Research Journal of Applied Sciences, Engineering and Technology 5(17): 4264-4267, 2013
ISSN: 2040-7459; e-ISSN: 2040-7467
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Submitted: July 07, 2012 Accepted: August 15, 2012 Published: May 01, 2013

Production Possibility of Production Plans in DEA with Imprecise Input and Output

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Abstract: Data Envelopment Analysis (DEA) is a mathematical approach to evaluate the relative efficiency of Decision Making Units (DMUs). DEA models create an efficient frontier using the best observed data. This frontier bounds all feasible production plans named Production Possibility Set. Traditional DEA models require crisp input and output data, but in many situations the data are not precisely available. When the data of the DMUs are imprecise, the location of the efficient frontier cannot be determined precisely therefore the boundary of the production possibility set is imprecise. This paper assumes that some of the data are in the interval format available and then considers the production possibility set as a fuzzy set that all the production plans belong to this set with different degrees of membership and a membership function for the production plans related to the fuzzy production possibility set is derived under a geometrical approach.

Keywords: Data envelopment analysis, fuzzy set, production possibility set

INTRODUCTION

Data Envelopment Analysis (DEA) is a method to evaluate the relative efficiency of a set of Decision Making Units which consume multiple inputs to produce multiple outputs.

DEA models evaluate the relative efficiency of DMUs by creating a production frontier using the best practice of observed data. Charnes *et al.* (1978) introduced the first DEA model (CCR) under Constant Return to Scale (CRS) assumption, into operation research literature. Banker *et al*. (1984) extended the CCR model using Variable Returns to Scale (VRS) assumption (BCC).

The traditional DEA models such as CCR, BCC and Slack-Based Measure (SBM) are limited to precise inputs and outputs, and a few changes in data may change the production frontier significantly. But, in real world problems the data are not often available precisely.

A comprehensive literature review on DEA models with imprecise data (IDEA) can be found in Zhu (2003). Zhu (2003) classifies the uncertain data into three groups: interval data, ordinal data and interval data ratio.

Many researchers applied fuzzy set theory (Zadeh, 1965) to enter the imprecision of the data in DEA models.

Sengupta (1992) proposed a fuzzy mathematical programming approach which incorporated fuzziness into a DEA model by defining tolerance levels on objective function and constraint violations.

Lertworasirikul *et al*. (2003) used fuzzy theory to enter the imprecision of data into DEA models (fuzzy DEA).

Nojehdehi *et al*. (2011) used a geometrical approach in order to build a fuzzy efficient frontier set when one of inputs and outputs of the DMUs have imprecise values.

In this study we consider the production possibility set as a fuzzy set that all production plans belong to this set with different degree of membership and we derive a membership degree for production plans under a geometrical approach.

PRILIMINARIES

Assume that there are n DMUs to be evaluated, indexed by j = 1,..,n and each DMU is assumed to consume m different inputs to produce s different outputs. Let $X_j = (x_{1j}, x_{2j},\ldots,x_{mj})$ and $Y_j = (y_{1j}, y_{2j},\ldots,y_{mj})$ be, respectively the inputs and outputs vectors of DMU_j that all components of these vectors have non-negative value and each DMU has at least one strictly positive input and output. If the vector (X, Y) indicates a production plan then the production possibility set in a CCR model is defined as follows:

In other words, TC includes all feasible production plans. The CCR model creates its production frontier

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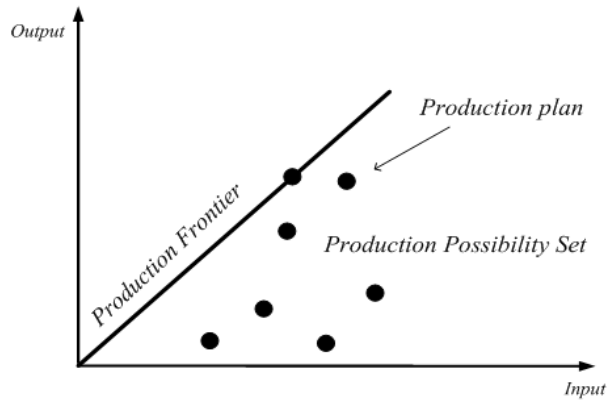

Fig. 1: Production frontier and TC in a CCR model

using linear combination of the existing production plans.

Especially when the DMUs have only one input and output, the production possibility set and production frontier in a CCR shown in Fig. 1.

FUZZY PRODUCTION POSSIBILITY SET

When the data of decision making units are in the interval format the position of production frontier cannot be determined precisely, the efficiency frontier locates somewhere between an upper and alower frontier. The upper and lower frontiers (Nojehdehi *et al*., 2011) that are the boundaries of efficiency frontier can be defined as follows:

- Upper frontier is the frontier created by replacing the maximum values of outputs and minimum values of inputs in a classic DEA models
- Lower frontier is the frontier created by replacing the minimum values of fuzzy outputs and maximum values of fuzzy inputs in a classic DEA models

In order to present a method to evaluate the membership degree of a production plan to the production possibility set we need to define some new variables as follows:

$$Tc = \{(x,y)\} \mid x \geq \sum_{j=1}^{n} \lambda_j y_j \ , \lambda_j \geq 0, j = 1, \ldots, n \qquad (1)$$

θ^u = The efficiency score of the production plans related to the optimistic frontier. θ^u is obtained by substituting the minimum value of inputs and maximum value of outputs in a classic DEA model.

θ^l = The efficiency score of the production plans related to the pessimistic frontier. θ^l is obtained by substituting the maximum value of inputs and minimum value of outputs in a classic DEA model.

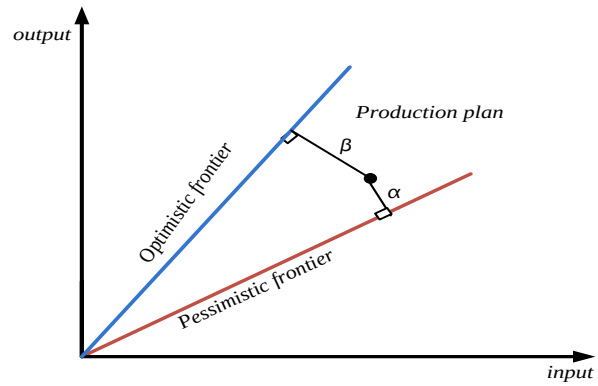

Fig. 2: Representation of lower and upper frontiers in a CCR model

Two dimensional approach: To evaluate the membership degree of production plans related to production possibility set, firstly we assume that all decision making units have only one interval input and one interval output. In this case, we need to consider the following properties that the membership function must have:

- The production plans located below the lower frontier or on the lower frontier, belong completely to the production possibility set and the membership function must attribute unitary value to such production plans.
- The membership function must appropriate an intermediate membership degree to those production plans whose locations are between the frontiers (frontier region).

According to above properties of membership function, it is obvious that the distance of the production plans from the frontiers plays an important role to derive the membership degree for production plans located in the frontier region. Figure 2 represents a production plan in the frontier region and the distance of the production plan from the upper and lower frontier in a two dimensional space.

Considering above information about the membership degree, we appropriate the ratio $\varphi(\alpha, \beta) = \frac{\beta}{\alpha+\beta}$ as the membership degree of those production plans located in the frontier region. This ratio has two important properties as follows:

- For the production plans with the same distance from the lower frontier, the membership degree for the production plan with longer distance from the

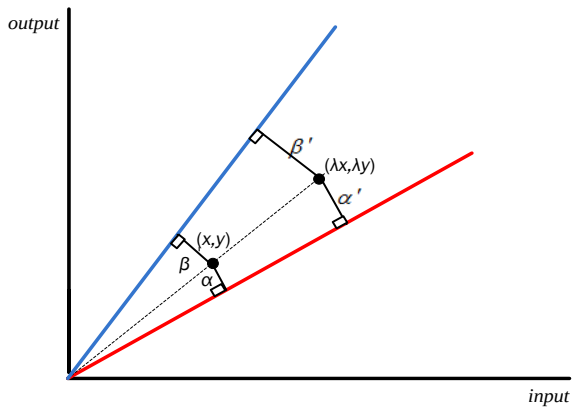

Fig. 3: Representation of production plans (x,y) and (λx, λy) in a CCR model

upper frontier is greater than the others. In other words $\frac{\partial\varphi}{\partial\beta} > 0$

- For the production plans with the same distance from the upper frontier, the membership degree for the production plan with longer distance from the lower frontier is smaller than the others. In other words $\frac{\partial\varphi}{\partial\alpha} > 0$

Theorem: suppose that λ is a positive real number, (λ>0), and (x , y) is a production plan placed in the region between two frontiers. The membership degrees of (x , y) and (λx, λy) are the same.

Proof: Let α and β be, respectively the distances of production plan from the lower and upper frontiers, (λx, λy) is a production plan located on the line passing through the point (x , y) and the origin. Let α′and β′ be respectively, the distances of (λx, λy) from the lower and upper frontier. Considering Fig. 3 and Thales theorem following result can be obtained: $\frac{\alpha}{\alpha'} = \frac{\beta}{\beta'}$ Therefore:

$$\frac{\alpha}{\alpha+\beta} = \frac{\alpha'}{\alpha'+\beta'}$$

so the membership degrees of (x, y) and (λx, λy) are the same.

Algebraic calculations of membership degree: All previous calculations are based on geometrical definitions which is feasible for very simple models. To find an expression for the membership degree that can be appropriate for all models with single interval input and output, we need to convert the geometrical terms to algebraic terms that might be derived from the classic DEA models.

Table 1: Information of 5 DMUs

DMU	DMU1	DMU2	DMU3	DMU4	
Input	[1,2]	[3,6]	[0.5,2]	[4,5]	[2.5,4]
Output	[3,5]	[12,30]	[1,2.5]	[24,39]	[20,23]

Assume that there are n DMUs to be evaluated, indexed by j = 1 ,…, n and each DMU is assumed to consume *m* different inputs to produce *s* different outputs. Let $X_j = (x_{1j}^l, x_j^u)$ and $Y_j = (y_j^l, y_j^u)$ and, respectively, indicate the input and output of DMUj.

It should be noticed that, due to the output orientation model, the inefficient production plans produce an efficient score greater than one. Hence for the production plans located below or on the lower frontier $\theta^l \geq 1$ also for the production plans located in the frontier region, $\theta^l < 1$ and $\theta^l > 1$

According to above considerations, the membership function of production plans is as follows:

$$\mu(X,Y) = \begin{cases} 1 \ \theta^l > 1 \\ 0 \ \theta^u = 1 \\ \frac{\beta}{\alpha+\beta}\theta^u > 1 \ \ and \ \theta^l < 1 \end{cases} \quad (2)$$

To obtain the values of α and β, firstly we need to measure the magnitude of the vectors (*h*, *t*) and (*k*, *l*) computed by the following linear programming's:

$$\begin{aligned} &\min h + t \\ &\text{s.t.} \\ &\textstyle\sum_{j=1}^{n} \lambda_j x_j^u \leq x + h \\ &\textstyle\sum_{j=1}^{n} \lambda_j y_j^l \geq y - t \\ &\lambda_j \geq 0 \ \ j = 1, \ldots, n \\ &h \geq 0, t \geq 0 \end{aligned} \quad (3)$$

and

$$\begin{aligned} &\min k + l \\ &\text{s.t.} \\ &\textstyle\sum_{j=1}^{n} \lambda_j x_j^l \geq x - k \\ &\textstyle\sum_{j=1}^{n} \lambda_j y_j^u \leq y + l \\ &\lambda_j \geq 0 \ \ j = 1, \ldots, n \\ &k \geq 0, l \geq 0 \end{aligned} \quad (4)$$

hence the values of α and β can be calculated as follows:

$$\alpha = \sqrt{h^2 + t^2} \quad (5)$$

$$\beta = \sqrt{k^2 + l^2} \quad (6)$$

Numerical example: Table 1 shows the information of five DMUs with single interval input and single interval output.

Table 2: Result of membership calculations for 6 different production plan

Production plan	θ^u	θ^l	μ
(0.5,5)	1.00	0.50	0.00
(1.5,8)	1.875	0.937	0.125
(1,4)	2.5	1.25	1.00
(3,15)	2.00	1.00	1.00
(4,23)	1.74	0.92	0.265
(6,54)	1.11	0.555	0.886

Table 2 shows the result of membership degrees calculations for 6 production plans in the production possibility set created by the DMUs presented in the Table 1 in the CCR model.

Extension of the idea: Assume that there are n DMUs to be evaluated, indexed by j = 1,…, n and each DMU is assumed to consume m different inputs to produce s different outputs. Let $X_j = (x_{1j}, x_{2j},\ldots,x_{mj})$ and $Y_j = (y_{1j}, y_{2j},\ldots,y_{mj})$ and be respectively the inputs and outputs vectors of DMUj such that Xij = (x_{ij}^l, x_{ij}^u) and $Y_{rj} = (y_{rj}^l, y_{rj}^u)$ and (j = 1,…, n, I =1,…, m and r = 1,…, s) and

In order to use Eq. 2 as the membership function of a production plans related to multidimensional production possibility set we compute the linear programming presented in (7) and (8):

$$\begin{aligned}
&\min \{\textstyle\sum_{i=1}^{m} k_i + \sum_{r=1}^{s} l_r\}\\
&\text{s.t.}\\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij}^u \geq x_i + k_i \qquad (i = 1,\ldots, m)\\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj}^l \leq y_r + l_r \qquad (r = 1,\ldots, s)\\
&\lambda_j \geq 0 \qquad j = 1,\ldots, n\\
&k_i \geq 0,\ l_r \geq 0
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\min \{\textstyle\sum_{i=1}^{m} h_i + \sum_{r=1}^{s} t_r\}\\
&\text{s.t.}\\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij}^u \leq x_i + h_i \qquad (i = 1,\ldots, m)\\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj}^l \geq y_r - t_r \qquad (r = 1,\ldots, s)\\
&\lambda_j \geq 0 \qquad j = 1,\ldots, n\\
&h_i \geq 0,\ t_r \geq 0
\end{aligned} \tag{8}$$

so the value of α and β can be calculated as follows:

$$\alpha = \sqrt{\textstyle\sum_{i=1}^{m} h_i^2 + \sum_{r=1}^{s} t_r^2} \tag{9}$$

$$\beta = \sqrt{\textstyle\sum_{i=1}^{m} k_i^2 + \sum_{r=1}^{s} l_r^2} \tag{10}$$

CONCLUSION

Ordinary DEA models require accurate data to evaluate the relative efficiency of the production plans. However in real world problems accurate data may not be available. When the data of decision making units are in the interval format available, the location of the efficient frontier cannot be determined precisely and it may be placed between an upper and a lower frontier. The efficient frontier is the boundary of production possibility set therefore the indefinite location of efficiency frontier enters the imprecision to the production possibility set. In this paper we consider the production possibility set as a fuzzy set which all the production plans are assumed as its members with different degrees of membership. we introduce a membership function for the production plans regarding the production possibility set based on a geometrical approach. The shortest distances of production plans from the upper and the lower frontier are measured to derive the membership function. Firstly propose the membership function for the models with single interval input and output and then we extended the idea for the models with multiple interval inputs and outputs.

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