Riemann Boundary Value Problems for Koch Curve

Zhengshun Ruan and Aihua Luo
Zhengshun Ruan and Aihua Luo

Abstract: In this study, when $L$ is substituted for Koch curve, Riemann boundary value problems was defined, but generally speaking, Cauchy-type integral is meaningless on Koch curve. When some analytic conditions are attached to functions $G(z)$ and $g(z)$, through the limit function of a sequence of Cauchy-type integrals, the homogeneous and non-homogeneous Riemann boundary problems on Koch curve are introduced, some similar results was attained like the classical boundary value problems for analytic functions.

Keywords: Holder condition, index, koch curve, riemann boundary value problem

INTRODUCTION

In the classical boundary value problems, let $L$ be an arc-wise smooth and have no sharp points in the complex plane, $G(z)$, $g(z)$ satisfy Holder condition of order $a (0 < a < 1)$ and $G(t)$ is $\Phi$- ($t$) + $g(t)$ ($t \not\equiv \Phi$), a detailed discussion of the Riemann boundary problems $\Phi$ ($t$) = $G(t)$ $\Phi$ ($t$) + $g(t)$ ($t \in L$) was presented, a complete result was attained. But when $L$ is substituted for Koch curve, functions $H^2$, $H^3$, $H^4$ (in the interior of $K_0$) and $H^0$ are defined, but generally speaking, Cauchy-type integral is meaningless on Koch curve.

In this study, when $L$ is substituted for Koch curve, Riemann boundary value problems was introduced, some similar results was attained like the classical boundary value problems for analytic functions.

Let $K_0$ be the curve of a equilateral triangle whose vertices are $z_k - e^{\frac{2\pi k}{3}} (k = 0, 1, 2)$, choosing the one-third of each side as the base, three equilateral triangles are attained from the outside of $K_0$, getting rid of these bases (the end point is kept), a polygonal curve $K_1$ is obtained. These steps are repeated all the time, the polygonal curve $K_n$ is obtained which consist of lines $I_{n,j}, j - 1, 2, 3, ... 4^n, 3, 4^n$ bumper to bumper, length of each line is $|I_{n,j}| = \frac{\sqrt{3}}{3^n}$. The Koch Curve $K$ is obtained when $n$ approaches infinity and the Hausdorff dimension of $K$ is $S = \log_2 \frac{3}{2}$. By using the fractal geometry: mathematics foundings and applications, Ruan and Ai-hua (2011) study a class of boundary value problem for analytic functions bounded by the koch curve. Muskhelishvili (1953) study the singular integral equations. Jian-ke (1993) have a research of the estimation of the hausdorff dimension of koch curve and sierpinski mat.

For $K$ and each $K_n$, oriented positively is counterclockwise, denoted by $E^+$ (or $E^-$) and $E_n^+$ (or $E_n^-$) the interior zone and exterior zone of $K$ and each $K_n$, then $E_n^+ \subseteq E_{n+1}^+$, $E_{n-1}^- \subseteq E_n^- (n \in N)$.

Let $\delta > 0$, $K_\delta^+ = \bigcup \{z \in E^+ | |z - t| < \delta \}$, $K_\delta^-$ is defined the $\delta$-neighborhood from the left (closed region). From the construction of $K$, each $E_n^1$, $E_n^+$ consist of closed region of $3 \times 4^n$ equilateral triangles $\frac{\sqrt{3}}{3^n}$ and each $E_n^- \subseteq E_{n+1}^+$, $E_{n-1}^- \subseteq E_n^-$. The boundaries of these triangular areas are denoted by $\Delta_{n,j} (j = 1, 2, ... 3 \times 4^n)$, $\Delta_{n,j}^+$ is the interior zone. When $\delta > 0$ is fixed, there exists $N_0 \in N$, such that $E_n^2 - E_{n0}^- = \bigcup_{n \in N \backslash N_0} \bigcup_{j = 1}^{3 \times 4^n} \Delta_{n,j}^+$ and each $\Delta_{n,j} (j = 1, 2, ... 3 \times 4^n)$ lies in the interior of $K_\delta^+$. In the following, $N_0$ is fixed and $\delta > 0$ and small sufficiently such that $0 \in K_\delta^+$. The boundaries of these triangular areas are denoted by $\Delta_{n,j} (j = 1, 2, ... 3 \times 4^n)$, $\Delta_{n,j}^+$ is the interior zone. When $\delta > 0$ is fixed, there exists $N_0 \in N$, such that $E_n^2 - E_{n0}^- = \bigcup_{n \in N \backslash N_0} \bigcup_{j = 1}^{3 \times 4^n} \Delta_{n,j}^+$ and each $\Delta_{n,j} (j = 1, 2, ... 3 \times 4^n)$ lies in the interior of $K_\delta^+$. In the following, $N_0$ is fixed and $\delta > 0$ and small sufficiently such that $0 \in K_\delta^+$.

Definition: Assume $0 < a < l$, If $f$ satisfy Holder condition of order $a (0 < a < l)$, which is $|f(z_1)| < M|z_1-z_2|^{a}$ for $M > 0$ and $f(z)$ is an analytic function in interior of $K_\delta^+$, then $f \in Ah^a (K_\delta^+)$.

Lemma 1: $t_0 \in K$ if and only if there exists $N (t_0) \in N$ such that $t_0 \in \bigcap_{n \in N \backslash N_0} K_n$.

Lemma 2: Assume:

$s - 1 < a < 1$, $s = \log_3 4$, $f \in Ah^a (K_\delta^+)$,

Then,
\[ \Phi(z) = \lim_{n \to \infty} \Phi_n(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_n} \frac{\phi(t)}{t - z} \, dt \] (1)

For any \( t_0 \in \mathbb{K} \), \( \Phi^+(t_0) = \lim_{z \to t_0 \in \mathbb{E}^+} \Phi(z) \) and \( \Phi^-(t_0) = \lim_{z \to t_0 \in \mathbb{E}^-} \Phi(z) \). Both exist, satisfied:

\[ \Phi^+(t_0) - \Phi^-(t_0) = \varphi(t_0) \] (2)

From the proof of Lemma 2, we can see that:

\[ \Phi(z) = \Phi_{a_+}(z), \quad (n \geq N(t_0), \quad z \in \mathbb{E}^+ \cup \mathbb{E}^-) \] (3)

and

\[ \Phi^\pm(t_0) = \Phi^\pm_{a_+}(t_0), \quad (n \geq N(t_0)) \] (4)

Now the Riemann boundary value problems for Koch Curve is considered, the analytic function \( \Phi(z) \) defined in \( \mathbb{E}^+ \) and \( \mathbb{E}^- \), satisfying the condition of Riemann boundary value:

\[ \Phi^+(t) = G(t) \Phi^-(t) + g(t) \quad (t \in \mathbb{K}) \] (*

where, \( G, \ g \in AH^a_k (k^+ \mathbb{K}), S - l < a < l \) \( G(z) \neq 0 \) \( z \in k^+ \mathbb{K} \) and \( \Phi \in \mathbb{R}_0 \) (that is \( \Phi(\infty) \) is finite).

In this study, when \( L \) is substituted for Koch curve, we define the Riemann boundary value problem. Cauchy-type integral is meaningless on Koch curve. Moreover, when some analytic conditions are attached to functions \( G(z) \) and \( g(z) \), through the limit function of a sequence of Cauchy-type integrals, the homogeneous and non-homogeneous Riemann boundary problems on Koch curve are introduced, some similar results was attained like the classical boundary value problems for analytic functions.

**RIEMANN BOUNDARY VALUE PROBLEMS AND THEIR SOLUTIONS**

The simplest Riemann boundary value problem is the jump problem: \( G(t) = 1 \) in (*), i.e., \( \Phi(t^-) - \Phi(t^+) = g(t) \) \( (t \in \mathbb{K}) \) (*). If \( \Phi(z) \) is required to have a pole of order \( m \geq 1 \) at most at infinity, then the problem is denoted by \( \Phi \in \mathbb{R}_m \), if \( \Phi(z) \) is required to equal to 0 of order \( m \geq 1 \), the problem is denoted by \( \Phi \in \mathbb{R}_m \).

**Theorem 1:** The solution of the jump problem (*), is given in following: its unique solution is given by (5) when solved in \( \mathbb{R}_m \): when \( m \geq 0 \), it has the general solution (6), when \( m \geq 2 \), it has the unique solution (5) if and only if m-2 conditions of solvability in (7) are fulfilled.

**Proof:** From (1), (2), (3) when solved in \( \mathbb{R}_m \):

\[ \Phi(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_n} \frac{g(t)}{t - z} \, dt \] (5)

is the unique solution of jump problem (*).

By the extended Liouville theorem, the general solution of (*), in \( \mathbb{R}_m (m \geq 0) \) is:

\[ \Phi(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_n} \frac{g(t)}{t - z} \, dt + P_m(z) \] (6)

where, \( P_m(z) \) is an arbitrary polynomial of degree \( m \). From (3), for any \( t_0 \in \mathbb{K} \), allowing for:

\[ \Phi(z) = \Phi_{a_}(z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{g(t)}{t - z} \, dt \quad (n \geq N(t_0), \quad z \in \mathbb{E}_n^{\mathbb{K}} \cup \mathbb{E}^-) \]

it has \( \Phi(z) = \sum_{k=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma_n} g(t) \, dt \right) z^{(k+1)}, \quad (n \geq N(t_0), \quad z \in \mathbb{G} \) for the region \( \mathbb{G} \) which satisfied the neighbourhood of \( z = \infty \) and \( \mathbb{G} \cap \mathbb{E} \). So if and only if:

\[ \lim_{n \to \infty} \int_{\gamma_n} g(t) t^k \, dt = 0 \quad (k = 0, 1, \ldots, m - 2) \] (7)

\( \Phi(z) \) defined by (5) has a zero point at infinity of order \( m \geq 2 \).

If \( g(t) = 0 \) in (*), then it is called a homogeneous Riemann boundary value problem, that is:

\[ \Phi^+(t) = G(t) \Phi^-(t) \quad (t \in \mathbb{K}) \] (*

**Theorem 2:** For the homogeneous Riemann boundary value problem (*), if the solutions are in \( \mathbb{R}_0 \), it has only the trivial solution when its index \( K < 0 \) and its solution is \( \Phi(z) = X(z) P_m(z) \) when \( K \geq 0 \), where \( P_m(z) \) is an arbitrary polynomial of degree \( m \).

**Proof:** Because \( G \in AH^a_k (k^+ \mathbb{K}) \) and taking value of no zero, there exists a integer \( K \), such that:

\[ \int_{2\pi i} \log G(z) \, dk = \int_{2\pi i} \log G(z) \, dk = K \quad (n \geq N_0) \]

called \( K \) is the index of (*), or (*).

Assume \( G_0(z) = z^k G(z) \), then \( G_0 \in AH^a_k (k^+ \mathbb{K}) \), \( G_0(z) \neq 0 \) \( (z \in k^+ \mathbb{K}) \),

by introducing a new function \( \psi(z) \):
\[ \Psi(z) = \begin{cases} 
\Phi(z), & z \in E^+ \\
\frac{1}{2\pi i} \int_{\kappa} \frac{\log G_0(t)}{t - z} dt, & z \notin E^+ \end{cases} \]

then (*)_2 becomes a Riemann boundary value problem for \( \psi(z) \psi^-(t) = G_0(t) \psi^+(t) (t \in K) \) with index 0. Since \( \Phi \in R_\kappa \), its solution \( \psi \in R_{-\kappa} \). If we set:

\[ \Gamma_n(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\kappa} \frac{\log G_0(t)}{t - z} dt, (z \notin K, n \geq N_0) \]

\[ \Gamma(z) = \lim_{n \to \infty} \Gamma_n(z) \]

which satisfied \( \Gamma^+(t) - \Gamma^-(t) = \log G_0(t) (t \in k) \) and \( \Gamma(\infty) = 0 \). Introducing \( X_0(z) = e^{\Gamma(z)} \), then \( X_n(z) \) is analytic in \( E^+ \cup E^- \) and \( X_0(z) = 1 \), \( X_0(z) \) and \( X^+(t) (t \in K) \) have no zero points, also:

\[ X_0^+(t) = G_0(t) X_0^-(t) = G(t) t^k X_0^-(t) (t \in K) \]

and call it canonical function of (*)_2 or (*) , then:

- \( X(z) \) satisfy \( X(z) \) is analytic in \( E^+ \cup E^- \) and \( X(z) \neq 0 \) \( (z \in E^+ \cup E^-) \)

- \( X^+(t) (t \in K) \) exists and \( X^+(t) \neq 0 (t \in K) \)

- \( X(z) \) has finite order -\( K \) at \( z = \infty \)

\[ X^+(t) = G(t) X^-(t) (t \in K) \]

Considering (9), (*)_2 becomes a jump problem:

\[ \Phi^+(t) - \Phi^-(t) = g(t) X^+(t) - X^-(t) = 0 \quad (t \in K) \quad (10) \]

From the process in jump problem, the results are attained from solving \( \frac{\Phi(t)}{X(t)} \) in \( R_\kappa \).

Let us consider the non-homogeneous Riemann boundary value problem (*), from the index, the canonical function \( X(z) \) of its corresponding homogeneous problem in (8), Considering (9), (*) becomes a jump problem:

\[ \Phi^+(t) - \Phi^-(t) = g(t) X^+(t) - X^-(t) = 0 \quad (t \in K) \quad (11) \]

solving \( \frac{\Phi(t)}{X(t)} \) in \( R_\kappa \). Since \( g \in AH^a(K^+) \), but \( \frac{g(0)}{X^+(0)} \) is may not a function in \( \frac{g(t)}{X^+(t)} \). For any \( t_0 \in K \), log \( G_0 \in AH^a(K^+) \), therefore \( \Gamma(z) = \Gamma_n(z) = \int_{-\kappa \pi}^{\kappa \pi} \log G_0(t) dt \)

(\( z \in E^+ \)).

By the Privalov theorem, \( \Gamma_n(z) \) and \( \Gamma_n^+(t) \) are analytic and satisfy Holder condition of order \( a \) in \( E^+ \), since \( X(z) = e^{\Gamma(z)} \)

(\( z \in E^+, n \geq N(t_0) \)), \( X(z) \) and \( X_n^+(t) \) are analytic and satisfy Holder condition of order \( a \) in \( E^+ \).

Introducing \( n \geq N_0 \):

\[ X^+(z) = \begin{cases} 
X^+(z), & \text{if } z \in K \cap K_n; \\
(1) & \text{if } z \in E^+ \setminus K \cap K_n \end{cases} \]

(12)

then \( \frac{g(t)}{X(t)} \) holds the condition of theorem 1, we have:

**Theorem 3:** For the non-homogeneous Riemann boundary value problem (*), if the solutions are in \( R_\kappa \), its general solution is:

\[ \Phi(z) = \lim_{n \to \infty} \frac{X(z)}{2\pi i} \int_{K} \frac{g(t)}{X^+(t) - z} dt + X(z) P_m(z) \]

when, \( k \geq 0 \), where \( P_m(z) \) is an arbitrary polynomial of degree \( m \). Its unique solution is:

\[ \Phi(z) = \lim_{n \to \infty} \frac{X(z)}{2\pi i} \int_{K} \frac{g(t)}{X^+(t) - z} dt + X(z) P_m(z) \]

(13)

when, \( K = -1 \), its unique solution is (13) if and only if:

\[ \lim_{n \to \infty} \int_{K} \frac{g(t)}{X^+(t) - z} dt = 0 \quad (k = 0, 1, \ldots, \kappa - 2) \]

is satisfied when \( K \leq 2 \).

**Corollary:** For the non-homogeneous Riemann boundary value problem (*), if the solutions are in \( R_{-1} \), corresponding problem (11) is solved in \( R_{-1} \), when \( K \geq 1 \), its general solution is:

\[ \Phi(z) = \lim_{n \to \infty} \frac{X(z)}{2\pi i} \int_{K} \frac{g(t)}{X^+(t) - z} dt + X(z) P_m(z) \]

its unique solution is (13) if and only if:

\[ \lim_{n \to \infty} \int_{K} \frac{g(t)}{X^+(t) - z} dt = 0 \quad (k = 0, 1, \ldots, \kappa - 1) \]

is satisfied when \( K \leq 1 \).
ACKNOWLEDGMENT

This study was supported by the Special Fund for Basic Scientific Research of Central Colleges, South-Central University for Nationalities, CZQ12016.

REFERENCES


