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Coapproximation in Probabilistic Normed Spaces

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Abstract: In this article, we studied the concept of best coapproximation in probabilistic normed spaces. We defined the best coapproximation on these spaces and generalized some definitions such as set of P-best coapproximation, P-coproximinal set, P-coapproximately compact and orthogonality relative to any set and then proved some interesting theorems.

Key words: P-best coapproximation, P-coproximinal, P-coChebyshev, probabilistic normed spaces

INTRODUCTION

Menger (1942) introduced the notion of probabilistic metric spaces. The idea of Menger was to use distribution function in stead of non negative real numbers as values of the metric. The concept of probabilistic normed spaces (briefly, PN-spaces) was introduced by Sertnev (1963). Ghazal (2010) considered the best approximation and co-approximation in normed spaces.

Shams and Vaezpour (2009) and Khorasani and Abrishami Moghaddam (2010) have introduced the concept of p-best approximation in probabilistic normed and 2-normed spaces. The main aim of this paper is to investigate best coapproximation in probabilistic normed spaces. In the sequel after an introduction to probabilistic normed spaces, we define the concept of best coapproximation in probabilistic normed space and generalized some definitions such as set of best coapproximation, coproximinal set and coapproximatively compact set.

Chang *et al*. (2001) defined some notions in probabilistic normed spaces as followes:

A distance distribution function (briefly, d.d.f .), is a function F defined from extended interval $[0,+\infty]$ into the unit interval I = [0,1], that is non decreasing and left continuous on $(0,+\infty)$ such that F(0) = 0 and $F(+\infty) = 1$. The family of all d.d.f's will be denoted by Δ^+ and we denote.

$$D^+ = \{F\in\Delta^+ | lim_{t\to\infty} F(t) = 1\} and\ D^+_0 = \{F\in\Delta^+ | F^{-1}(1)\neq\varnothing\}.$$

By setting $F\leq G$ whenever $F(t)\leq G(t)$, for all $t\in R^+$, one introduces a natural ordering in D^+. If $a\in R^+$ then H will be an element of D^+, defined by $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ if $t > 0$. It is obvious that $H\geq F$ if $t > 0$ for all $F\in D^+$.

A t-norm τ is a two place function $\tau: I\times I\to I$ which is associative, commutative, non decreasing in each place and such that $\tau(a,1) = a$, for all $a\in[0,1]$.

A triangle function is a mapping $(T:\Delta^+\times\Delta^+\to\Delta^+)$ which is associative, commutative, non decreasing and for which H is the identity, that is, T (H,F) = F, for every $F\in D^+$.

Definition 1: A probabilistic normed space (shortly, PN-space) is an ordered pair $(V,\mathfrak{I})$, where V is a real linear space, $\mathfrak{I}$ is a mapping from V into D^+ (We shall denote the distribution function $\mathfrak{I}(x)$ by F_x) satisfying the following conditions:

(PN-1)$F_x(t) = 1$ for all t>0 if and only if x = 0
(PN-2)$F_x(0) = 0$
(PN-3)$F_{\alpha x}(t) = F_x(t/|\alpha|)$, for every $t > 0$, $\alpha \neq 0$ and $\alpha \in R$
(PN-4)$F_x(t_1) =1$, $F_y(t_2) =1$, then $F_{x+y}(t_1+t_2) = 1$ for all $x,y \in V$

Definition 2: A Menger PN-space is a triple $(V, \mathfrak{I}, \tau)$, where V is a PN-space and τ is a t-norm satisfy the following conditions:

(PN-4_m) $F_{x+y}(t_1+t_2)\geq\tau(F_x(t_1), F_y(t_2))$
for all x,y $\in$ V and $t_1,t_2 \in R^+$

Theorem 3: Let $(V, \mathfrak{I})$ be a PN-space and $\mathfrak{I}$ take its values in D^+_0. If we define the functions $\|.\|$ and $\|.\|_\alpha$, $\alpha\in(0,1]$, as follows, respectively:

$$\|x\| = \inf_t \{t \geq 0, F_x(t) = 1\}$$

$$\|x\|_\alpha = \inf_t \{t \geq 0, F_x(t) > 1-\alpha\}$$

for all $x\in V$ and $\alpha \in(0, 1]$, then:

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- $\|.\|$ is a norm on V and so $(X, \Im, \|.\|)$ is a normed linear space
- $\|.\|_\alpha$ is monotone decreasing in $\alpha \in (0.1]$ and

$$\|x\| = \lim_{\alpha} \|x\|_\alpha = \sup_{\alpha \in (0.1]} \|x\|_\alpha .$$

Theorem 4: Let $(V, \Im)$ be a PN-space and $\Im$ satisfy the following condition:

(PN-B): For any x, y $\in$ V and $t_1, t_2 > 0, \lambda > 0$, if $F_x(t_1) > 1-\lambda$ and $F_y(t_2) > 1-\lambda$, then $F_{x+y}(t_1 + t_2) > 1-\lambda$. Then:

- For each $\alpha \in (0, 1]$, the function ρ_α defined by:

$$\rho_\alpha(x) = \inf_t \{t \geq 0, F_x(t) > 1-\alpha\}$$

  is a semi-norm and the family of semi-norms $\{\rho_\alpha, \alpha \in (0, 1]\}$ can separate the points of V, i.e., $x \neq 0$, then there exists an $\alpha \in (0,1]$ such that $\rho_\alpha(x) \neq 0$,

- $(V, \Im)$ is a locally convex topological space induced by the family of semi-norms

  $\{\rho_\alpha, \alpha \in (0,1]\}$ and, for each $x \in V$

  $u(x) = \{U(x, \alpha_1, \alpha_2, \ldots, \alpha_n, \varepsilon): \varepsilon > 0, \alpha_1, \alpha_2, \ldots, \alpha_n \in (0, 1]$, n is any positive integer$\}$ is the basis of neighborhoods of x, where:

  $U(x, \alpha_1, \alpha_2, \ldots, \alpha_n, \varepsilon) = \{y \in V: \rho_{\alpha_i}(x-y) < \varepsilon, \alpha_i \in (0, 1], i = 1, \ldots, n\}$

- The topology induced by the basis u(x) of neighborhoods of x coincides with the topology induced by the following basis of neighborhoods of x:

  $N(x) = \{N_x(\varepsilon, \alpha): \varepsilon > 0, \alpha > 0\}, \alpha \in (0, 1]$

  where

  $N_x(\varepsilon, \alpha) = \{y \in V: F_{x-y}(\varepsilon) > 1-\alpha\}$

P-BEST COAPPROXIMATION IN PROBABILISTIC NORMED SPACE

Definition 1: Let A be a nonempty subset of a PN-space $(V, \Im)$. For $t > 0$, an element $a_0 \in A$ is called a P-best coapproximation to $x \in V$ from A if for every $a \in A$:

$$F_{a_0 - a}(t) \geq F_{x-a}(t)$$

The set of all such elements a_0 that called a P-best coapproximation to $x \in V$, is denoted by $R^t_A(x)$, i.e.,

$R^t_A(x) = \{a_0 \in A: F_{a_0 - a}(t) \geq F_{x-a}(t)$ for all $a \in A$, $t > 0\}$.

Putting

$\breve{A} = \{x \in V : F_a(t) \geq F_{a-x}(t)$ *for all* $a \in A, t > 0\} = (R^t_A)^{-1}(\{0\})$,

it is clear $a_0 \in R^t_A(x)$ if and only if $x - a_0 \in \breve{A}$.

Definition 2: Let $(V, \Im)$ be a PN-space. For $t > 0$, the nonempty subset $A \subset V$ is called P-coproximinal set if $R^t_A(x)$ is non-void for every $x \in V$ and A is called P-coChebyshev set if for every $x \in V$ the set $R^t_A(x)$ contains exactly one element.

Definition 3: Let A be a nonempty subset of a PN-space $(V, \Im)$, and $\{x_n\}$ be a sequence of V.

- Then the sequence $\{x_n\}$ is said to be P- convergent to $x \in V$ and denoted by $x_n \to^p x$, if $\lim_{n \to \infty} F_{x_n - x}(t) = 1$, for all $x \in V$ and $t > 0$.
- The set A is closed if and only if, whenever $\{a_n\}$ is a sequence of points in A that P-converging to $x \in V$, then x is also in A. Also we recall that a subset $A \subseteq X$ is called compact if each sequence $\{a_n\}$ in A has a subsequence $\{a_{n_k}\}$ that P-converging to an element $a_0 \in A$.

Theorem 4: Let A be a nonempty subset of a PN-space $(V, \Im)$. Then for $t > 0$:

- $R^t_{A+y}(x+y) = R^t_A(x) + y$, for every $x, y \in V$.
- $R^{|\alpha|t}_{\alpha A}(\alpha x) = \alpha R^t_A(x)$, for every $x \in V$ and any scaler $\alpha \in R \backslash \{0\}$.
- A is P-coproximinal (respectively P-coChebyshev) if and only if A+y is P-coproximinal (respectively P-coChebyshev) for every $y \in V$.

Proof:

- For any $x, y \in V$ and $t > 0$, let $a_0 \in R^t_{A+y}(x+y)$ if and only if, $F_{a_0 - (a+y)}(t) \geq F_{x+y-(a+y)}(t)$ for all $(a+y) \in A+y$ if and only if, $F_{(a_0 - y) - a}(t) \geq F_{x-a}(t)$ for all $a \in A$ if and only if, $(a_0 - y) \in R^t_A(x)$ i.e., $a_0 \in R^t_A(x) + y$.
- Let $a_0 \in R^{|\alpha|t}_{\alpha A}(\alpha x)$, for any $x \in V$, $t > 0$ and $\alpha \in R \backslash \{0\}$ if and only, $F_{a_0 - \alpha a}(|a|t) \geq F_{\alpha x - \alpha a}(|\alpha|t)$ for all $a \in A$ if and only, $F_{\frac{1}{\alpha} a_0 - a}(t) \geq F_{x-a}(t)$ if and only, $\frac{1}{\alpha} a_0 \in R^t_A(x)$ if and only, $a_0 \in \alpha R^t_A(x)$. Therefore, $R^{|\alpha|t}_{\alpha A}(\alpha x) = \alpha R^t_A(x)$.
- Is an immediate consequence of (i).

Theorem 5: Let (V, $\mathfrak{F}$, τ) be a Menger PN-space and A be a convex subset of V. Then for t>0 and x ∈ V, $R^t_A(x)$ is a convex subset of A (for $R^t_A(x) \neq \emptyset$).

Proof: Let a_1, $a_2 \in R^t_A(x)$, t > 0 , and x ∈ V, then:

$$F_{a-a_1}(t) \geq F_{x-a}(t) \text{ and } F_{a-a_2}(t) \geq F_{x-a}(t) \textit{ for all } a \in A$$

Now for $\lambda \in (0,1)$ we have:

$$F_{a-(\lambda a_1+(1-\lambda)a_2)}(t) = F_{\lambda a-\lambda a_1+a-\lambda a-a_2+\lambda a_2}(t)$$
$$= F_{\lambda(a-a_1)+(1-\lambda)(a-a_2)}(t)$$
$$\geq \tau(F_{a-a_1}(\lambda t/\lambda),\ F_{a-a_2}((1-\lambda)t/(1-\lambda))$$
$$\geq \tau(F_{x-a}(t), F_{x-a}(t)) = F_{x-a}(t)$$

So for each $\lambda \in (0,1)$, we have $F_{a-(\lambda a_1+(1-\lambda)a_2)}(t) \geq F_{x-a}(t)$ and then $\lambda a_1+(1-\lambda)a_2 \in R^t_A(x)$. Hence $R^t_A(x)$ is a convex.

Theorem 6: Let (V, $\mathfrak{F}$, τ) be a Menger PN-space and A be a subset of V. If $a_0 \in R^t_A(x)$ and $(1-\lambda)x+\lambda a_0 \in A$, for x ∈ V and every scaler λ≠0, then $(1-\lambda)x+\lambda a_0 \in R^t_A(x)$.

Proof: Let $a_0 \in R^t_A(x)$, t > 0 and x ∈ V, then:

$$F_{a-a_0}(t) \geq F_{x-a}(t) \textit{ for all } a \in A$$

Then for, λ≠0:

$$F_{a-[(1-\lambda)x+\lambda a_0]}(t) = F_{a-(1-\lambda)x-\lambda a+\lambda a-\lambda a_0}(t)$$
$$= F_{(1-\lambda)a-(1-\lambda)x+\lambda(a-a_0)}(t)$$
$$= F_{(1-\lambda)(a-x)+\lambda(a-a_0)}(t)$$
$$\geq \tau(F_{a-x}((1-\lambda)t/(1-\lambda)),\ F_{a-a_0}(\lambda t/\lambda))$$
$$\geq \tau(F_{a-x}(t), F_{x-a}(t)) = F_{x-a}(t),$$

for all a ∈ A, thus $(1-\lambda)x+\lambda a_0 \in R^t_A(x)$.

Example 7: Let V = R^2. Define $\mathfrak{F}$: $R^2 \to D^+$ as:

$$F_{(x_1,x_2)}(t) = (\exp(\sqrt{x_1^2+x_2^2}\ /\ t))^{-1}$$

Then (V, $\mathfrak{F}$, τ) is a Mener PN-space where $\tau(F(t), G(t)) = F(t).G(t)$ for every F and G in D^+.

Let $A = \{(x_1, x_2) \in R^2 | -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x^2_1\}$ and $x = (0, 2)$. Then for every $t > 0$, (1, 1), (-1, 1)$\in R^t_A(0, 2)$.

Theorem 8: For t > 0, let A be a P-coproximinal subspace of a PN-space (V,$\mathfrak{F}$). Then:

- if $\breve{A}$ is a compact set then $R^t_A(x)$ is compact, for every x ∈ V.
- if $\breve{A}$ is a closed set then $R^t_A(x)$ is closed, for every x ∈ V.

Proof:

- Suppose $x \in V$ and $\{a_n\}$ is a sequence in $R^t_A(x)$. Since $x-a_n \in \breve{A}$ and $\breve{A}$ is a compact set, there is a subsequence $\{x-a_{n_k}\}$ that convergence to $u_0 \in \breve{A}$. Since
  $x-u_0 = a_0$, therefore $a_0 \in R^t_A(x)$.
- It is similar to (1).

Definition 9: Let (V, $\mathfrak{F}$, τ) be a Menger PN-space and A be a subset of V. An element $x \in A$ is said to be orthogonal to an element $y \in V$, and we denote $x \perp^t_B y$, if $F_{x+\lambda y}(t) \leq F_x(t)$ for all scalar λ ∈ R, λ≠0 and $t > 0$.

Also, An element $x \in V$ is said to be orthogonal to A, and we denote $A \perp^t_B x$, if $y \perp^t_B x$, for all $y \in A$.

Theorem 10: Let (V, $\mathfrak{F}$, τ) be a Menger PN-space and A be a subspace of V. Then for $x \in V$, $y_0 \in R^t_A(x)$ if and only if $A \perp^t_B x-y_0$ for each $t > 0$.

Proof: Suppose $x \in V$ and $A \perp^t_B x-y_0$. Then $F_{a+\lambda(x-y_0)}(t) \leq F_a(t)$ for all $a \in A$ and all scalar λ ∈ R , λ≠ 0 and $t > 0$, if and only if $F_{x-y_0+\lambda_a^{-1}}(t/|\lambda|) \leq F_{\lambda_a^{-1}}(t/|\lambda|)$, if and only if $F_{x-y_0+a'}(t/|\lambda|) \leq F_{a'}(t/|\lambda|)$ and for $a' = \lambda^{-1}a$ and $y_0 - a' = a''$, if and only if $F_{x-a''}(t/|\lambda|) \leq F_{y_0-a''}(t/|\lambda|)$ for every λ≠0, if and only if $F_{a-y_0}(t) \geq F_{x-a}(t)$ for all $a \in A$ and for each $t > 0$, if and only if $y_0 \in R^t_A(x)$.

Remark 11: Let (V, $\mathfrak{F}$, τ) be a Menger PN-space and A be a subspace of V.

$$(R^t_A)^{-1}(\{0\}) = \{x \in V: F_a(t) \geq F_{a-x}(t) \textit{ for all } a \in A$$
$$t > 0\} = \{x \in V: A \perp^t_B x\}$$
$$\breve{A} = \{x \in V: A \perp^t_B x\}$$

Theorem 12: Let A be subspace of a Menger PN-space (V, $\mathfrak{F}$, τ), then $\breve{A} \cap A = \{0\}$.

Proof: Let $a \in \breve{A} \cap A$, we show that $a = 0$. To see this, we have $a \in \breve{A}$, then $A \perp^t_B a$ and $a \in A$, this implies that $h \perp^t_B a$ for all $h \in A$. Therefore, $F_{h+\lambda a}(t) \leq F_h(t)$ for all $h \in A$, $t > 0$ and all scalar λ. Now, if we choose λ = -1/3 and $h = a$, then $F_{a-(1/3)a}(t) \leq F_a(t)$, and so $F_{(2/3)a}(t) = F_a(3/2t) \leq F_a(t)$. Hence $a = 0$. i.e., $\breve{A} \cap A \subseteq \{0\}$. But $\{0\} \subseteq \breve{A} \cap A$, together, we get $\breve{A} \cap A = \{0\}$.

Theorem 13: For $t > 0$, let A be a P-coproximinal subspace of a Menger PN space (V, $\mathfrak{I}$, τ). If $\check{A}$ is a convex set, then A is P- coChebyshev, for every $x \in V$.

Proof: Suppose $t > 0$, $x \in V$ and $a_1, a_2 \in R^t_A(x)$; then $x-a_1, x-a_2 \in \check{A}$. Put $\check{a}_1 = x-a_1$ and $\check{a}_2 = x-a_2$ and let us have $x = a_1+\check{a}_1 = a_2+\check{a}_2$. Since $1/2(\check{a}_1-\check{a}_2) \in \check{A}$, it follows that $a_1-a_2 \in \check{A} \cap A=\{0\}$; then $a_1 = a_2$.

Theorem 14: For $t > 0$. Let (V, $\mathfrak{I}$, τ) be a Menger PN-space, A and $\acute{A}$ be subspaces of V, such that $A \subseteq \acute{A}$, and let $x \in V$. Then:

$$R^t_A(R^t_{\acute{A}}(x)) \subseteq R^t_A(x)$$

Proof: Suppose $a_0 \in R^t_A(R^t_{\acute{A}}(x))$, then $a_0 \in R^t_A(\acute{a}_0)$ for $a'_0 \in R^t_{A'}(x)$, so $\acute{A} \perp^t_B (x-\acute{a}_0)$, and $A \perp^t_B (\acute{a}_0-a_0)$. Thus, $F_{a'+\lambda(x-a'_0)}(t) \le F_{\acute{a}}(t)$ for all $\lambda \in R$ and $\acute{a} \in \acute{A}$, and $F_{a+\lambda(a'_0-a_0)}(t) \le F_a(t)$ for all $\lambda \in R$ and $a \in A$. Now since, $a+\lambda(\acute{a}_0-a_0) \in \acute{A}$ for $\lambda \in R$ and $a \in A \subseteq \acute{A}$, therefore,

$$F_{a+\lambda(x-a_0)}(t) = F_{a+\lambda(a'_0-a_0)+\lambda(x-a'_0)}(t) \le F_{a+\lambda(a'_0-a_0)}(t) \le F_a(t),$$

since $F_{a+\lambda(x-a_0)}(t) \le F_a(t)$, so, $a \perp^t_B (x-a_0)$ for all $a \in A$, then $A \perp^t_B (x-a_0)$, i.e., $a_0 \in R^t_A(x)$. Hence $R^t_A(R^t_{A'}(x)) \subseteq R^t_A(x)$.

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Corollary 15: For $t > 0$. Let (V, $\mathfrak{I}$, τ) be a Menger PN-space, and A be subspace of V. Then $R^t_A(x) = A \cap (x-\check{A})$.

Proof: Let $a_0 \in A \cap (x-\check{A})$, if and only if $a_0 \in A$, and $a_0 \in (x-\check{A})$, if and only if $a_0 \in A$, and $a_0 = x-\check{a}$, where $\check{a} \in \check{A}$, if and only if $a_0 \in A$, and $\check{a} = x-a_0 \in \check{A}$, if and only if $a_0 \in R^t_A(x)$. Therefore, $R^t_A(x) = A \cap (x-\check{A})$.

Theorem 16: Let A be subspace of a Menger PN-space (V, $\mathfrak{I}$, τ), then:

- A is a P-coproximinal subspace if and only if $V = A+\check{A}$.
- A is a P-coChebyshev subspace if and only if $V = A \oplus \check{A}$.

Proof:

- ($\Rightarrow$)Let $t > 0$, assume that A is P-coproximinal, and let $x \in V$ and $a_0 \in R^t_A(x)$. Then, $x-a_0 \in \check{A}$. Now, $x = a_0+(x-a_0) \in A+\check{A}$. Hence, $V = A+\check{A}$.

  ($\Leftarrow$) Let $t > 0$, $V = A+\check{A} = \{a+y: a \in A, y \in \check{A}\}$, and $x \in V$. Then $x = a_0+y$, where $a_0 \in A$, $y \in \check{A}$. Since $y \in \check{A} = R^t_A(0)$, then $0 \in R^t_A(y)$. Since $x = a_0+y$, then $y = x-a_0$, so $R^t_A(y) = R^t_A(x-a_0)$, this implies that $0 \in R^t_A(y) = R^t_A(x-a_0)$.

  Then $F_{0-(x-a_0)}(t) \ge F_{a-(x-a_0)}(t)$, so $F_{a_0-x}(t) \ge F_{(a+a_0)-x}(t)$ where $(a+a_0) \in A$; hence $a_0 \in R^t_A(x)$. Therefore A is P- coproximinal.
- ($\Rightarrow$) Suppose that A is P-coChebyshev subspace and $x \in V$, $x = a_1+\check{a}_1 = a_2+\check{a}_2$, where $a_1, a_2 \in A$ and $\check{a}_1, \check{a}_2 \in \check{A}$.

  We show that $a_1 = a_2$, and $\check{a}_1 = \check{a}_2$, since $x = a_1+\check{a}_1 = a_2+\check{a}_2$, then $x-a_1 = \check{a}_1$, $x-a_2 = \check{a}_2 \in \check{A}$, this implies that $a_1, a_2 \in R^t_A(x)$.

  Therefore, $a_1 = a_2$ because A is P-coChebyshev, it follows that $\check{a}_1 = \check{a}_2$. Thus $V = A \oplus \check{A}$.

  ($\Leftarrow$) Let $V = A \oplus \check{A}$ and suppose for $x \in V$, there exist $a_1, a_2 \in R^t_A(x)$. We show $a_1 = a_2$.

  Since $a_1, a_2 \in R^t_A(x)$, then $x-a_1, x-a_2 \in \check{A}$ and therefore, $x = a_1+\check{a}_1 = a_2+\check{a}_2$, where $\check{a}_1 = x-a_1$ and $\check{a}_2 = x-a_2$. Since $V = A \oplus \check{A}$, then $a_1 = a_2$ and $\check{a}_1 = \check{a}_2$. Hence A is P-coChebyshev.

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