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Feasible and Descent Direction Method for Continuous Equilibrium Network Design Problem

[1,3]Aihua Luo, [2]Zhengshun Ruan, [1]Chongchao Huang and [4]Hua Chen
[1]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
[2]School of Science, Wuhan Institute of Technology, Wuhan 430073, China
[3]School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China
[4]School of science, Hubei University of Technology, Wuhan 430068, China

Abstract: In this study, we firstly express the stochastic user equilibrium traffic assignment problem in asymmetric traffic network as variation inequality model and then formulate ntinuous network design problem as mathematical program with equilibrium constraints. When path flow travel cost function is continuous, differentiable and strong monotone, the solution of variational inequality follows logit assignment principle and is unique. So mathematical program with equilibrium constraints can be written as an implicit optimization problem and the gradient of objective function is received by sensitivity analysis. A feasible and descent direction method is addressed where the direction can be computed with the sign of gradient and the step size can be calculated by operation of comparison. Finally, numerical experiments are conducted and calculation results show high efficiency of the proposed method in solving asymmetric equilibrium network design problem.

Keywords: Continuous network design, feasible and descent direction method, stochastic user equilibrium

INTRODUCTION

For a city traffic road network, the affection of neighboring link flow and opposite link flow is not neglected. Link flow travel cost is obviously impacted by the capacity and flow of downstream link. Therefore, the Jaccobi matrix of link flow travel cost function should be asymmetric and it is more reasonable for considering traffic assignment problem on asymmetric traffic road network. So Stochastic User Equilibrium (SUE) where user route choice follows logit assignment principle should be presented as Variational Inequality problem (VI) instead of nonlinear program and Continuous Network Design Problem (CNDP) which is to determine link capacity expansions should be formulated as Mathematical Program with Equilibrium Constraints (MPEC) instead of bilevel program.

In past decades, a lot of rich work has been done for CNDP. Meng, Yang and Bell developed a bilevel program for CNDP, while Lawphongpanich formulated CNDP as MPEC and Ban formulated CNDP as Mathematics Program with Complementarity Constraints (MPCC). Friesz, Tobin, Cho and Mehta were the first ones who presented gradient-based methods for CNDP. Suh, Kim and Patrikson employed the sensitivity analysis to solve CNDP. Due to the non-differentiability of the perturbed solutions in the equilibrium constraints with respect to the decision variables, Satish, Yang characterized the optimality conditions and derived the corresponding solution methods where the non-smooth approaches have been considered. Based on this conclusion, Chiou presented a series of methods based on subgradient, such as generalized bundle subgradient projection method. Yang and Bell (1998) proposes the models and algorithms for road network design: A review and some new developments. Meng and Yang (2001) study the equivalent continuously differentiable model and a locally convergent algorithm for the continuous network design problem. Lawphongpanich and Hearn (2004) have a research of an MPEC approach to second-best toll pricing. Mathematical Programming. Ban *et al*. (2006) analyzes a general MPCC model and its solution algorithm for continuous network design problem. Mathematical and Computer Modelling. Patriksson (2008) study the applicability and solution of bilevel optimization model in transportation science: A study on the existence, stability and computation of optimal solutions to stochastic mathematical programs with equilibrium constraints. Chiou (2009) proposes a subgradient optimization model for continuous road network design problem. Tobin and Friesz (1988) analyzes the sensitivity for equilibrium network flows. Patriksson (2004) has a research of the sensitivity

Corresponding Author: Hua Chen, School of science, Hubei University of Technology, Wuhan 430068, China

analysis of traffic equilibria. Josefsson and Patriksson (2007) have a research of the sensitivity of separable traffic equilibrium with application to bilevel optimization in network design. Connors *et al*. (2007) study the sensitivity of the variable demand probit stochastic user equilibrium with multiple user-classes.

In this study, assume user route behavior follows logit assignment principle in asymmetric road network, we formulate the stochastic user equilibrium traffic assignment problem with a Parametric Variational Inequality (PVI) and the gradient of variable of interest with respect to link capacity expansion can be computed by the first order sensitivity for KKT condition of PVI. In fact, relying on the result of analyzing KKT condition of PVI, the gradient can be directly computed with simplified formula. Then a feasible and descent direction is present for MPEC where the direction can be received by the sign of gradient of objective function and the step size can be computed by the comparison operation. The character for computing direction and step size in brief makes algorithm win high efficiency which is embodied with numerical experiments and comparative analysis on an example network.

The organization of this study is as follows, in the next section a PVI model for SUE and an MPEC for CNDP are given. The first order sensitivity analysis for obtaining the gradient is conducted. And then a feasible and descent direction method is proposed for solving MPEC. Following, numerical calculations are conducted a good computational results are obtained on an example road network. Conclusion and further research work are concluded in the end.

PROBLEM FORMULATION

In this section, CNDP is presented with a mathematical program with equilibrium constraints. Firstly, SUE is expressed in terms of Variational Inequality (VI) where user's route choice is assumed to follow logit assignment principle. Then, first-order sensitivity analysis is conducted for which the gradient of variables of interests is conducted. Finally, an MPEC formulation is presented.

The following Notation will be used:

$G(N, A)$	: Directed road network, where N is set of nodes and A is set of links
W	: Set of OD pairs
R_w	: Set of paths between OD w
y_a	: Link capacity expansion on link a
l_a, u_a	: Bounds of link capacity expansion on link a
$G_a(y_a)$	: Investment cost on link a
θ	: Conversion factor from investment cost to travel time cost.
f	: Vector of link flow
h	: Vector of path flow
t	: Vector of link flow travel cost
C	: Vector of path flow travel cost
q	: Vector of travel demand
Δ	: Link-path incidence matrix
Γ	: OD-path incidence matrix

For traffic assignment problem of stochastic user equilibrium, a variational inequality model can be expressed as follows:

$$[\ln h_k^w + \alpha C_k^w (\sum_w \sum_k h_k^w \delta_{ak}^w)](\overline{h_k^w} - h_k^w) \geq 0,$$
$$\forall \overline{h_k^w} \in K = \left\{ h_k^w \middle| \sum_{k \in R_w} h_k^w = q_w, p_k^w > 0 \right\} \quad (1)$$

α is a positive dispersion parameter, which reflects an aggregate measure of drivers' perception of travel costs. Higher values of α indicate that drivers have a more accurate perception of travel time and consequently tend to choose the least-cost path.

Theorem 1: Assume path flow travel cost function C(h) is continuous, differentiable and monotone, the solution of (1) follows logit assignment principle.

Proof: The KKT conditions of (1) are:

$$\ln h_k^w + \alpha C_k^w + u_w - v_k^w = 0 \quad (2)$$

$$\sum_k h_k^w = q_w \quad (3)$$

$$h_k^w > 0, \quad v_k^w \geq 0, \quad h_k^w v_k^w = 0 \quad (4)$$

From (4), $v_k^w = 0$. With (2), $h_k^w = e^{-\alpha C_k^w - u_w}$:

$$q_w = \sum_{k \in R_w} h_k^w = e^{-u_w} \sum_{k \in R_w} e^{-\alpha C_k^w}, \frac{h_k^w}{q_w} = \frac{e^{-\alpha C_k^w}}{\sum_{k \in R_w} e^{-\alpha C_k^w}}$$

So $h_k^w = q_w \frac{e^{-\alpha C_k^w}}{\sum_{k \in R_w} e^{-\alpha C_k^w}}$

This means the solution of (1) follow logit assignment principle.

In term of (1), traffic assignment problem with link capacity expansions can be concluded as a parametric variational inequality:

$$[\ln h_k^w(y)+\alpha C_k^w(h(y),y)][\overline{h_k^w}(y)-h_k^w(y)]\geq 0$$
$$\forall \overline{h_k^w}(y,\lambda)\in K(y,\lambda)=\{h_k^w(y,\lambda)\,|\,\textstyle\sum_k h_k^w(y,\lambda)=q_w$$

$$h_k^w(y,\lambda)>0\} \tag{5}$$

where,

$$C_k^w \textstyle\sum_a \delta_{ak}^w c_a\,(f(y,\lambda),y,\lambda)$$

$$f_a(y,\lambda)=\textstyle\sum_k\sum_k \delta_{ak}^w p_k^w\,(y,\lambda)$$

The KKT conditions of (5) are:

$$\begin{aligned}&\ln h_k^w(y)+\alpha C_k^w(h(y),y)+u_w-v_k^w=0\\&\sum_{k\in R_w} h_k^w(y)=q_w\end{aligned} \tag{6}$$

$$h_k^w(y)>0,\quad v_k^w\geq 0,\quad h_k^w(y)v_k^w=0$$

Simplify (6):

$$\begin{aligned}&\ln h_k^w(y)+\alpha C_k^w(h(y),y)+u_w=0\\&\sum_{k\in R_w} h_k^w(y)=q_w\end{aligned} \tag{7}$$

Introduce:

$$H(h,u,y)=\begin{pmatrix}\ln h+\alpha\Delta^T t(f,y)+\Gamma^T u\\ \Gamma h-q\end{pmatrix} \tag{8}$$

where,

$$C_k^w(h(y),y)=\Delta^T t(f(y),y)\,,\ f=\Delta h\,.$$

Denote $z=\left(h^T,u^T\right)^T$, the first order sensitivity analysis of equations (8) for y can be derived by:

$$\nabla_z H\nabla_y z+\nabla_y H=0 \tag{9}$$

where,

$$\nabla_z H=\begin{pmatrix}H^{-1}+\alpha\Delta^T\nabla_f t(\Delta h,\beta)\Delta & \Gamma^T\\ \Gamma & 0\end{pmatrix}=\begin{pmatrix}A & \Gamma^T\\ \Gamma & 0\end{pmatrix}$$

$$\nabla_y H=\begin{pmatrix}\alpha\Delta^T\nabla_y t(f,y)\\ 0\end{pmatrix},H^{-1}=diag(\cdots,\frac{1}{h_k^w},\cdots)$$

From (9), $\nabla_y z=-\nabla_z H^{-1}\nabla_y H$

$$=-\begin{pmatrix}A & \Gamma^T\\ \Gamma & 0\end{pmatrix}^{-1}\begin{pmatrix}\alpha\Delta^T\nabla_y t(f,y)\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}A^{-1}(I-\Gamma^T(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1}) & A^{-1}\Gamma^T(\Gamma A\Gamma^T)^{-1}\\ (\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1} & -(\Gamma A^{-1}\Gamma^T)^{-1}\end{pmatrix}\begin{pmatrix}\alpha\Delta^T\nabla_y t(f,y)\\ 0\end{pmatrix}$$

This means $\nabla_y h=$

$$-\alpha A^{-1}(I-\Gamma^T(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1})\Delta^T\nabla_y c(f,y)$$
$$\nabla_y u=-\alpha(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1}\Delta^T\nabla_t t(f,y).$$

Because $f=\Delta h,\nabla_y f=\Delta\nabla_y h=\alpha\Delta A^{-1}$

$$(I-\Gamma^T(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1})\Delta^T\nabla_y t(f,y) \tag{10}$$

An optimization model for CNDP can be formulated as:

$$\begin{aligned}&\min\quad Z(f,y)=\sum_a t_a(f(y),y)f_a(y)+\theta\sum_a G_a(y_a)\\&s.t\quad l_a\leq y_a\leq u_a,\quad \forall a\in A\\&\qquad f_a(y)=\Delta h(y)\\&\qquad h(y)\in S(y)\end{aligned} \tag{11}$$

where $S(y)$ is the solution set (5).

Following the results in sensitivity analysis, the first-order partial derivatives can be obtained by (10). Now the model (11) can be re-expressed as a single-level problem:

$$\begin{aligned}&\min\quad Z=Z(y)\\&l\leq y\leq u\end{aligned} \tag{12}$$

For (12), the objective function Z(y) has no specific form. However, the gradient can be derived by sensitivity analysis:

$$\nabla Z(y^k) = \nabla_y Z(f^k, y^k) + \nabla_f Z(f^k, y^k) \nabla_y f \quad (13)$$

FEASIBLE AND DESCENT DIRECTION METHOD FOR CNDP

Due to the sensitivity analysis, a feasible and descent direction method for traffic network design with capacity expansions in (11) can be established.

Supposing y^k is current iterative point of (11), introduce the following steps:

$$A' = \{a \mid y_a^k = l_a, \forall a \in A\}, A'' = \{a \mid y_a^k = u_a, \forall a \in A\}.$$

Let $d = y - y^k$,

a linear program can be concluded:

$$\begin{aligned} \min \quad & \nabla Z(y^k)^T d \\ s.t. \quad & d_a \ge 0, \quad \forall a \in A' \\ & d_a \le 0, \quad \forall a \in A'' \\ & -1 \le d_a \le 1, \quad \forall a \in A \end{aligned} \quad (14)$$

Theorem 2: The solution of LP (14) is $d^k = (\ldots, d^k_a, ..)$,

where,

$$d_a^k = \begin{cases} 0, & if\,(\nabla z(y^k))_a > 0, a \in A' \\ & \text{or}\,(\nabla z(y^k))_a \le 0, a \in A'' \\ 1, & if\,(\nabla z(y^k))_a \le 0, a \in A \setminus A'' \\ -1, & \text{el se} \end{cases} \quad (15)$$

If $\nabla z(y^k)^T d^k \ne 0$, then d^k is a decreasing and feasible direction.

Proof: Considering (14), we can receive the solution d^k by analyzing the sign of $\nabla z(y^k)$. If $(\nabla z(y^k))_a$:

$0,\ 0 \le d_a \le 1$, then $d_a^k = 0,\ \forall a \in A'$; If $(\nabla z(y^k))_a \le 0,\ 0 \le d_a \le 1$, then $d_a^k = 1, \forall a \in A'$; If $(\nabla z(y^k))_a > 0,\ -1 \le d_a \le 0$, then $d_a^k = -1,\ \forall a \in A''$; If $(\nabla z(y^k))_a \le 0,\ -1 \le d_a \le 0$, then $d_a^k = 0, \forall a \in A''$; If $(\nabla z(y^k))_a > 0,\ -1 \le d_a \le 1$, then $d_a^k = -1, \forall a \in A \setminus A' \cup A''$; If $(\nabla z(y^k))_a \le 0,\ -1 \le d_a \le 1$, then $d_a^k = 1,\ \forall a \in A \setminus A' \cup A''$

Summarize the above analysis, we can get (15).

Because $d = 0$ is feasible solution of (9), $\nabla z(y^k)^T d^k \le 0$. If $\nabla z(y^k)^T d^k = 0$, by Farkas theorem, y^k is the optimal solution. If $\nabla z(y^k)^T d^k < 0$, d^k is a descent and feasible direction.

Theorem 3: Considering (11), do linear search with d^k at y^k, the optimal step size is:

$$\bar{\alpha} = \min\left\{ \min_{a \in A \cup A'} \{u_a - l_a\}, \min_{a \in A \setminus A \cup A'} \{y_a^k - l_a, u_a - y_a^k\} \right\} \quad (16)$$

Proof: When $y_a^k = l_a, \forall a \in A'$, $l_a \le y_a^k + \alpha d_a^k \le u_a$, then $0 \le \alpha d_a^k \le u_a - l_a$; If $d_a^k = 0, \alpha \ge 0$; If $d_a^k = 1$ $0 \le \alpha \le u_a - l_a$. When $y_a^k = u_a, \forall a \in A''$, $l_a \le y_a^k + \alpha d_a^k \le u_a$, then $l_a - u_a \le \alpha d_a^k \le 0$; If $d_a^k = 0, \alpha \ge 0$; If $d_a^k = -1, 0 \le \alpha \le u_a - l_a$.

When $l_a < y_a^k < u_a$ $\forall a \in A \setminus A' \cup A''$, $l_a \le y_a^k + \alpha d_a^k \le u_a$, then $l_a - y_a^k \le \alpha d_a^k \le u_a - y_a^k$; If $d_a^k = 1$ $0 \le \alpha \le u_a - y_a^k$; If $d_a^k = -1, 0 \le \alpha \le y_a^k - l_a$.

Conclude the above analysis; we can receive (16).

Due to theorem 2, 3, a feasible and descent direction scheme for CNDP is established in the following steps.

Step 1: Set initial parameters $y^0, \theta, \alpha, k = 0$;

Step 2: Solve (5) and let h^k be the solution, $f^k = \Delta h^k$.

Step 3: According to (10), $\nabla_y f$ is obtained. Compute $\nabla z(y^k)$ with (13) and d^k with (15).

Step 4: If $\nabla z(y^k)^T d^k = 0$, then stop, y^k, f^k is optimal solution; otherwise continue.

Step 5: Compute step size $\overline{\alpha}$ with (16). Let $y^{k+1} = y^k + \overline{\alpha} d^k$, $\sigma = c\sigma, k = k+1$, and then go to step 2.

NUMERICAL CALCULATIONS

In this section, numerical computations are conducted by feasible and descent direction method in an example network which is shown in Fig. 1. In this traffic network, capacity of links 1, 2, 3, 4 need adjustment. Computational results are concluded in Fig. 2, 3 and 4. As it observed in experiments result, with the increasing of θ, investing cost become higher. Feasible and descent method receives promising results and show faster convergence.

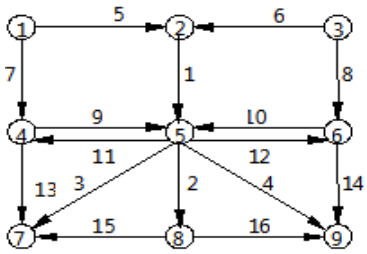

Fig. 1: Testing network

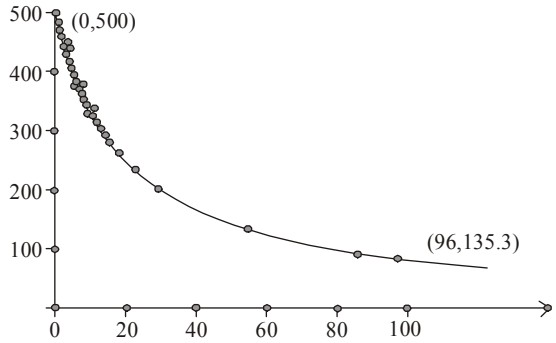

Fig. 2: Travelling cost

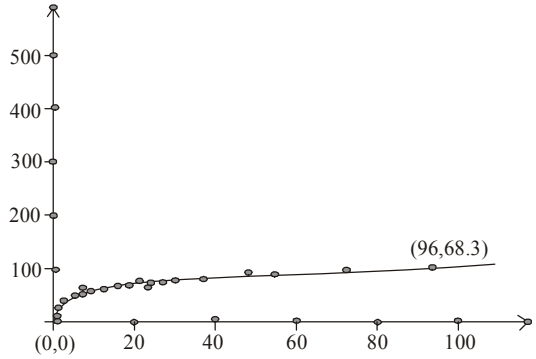

Fig. 3: Investing cost

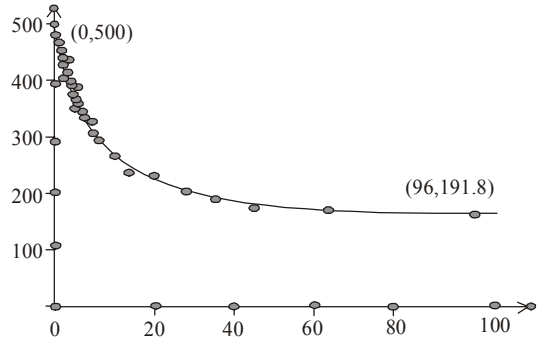

Fig. 4: Value of objective function Z

CONCLUSION

This study presents a new model for CNDP based on SUE which is expressed as an MPEC program. A feasible and descent scheme is proposed to effectively search for optimal solution. In each iteration, the feasible and descent direction can be easily concluded by sign judgment and the optimal step size can be receive by comparison operation. Numerical experiments are conducted on example network, where good performance shown in solving CNDP.

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