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Riemann Boundary Value Problems for Koch Curve

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Abstract: In this study, when L is substituted for Koch curve, Riemann boundary value problems was defined, but generally speaking, Cauchy-type integral is meaningless on Koch curve. When some analytic conditions are attached to functions G (z) and g (z), through the limit function of a sequence of Cauchy-type integrals, the homogeneous and non-homogeneous Riemann boundary problems on Koch curve are introduced, some similar results was attained like the classical boundary value problems for analytic functions.

Keywords: Holder condition, index, koch curve, riemann boundary value problem

INTRODUCTION

In the classical boundary value problems, let Lbe an arc-wise smooth and have no sharp points in the complex plane, G (z), g (z) satisfy Holder condition of order a (0‹a‹1) and G (t) ≠ 0 (t∈ L), a detailed discussion of the Riemann boundary problems Φ^+ (t) = G (t) Φ^- (t) + g (t) (t∈ L) was presented, a complete result was attained. But when L is substituted for Koch curve, the problems and results will change corresponding.

Let K_0 be the curve of a equilateral triangle whose vertices are $z_k - e^{2/3k_{Л}i}$ (k – 0, 1, 2), choosing the one-third of each side as the base, three equilateral triangles are attained from the outside of K_0, getting rid of these bases (the end point is keeped), a polygonal curve K_1 is obtained. These steps are repeated all the time, the polygonal curve K_n is obtained which consist of lines $I_{n,j}$, j-1, 2, ..3×4^n bumper to bumper, length of each line is $|I_{nj}| = \frac{\sqrt{3}}{3^{\square}}$. The Koch Curve K is obtained when n approaches infinity and the Hausdorff dimension of K is S = $\log_3 4$. Falconer (1990) have a research of the fractal geometry: mathematics foundings and applications. Ruan and Ai-hua (2011) study a class of boundary value problem for analytic functions bounded by the koch curve. Muskhelishvili (1953) study the singular integral equations. Jian-ke (1993) have a research of the boundary value problems for analytic functions. Wang and Gong (2001) study the stability about intergral curves of singular intergrals with Cauchy Kernel. Wang and Gong (1999) analyze the stability about intergral curves of singular intergrals and cauchy-type integrals. Qu and Su (2000) have a research of the estimation of the hausdorff himension of koch curve and sierpinski mat.

For K and each K_n, oriented positively is counterclockwise, denoted by E^+ (orE^-) and E_n^+ (or E_n^-) the interior zone and exterior zone of K and each K_n, then $E_n^+ \subset E_{n+1}^+$, $E_{n+1}^- \subset E_n^-$ (n∈N).

Let δ>0, $K_\delta^+ = \bigcup_{tek\{} z \in E^{+||z-t|\leq}\delta\}$, K_δ^+ is defined the δ-neighborhood from the left (closed region). From the construction of K, each $\overline{En1^+}$-E_n^+ consist of closed region of 3×4^n equilateral triangles non-intersecting to each other with side length $\frac{\sqrt{3}}{3^{n+1}}$. The boundaries of these triangular areas are denoted by $\Delta_{n,j}$(j = 1, 2, …, 3×4^n), $\Delta_{n,j}^+$ is the interior zone. When δ> *0* is fixed, there exists $N_0 \in N$, such that $\overline{E^+}$-E_{N0}^+ = $\bigcup_{n\geq N_0}\bigcup_{j-1}^{3\times4n}\overline{\Delta_{n,J}^+}$ and each $\Delta_{n,j}$ (j = 1, 2, …, 3×4^n) lies in the interior of K_δ^+. In the following, N_0 is fixed and δ> 0and small sufficiently such that 0 ∉ K_δ^+.

Definition: Assume $0< a < 1$, If φ satisfy Holder condition of order a ($0< a < 1$), which is | φ (z_1)- φ (z2)|≤ $M|z_1-z_2|^a$, (z_1, z_2∈ K_δ^+) for M> 0 and φ (z) is an analyitc function in interior of K_δ^+, then φ ∈ AH^a (K_δ^+).

Lemma 1: t_0∈K if and only if there exists N (t_0) ∈ *N* such that $t_0 \in \bigcap_{n\geq N(t_0} K_n$.

Lemma 2: Assume:

$$s-1<\alpha<1, \quad s=\log_3 4, \quad \varphi \in AH^{\alpha}(K_\delta^+),$$

Then,

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$$\Phi(z)=\lim_{n\to\infty}\Phi_n(z)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{K_n}\frac{\varphi(t)}{t-z}dt \qquad (1)$$

For any $t_0 \in K$, $\Phi^+(t_0) = lim_{z\to t0zet+}\Phi(z)$ and $\Phi^+(t_0) = lim_{n\to\infty_{0z\in E^+}}\Phi(z)$ Both exist, satisfied:

$$\Phi^+(t_0)-\Phi^-(t_0)=\varphi(t_0) \qquad (2)$$

From the proof of Lemma 2, we can see that:

$$\Phi(z)=\Phi_n(z)\,,(n\geq N(t_0)\,,z\in E^+_{N(t_0)}\cup E^-) \qquad (3)$$

and

$$\Phi^{\pm}(t_0)=\Phi_n^{\pm}(t_0)\,,(n\geq N(t_0)) \qquad (4)$$

Now the Riemann boundary value problems for Koch Curve is considered, the analyitic function Φ (z) defined in E^+ and E^-, satisfying the condition of Riemann boundary value:

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t) \quad (t\in K) \qquad (\ *\ )$$

where, G, g$\in AH^a\,(k_\delta^+), S-1<a<1)$ G (z) $\neq$ 0 (z$\in k_\delta^+$) and $\Phi\in R_0$ (that is $\Phi(\infty)$ is finite).

In this study, when L is substituted for Koch curve, we define the Riemann boundary value problem. Cauchy-type integral is meaningless on Koch curve. Moreover, when some analytic conditions are attached to functions G (z) and g (z), through the limit function of a sequence of Cauchy-type integrals, the homogeneous and non-homogeneous Riemann boundary problems on Koch curve are introduced, some similar results was attained like the classical boundary value problems for analytic functions.

RIEMANN BOUNDARY VALUE PROBLEMS AND THEIR SOLUTIONS

The simplest Riemann boundary value problem is the jump problem: G (t) = 1 in (*), i.e., $\Phi^+(t)-\Phi^-(t)=g(t)$ (t∈K) $(*)_i$ If Φ (z) is required to have a pole of order m≥ *1* at most at infinity, then the problem is denoted by $\Phi\in R_m$, if Φ (z) is required to eaual to 0 of order m≥ *1* , the problem is denoted by $\Phi\in R_{-m}$.

Theorem 1: The solution of the jump problem $(*)_1$ is given in following: its unique solution is given by (5) when solved in R_{-1}; when m≥ *0*, it has the general solution (6), when m≥ *2*, it has the unique solution (5) if and only if m-2 conditions of solvability in (7) are fulfilled.

Proof: From (1)、(2)、(3)when solved in R_{-1}:

$$\Phi(z)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{K_n}\frac{g(t)}{t-z}dt \qquad (5)$$

is the unique solution of jump problem $(*)_1$.

By the extended Liouville theorem, the general solution of $(*)_1$ in R_m (m≥ *0*) is:

$$\Phi(z)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{K_n}\frac{g(t)}{t-z}dt+P_m(z) \qquad (6)$$

where, P_m (z) is an arbitrary polynomial of degree m. From (3), for any $t_0\in K$, allowing for:

$$\Phi(z)=\Phi_n(z)=\frac{1}{2\pi i}\int_{K_n}\frac{g(t)}{t-z}dt\,,\ (n\geq N(t_0),z\in E^+_{N(t_0)}\cup E^-)$$

it has $\Phi(z)=\sum_{k-0}^{+\infty}\left(\frac{1}{2\pi i}\int k_n\, g(t)t^k dt\right) z^{-(k+1)}$, (n≥ $N(t_0)$, z∈ G) for the reigon G whose satisfied the neighbourhood of z = ∞ and $\bar{G}\cap E$. So if and only if:

$$\lim_{n\to\infty}\int_{K_n}g(t)t^k dt=0 \quad (k=0,1,\cdots,m-2) \qquad (7)$$

Φ (z) defined by (5) has a zero point at infinity of order m≥ *2*.

If g (t) = ^{0} in (*), then it is called a homogeneous Riemann boundary value problem, that is:

$$\Phi^+(t)=G(t)\Phi^-(t) \quad (t\in K) \qquad (\ *\ )_2$$

Theorem 2: For the homogeneous Riemann boundary value problem $(*)_2$, if the solutions are in R_0, it has only the trivial solution when its index K< *0* and its solution is Φ (z) = X (z) P_k (z) when K ≥ *0*, where P_m (z) is an arbitrary polynomial of degree m.

Proof: Because G∈ AH^a (K_δ^+) and taking value of no zero, there exists a integer K, such that:

$$\frac{1}{2\pi i}[\log G(z)]_k=\frac{1}{2\pi i}[\log G(z)]_{KN}=K\ (n\geq N_0)$$

called K is the index of $(*)_2$ or (*).

Assume $G_0(z)=z^{-k}G(z)$, then $G_0\in AH^a(K_\delta^+)$, $G_0(z)\neq 0$ (z∈ K_δ^+),
by introducing a new function ψ (z):

$$\Psi(z)=\begin{cases}\Phi(z), & z\in E^{+}\\ z^{\kappa}\Phi(z), & z\in E^{-}\end{cases}$$

then $(*)_2$ becomes a Riemann boundary value problem for $\psi(z)\,\psi^{+}(t)=G_0(t)\,\psi^{-}(t)$ ($t\in K$) with index 0. Since $\Phi\in R_0$, its solution $\psi\in R_{-K}$. If we set:

$$\Gamma_n(z)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{K_n}\frac{\log G_0(t)}{t-z}dt\,,(z\notin K_n,n\geq N_0)$$

$\Gamma(z)=lim_n\to\infty\,\Gamma_n(z)$ is analystic in $E^{+}\cup E^{-}$, which satisfied $\Gamma^{+}(t)-\Gamma^{-}(t)=\log G_0(t)$ ($t\in k$) and $\Gamma(\infty)=0$. Introducing $X_0(z)=e^{\Gamma(z)}$ ($z\in E^{+}\cup E^{-}$), then $X_0(z)$ is analystic in $E^{+}\cup E^{-}$ and $X_0(\infty)=1$, $X_0(z)$ and $X^{\pm}(t)$ ($t\in K$) have no zero points, also:

$X_0^{+}(t)=G_0(t)\,X_0^{-}(t)=G(t)\,t^{-k}X_0^{-}(t)$ ($t\in K$). If we set:

$$X(z)=\begin{cases}X_0(z)=e^{\Gamma(z)}, & z\in E^{+}\\ z^{-\kappa}X_0(z)=z^{-\kappa}e^{\Gamma(z)}, & z\in E^{-}\end{cases}\qquad(8)$$

and call it canonical function of $(*)_2$ or (*), then:

- X (z) satisfy X (z) is analystic in $E^{+}\cup E^{-}$ and $X(z)\neq 0\quad(z\in E^{+}\bigcup E^{-})$
- $X^{\pm}(t)$ ($t\in K$) exists and $X^{\pm}(t)\neq 0$ ($t\in K$)
- X (z) has finite order -K at $z=\infty$

$$X^{+}(t)=G(t)X^{-}(t)\quad(t\in K)\qquad(9)$$

Considering (9), $(*)_2$ becomes a jump problem:

$$\frac{\Phi^{+}(t)}{X^{+}(t)}-\frac{\Phi^{-}(t)}{X^{-}(t)}=0\quad(t\in K)\qquad(10)$$

From the process in jump problem, the results are attained from solving $\frac{\Phi(t)}{X(t)}$ in R_k.

Let us consider the non-homogeneous Riemann boundary value problem (*), from the index, the canonical function X (z) of its corresponding homogeneous problem in (8), Considering (9), (*) becomes a jump problem:

$$\frac{\Phi^{+}(t)}{X^{+}(t)}-\frac{\Phi^{-}(t)}{X^{-}(t)}=\frac{g(t)}{X^{+}(t)}\quad(t\in K)\qquad(11)$$

solving $\frac{\Phi(t)}{X(t)}$ in R_k. Since $g\in AH^{a}(K_{\delta}^{+})$, but $\frac{g(t)}{X^{+}(t)}$ is may not a function in $\frac{g(t)}{X^{+}(t)}$. For any $t_0\in K$, $\log G_0\in AH^{a}(K_{\delta}^{+})$, therefore $\Gamma(z)=\Gamma_n(z)=\frac{1}{2\pi i}\int K_n\frac{\log G_0(t)}{t-z}dt$ ($z\in E_{n,}^{+}n\geq N(t_0)$).

By the Privalov theorem, $\Gamma_n(z)$ and $\Gamma_n^{+}(t)$ are analystic and satisfy Holder condition of order a in $\overline{E_n^{+}}$, since $X(z)=e^{\Gamma(z)}=e^{\Gamma_n(z)}$

($z\in E_n^{+},n\geq N(t_0)$), X (z) and $X_n^{+}(t)$ are analystic and satisfy Holder condition of order a in $\overline{E_n^{+}}$. Introducing $n\geq N_0$:

$$X^{+}(z)=\begin{cases}X^{+}(z), & z\in K\bigcap K_n;\\ X(z), & z\in\overline{E_n^{+}}\text{且}z\notin K\end{cases}\qquad(12)$$

then $\frac{g(z)}{X^{+}(z)}$ holds the condition of theorem 1, we have

Theorem 3: For the non-homogeneous Riemann boundary value problem (*), if the solutions are in R_0, its general solution is:

$$\Phi(z)=\lim_{n\to\infty}\frac{X(z)}{2\pi i}\int_{K_n}\frac{g(t)}{X^{+}(t)}\frac{dt}{t-z}+X(z)P_{\kappa}(z)$$

when, $k\geq 0$, where $P_m(z)$ is an arbitrary polynomial of degree m.. its unique solution is:

$$\Phi(z)=\lim_{n\to\infty}\frac{X(z)}{2\pi i}\int_{K_n}\frac{g(t)}{X^{+}(t)}\frac{dt}{t-z}\qquad(13)$$

when, K = -1, its unique solution is (13) if and only if:

$$\lim_{n\to\infty}\int_{K_n}\frac{g(t)t^{k}}{X^{+}(t)}dt=0\quad(k=0,1,\cdots,-\kappa-2)$$

is satisfied when $K\leq -2$.

Corollary: For the non-homogeneous Riemann boundary value problem (*), if the solutions are in R_{-1}, crossponding problem (11) is solved in R_{k-1}, when $K\geq 1$, its general solution is:

$$\Phi(z)=\lim_{n\to\infty}\frac{X(z)}{2\pi i}\int_{K_n}\frac{g(t)}{X^{+}(t)}\frac{dt}{t-z}+X(z)P_{\kappa-1}(z)$$

its unique solution is (13) when $\kappa=0$; its unique solution is (13) if and only if:

$$\lim_{n\to\infty}\int_{K_n}\frac{g(t)t^{k}}{X^{+}(t)}dt=0\quad(k=0,1,\cdots,-\kappa-1)$$

is satisfied when $K\leq -1$.

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