

X-Domatic Partition of Bipartite Graphs

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Abstract: In this study, we define X-domatic partition of a bipartite graph. X-domatic number of a unicyclic graph is less than or equal to 3 is proved. Nordhaus Gaddum type of results involving X-domatic number are obtained and the graphs for which $d_x(G) + d_x(\bar{G}) = 2P$, $d_x(G) + d_x(\bar{G}) = 2P - 1$, are characterized.

Keywords: X-domatic partition, X-dominating set, Y-dominating set

INTRODUCTION

Graph colouring and domination in graphs are two areas within graph theory which have been extensively studied. Graph coloring partitions the set of objects into classes according to certain rules. Similar to the concept of chromatic partition, (Cockayne and Hedetniemi, 1977) introduced the concept of domatic partition of a graph. The idea of k-rainbow domatic number was introduced by Sheikholeslami and Volkmann (2012). The concept X-domatic partition of a bipartite graph is an extension of the concept of domatic partition of a graph.

Let G be a graph. A subset S of V is a dominating set of G if for every $v \in V - S$ there exists a vertex $u \in S$ such that u and v are adjacent. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A partition $\{V_1, V_2, \dots, V_n\}$ of V is a domatic partition of G if every V_i is a dominating set. The maximum order of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. For further details on domatic number, the reader is referred to the paper on domatic number of graphs by Zelinka (1981).

We shall consider only connected bipartite graphs with bipartition (X, Y) where $|X| = P$; $|Y| = q$ and have no loops and multiple edges. Bipartite theory of graphs was proposed by Stephen and Renu (1986 a, b) in which concepts in graph theory have equivalent formulations as concepts for bipartite graphs. One such reformulation is the concept of X-dominating and Y-dominating set.

Two vertices $u, v \in X$ are X-adjacent if they are adjacent to a common vertex in Y . A subset D of X is an X-dominating set if every vertex in $X - D$ is X-adjacent to at least one vertex in D . The minimum cardinality of a X-dominating is called the X-domination number of G and is denoted by $\gamma_x(G)$. If x

$\in X$, then the X-degree of x is defined to be the number of vertices in X which are X-adjacent to x and is denoted by $d_x(x)$. Let $\delta_x(G) = \min \{d_x(x) : x \in X\}$.

Proposition 1: $\gamma_x(C_l) = \lfloor \frac{l}{6} \rfloor$ for every $l \geq 4$, $l \equiv 0, 2, 4, \pmod{6}$.

A subset $S \subseteq X$ which dominates all vertices in Y is called a Y-dominating set of G . The Y-domination number denoted by $\gamma_y(G)$ is the minimum cardinality of a Y-dominating set of G .

Vertices u, v, w are adjacent to a vertex $y \in Y$, we say that the vertices u, v, w are X-adjacent through same $y \in Y$. If the vertices u, w are adjacent to $y_1 \in Y$ and u, v are adjacent to $y_2 \in Y$, we say that the vertices u, v, w are X-adjacent through different $y \in Y$. A bipartite graph G is said to be X-complete if every vertex in $X(G)$ is X-adjacent to every other vertex in X (G).

X-domatic partition of a graph: An X-domatic partition of G is a partition of X , all of whose elements are X-dominating sets in G . The X-domatic number of G is the maximum number of classes of an X-domatic partition of G . The X-domatic number of a graph G is denoted by $d_x(G)$.

Remark 1: Since $\{X\}$ itself is a X-domatic partition of G , the existence of a X-domatic partition is guaranteed for any bipartite graph and so the parameter $d_x(G)$ is well defined. Further, obviously $1 \leq d_x(G) \leq P$ for any bipartite graph.

The X-domatic number of a bipartite graph is analogous to the concept of domatic number of an arbitrary graph. For more information on the domination number and on the domatic number and their invariants, the reader is referred to survey book by Haynes *et al.* (1998) and Zelinka (1988).

Theorem 1: In any graph G , $\gamma_x(G) + d_x(G) \leq P + 1$.

Proof: If $\gamma_x(G) = 1$ then $\gamma_x(G) + d_x(G) \leq P + 1$. If $\gamma_x(G) = k$ ($k \geq 2$) then the remaining $P-k$ vertices of X can be X -dominating sets and hence, $d_x(G) \leq P - k + 1$. Therefore, $\gamma_x(G) + d_x(G) \leq P + 1$.

Lemma 1: In any graph G , $d_x(G) \leq 1 + \delta_Y(G)$.

Proof: Let $x \in X$ be a vertex with minimum X -degree. Suppose $d_x(G) > 1 + \delta_Y(G)$. Since x has at least $(d_x(G) - 1)$ X -neighbours in X , $d_Y(x) \geq (d_x(G) - 1) > \delta_Y(G)$, a contradiction. Therefore, $d_x(G) \leq 1 + \delta_Y(G)$.

Theorem 2: If G is a connected unicyclic graph then $d_x(G) \leq 3$.

Proof: Since G is unicyclic, G contains a unique even cycle C_{2n} , where n is even. Let $G = C_{2n}$. We note that $d_x(C_6) = 3$ and $d_x(C_4) = 2$. Also, $\gamma_x(C_l) = \lfloor \frac{l}{6} \rfloor$ for every $l \geq 4$, $1 \equiv 0, 2, 4 \pmod{6}$. Therefore, $\gamma_x(C_l) \geq \frac{l}{6}$. Hence, $d_x(G) \leq \frac{P}{\gamma_x} \leq 3$.

In case $G \neq C_{2n}$, then G has a pendant vertex and therefore $d_x(G) \leq 1 + \delta_Y(G) = 2$.

Corollary 1: For a connected unicyclic graph G , $d_x(G) = 3$ if and only if $G = C_{2n}$ where $|X| \equiv 0 \pmod{3}$.

Proof: Assume $d_x(G) = 3$. When $|X| \leq 2$, G will have a vertex of full X -degree implying $d_x(G) = 2$, a contradiction. Hence, $|X| \geq 3$. G Cannot have a pendent X -vertex since in that case $d_x(G) \leq 2$. Hence, $G = C_{2n}$. Since $\gamma_x(C_l) = \lfloor \frac{l}{6} \rfloor$ for every $l \geq 4$, $1 \equiv 0, 2, 4 \pmod{6}$ and $d_x(G) = 3$, $|X|$ is a multiple of 3.

Conversely, if $|X| \equiv 0 \pmod{3}$ and $G = C_{2n}$ where $X = \{x_1, x_2, \dots, x_{3n}\}$ and $Y = \{y_1, y_2, \dots, y_{3n}\}$. The sets $X_1 = \{x_1, x_4, \dots\}$, $X_2 = \{x_2, x_5, \dots\}$ and $X_3 = \{x_3, x_6, \dots\}$ form three γ_x -sets and hence, $d_x(G) = 3$.

Corollary 2: If G is a connected unicyclic graph with $|X| \geq 3$, then $d_x(G) = 2$ if and only if $G \neq C_{2n}$ where $|X| \equiv 0 \pmod{3}$.

Proof: Let $d_x(G) = 2$ and $|X| \geq 3$. By the above corollary $G \neq C_{2n}$ where $|X| \equiv 0 \pmod{3}$. Conversely, let $G \neq C_{2n}$ where $|X| \equiv 0 \pmod{3}$. Since G is unicyclic, $d_x(G) \leq 3$. $d_x(G)$ cannot be three by the above corollary and $d_x(G)$ cannot be 1. Therefore, $d_x(G) = 2$. It is easy to prove the following.

Proposition 2: In any graph G , $d_x(G) = P$ if and only if G is X -complete.

Now we define the complement of a bipartite as defined by Sampathkumar and Pusphalatha (1989). Let $G = (X, Y, E)$ be a bipartite graph. We define the complement of G denoted by $\bar{G} = (X, Y, F)$ as follows:

- No two vertices in X are adjacent
- No two vertices in Y are adjacent
- $x \in X$ and $y \in Y$ are adjacent in \bar{G} if and only if $x \in X$ and $y \in Y$ are not adjacent in G

Proposition 3: In any graph G , $d_x(\bar{G}) = P$ if and only if for every $u, v \in X$, there exists $y \in Y$ such that y is not adjacent to u, v in G .

Proof: Let $d_x(\bar{G}) = P$. Every vertex in \bar{G} is X -adjacent to all other vertices. There exists $y \in Y$ such that u and v are adjacent to $y \in Y$ in \bar{G} . Then, $y \in Y$ is not adjacent to u and v are in G .

Conversely, suppose that for every $u, v \in X$, there exists $y \in Y$ such that y is not adjacent to u, v in G . Then y is adjacent to u, v in \bar{G} . Hence, u and v are X -adjacent in \bar{G} . Hence, $d_x(\bar{G}) = P$.

NORDHAUS-GADDUM TYPE RESULTS

Nordhaus-Gaddum type theorem establish bounds on $\theta(G) + \theta(\bar{G})$ for some parameter θ , where \bar{G} is the complement of G . Several Nordhaus-Gaddum type theorems were characterized for various domination parameters, some are rainbow vertex-connection number (Chen and Memngmeng, 2011) of a graph, rainbow connection number (Chen *et al.*, 2010) of graphs and k -rainbow domatic number (Meierling *et al.*, 2011) of a graph. Here we characterize graphs for which $d_x(G) + d_x(\bar{G})$ equal to $2P$ and $2P-1$. To characterize the graphs attaining the bounds, we define the following family of graphs.

Λ is the family of graphs such that every vertex in X is of X -degree $P-1$ and for any two vertices u and v in X , there exists $y \in Y$ such that u and v are not adjacent to y .

Λ_1 is the family of graphs with a vertex in Y of degree $|X|$ and a unique γ_Y -set of cardinality 2.

Λ_2 is the family of graph with $|X| = 2$ and at least one vertex in X and Y is of full degree.

Theorem 3: In any graph G , $d_x(G) + d_x(\bar{G}) = 2P$ if and only if $G \in \Lambda$.

Proof: If $d_x(G) + d_x(\bar{G}) = 2P$ then $d_x(G) = P$ and $d_x(\bar{G}) = P$. By proposition 2, we get every vertex in X is of X -degree $(P-1)$. Conversely, suppose that for every $u, v \in X$, there exists $y \in Y$ such that y is not adjacent to u, v in G . Then y is adjacent to u, v in \bar{G} . Hence, u and v are X -adjacent in \bar{G} . Hence, $d_x(\bar{G}) = P$. Converse is obvious.

Theorem 4: Let G be a graph with $|X| \geq 3$. Then $d_x(G) + d_x(\bar{G}) = 2P-1$ if and only if either G or \bar{G} is in $\{\Lambda_1, \Lambda_2\}$.

Proof: Let $G \in \{\Lambda_1, \Lambda_2\}$. Clearly, $d_x(G) = P, d_x(\bar{G}) = P - 1$. Hence, $d_x(G) + d_x(\bar{G}) = 2P-1$. If $\bar{G} \in \{\Lambda_1, \Lambda_2\}$, then $d_x(G) = P-1$ and $d_x(\bar{G}) = P$. Hence, $d_x(G) + d_x(\bar{G}) = 2P-1$.

Let $d_x(G) + d_x(\bar{G}) = 2P-1$. The possibilities are:

- $d_x(G) = P$ and $d_x(\bar{G}) = P-1$
- $d_x(G) = P-1$ and $d_x(\bar{G}) = P$

Case 1: Let $d_x(G) = P$ and $d_x(\bar{G}) = P-1$.

If $d_x(G) = P$, every vertex in X is of X -degree $P-1$. If every vertex in X is of X -adjacent to other vertices through different $y \in Y$, then $d_x(\bar{G}) = P$, a contradiction. Therefore, there exists a vertex $y \in Y$ of full degree.

Claim: $\gamma_Y(G) \leq 2$.

Suppose, $\gamma_Y(G) \geq 3$. Then any subset of X of cardinality 2 is not a Y -dominating set of G . Therefore, if $u, v \in X$, then there exists $y \in Y$ such that u and v are not adjacent with y . Hence, u and v are X -adjacent in \bar{G} and so $d_x(\bar{G}) = P \neq P-1$, a contradiction. Hence, $\gamma_Y(G) \leq 2$.

Sub case 1: If $\gamma_Y(G) = 1$, then a vertex in X is of full degree. Therefore, $d_x(\bar{G}) = 1 = P-1$. Hence, $P = 2$. Therefore, $G \in \Lambda_2$.

Sub case 2: $\gamma_Y(G) \leq 2$.

Let $S_1 = \{u_1, u_2\}$ be a γ_Y -set of G .

Claim: u_1 and u_2 are not X -adjacent in \bar{G} .

Suppose u_1 and u_2 are X -adjacent in \bar{G} . Then there exists $y \in Y(\bar{G})$ such that u_1 and u_2 are adjacent with $y \in Y(\bar{G})$. Therefore, u_1 and u_2 are not adjacent with y in G . Hence, y is not dominated by u_1 and u_2 in G , a contradiction.

Claim: The existence of γ_Y -set is unique in G .

Let S_1 and S_2 be two minimum Y -dominating sets

of G . Let $S_1 = \{u_1, u_2\}$ and $S_2 = \{u_3, u_4\}$. Therefore, u_1 and u_2 are not X -adjacent in \bar{G} . Similarly u_3 and u_4 are not X -adjacent in \bar{G} . If u_1, u_2, u_3 and u_4 are all distinct, then $\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}$ are all non X -dominating set in \bar{G} . Therefore, $d_x(\bar{G}) \leq P-2$, a contradiction. Let S_1 and S_2 have a common vertex. Let $u_2 = u_4$. Then $\{u_1\}, \{u_2\}, \{u_3\}$ are all non X -dominating set in \bar{G} . Therefore, $d_x(\bar{G}) \leq P-2$, a contradiction. Therefore, γ_Y -sets of cardinality 2 in G is unique. Therefore, $G \in \Lambda_1$.

Case 2: Let $d_x(G) = P-1$ and $d_x(\bar{G}) = P$.

Using the above argument, we get \bar{G} is in $\{\Lambda_1, \Lambda_2\}$.

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