two examples including a synchronous generator and a non-autonomous system show the effectiveness of this approach. Our future works are to apply the method to stabilization problem of chaotic electronic circuits.

ACKNOWLEDGMENT

The authors gratefully acknowledge financial support by Rajamangala University of Technology Isarn, Suranaree University of Technology and the Office of the Higher Education Commission under the NRU Project of Thailand.

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