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Shift-Relaxation Combined Approximations Method for Structural Vibration Reanalysis of Near Defective Systems

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Abstract: Structural reanalysis for defective systems and the near defective systems is discussed in this paper. Specially, by a relaxation factor embedded in the Combined Approximations (CA) approach and a frequency shift, the reanalysis problem of near defective systems with close eigenvalues can be transformed into one of defective systems with repeated ones, which is equal to the average of the close ones. Numerical examples show that the proposed method is effective and stable in reanalysis problem of near defective systems.

Keywords: Close frequencies, generalized modal theory, structural modification, vibration system

INTRODUCTION

Structural modification of systems is an important research issue and has a wide range of applications such as vibration suppression, system design and control. In the structural dynamic optimization, the multiple repeated analyses are ones of the most costly computations. The need for efficient and accurate reanalysis technique in modern structural design is crucial because the designs become larger and more complex.

In problems such as those of dynamic and symmetric structures, however, the corresponding matrices can have repeated eigenvalues. In structural dynamics reanalysis, it is by no means rare to encounter systems with multiple eigenvalues. Very often indeed, the geometric multiplicity of the eigenvalues is less than the algebraic multiplicity; systems of this type are called defective systems (Luongo, 1995). Defective systems, however, represent exceptional cases. More important, many systems with clusters of frequencies can occur in practical engineering, systems having close eigenvalues are encountered instead of perfectly coincident. For example, during system optimization, some originally separated frequencies can approach closer and closer. In these cases, if the associated eigenvectors make groups of nearly parallel vectors, the system can be classified as near defective system. From the view point of mathematics, the close eigenvalues of near defective systems are distinct, but the dynamic characteristic is still defective. Vibration modes with close frequencies, i.e. clusters of frequencies, often occur in some structural systems, including large space structures, multi span beams and some nearly periodic structures and symmetric structures (Chen, 2007). Therefore it is of great importance to research about the reanalysis problem of vibration modes of near defective systems for practical engineering. Chen treated the control problem of near defective systems well using the generalized modal theory based on the invariant subspace recursive method, but the number of actuators cannot less than that of closed eigenvalues (Chen *et al*., 2001; Chen and Chen, 2003).

In order to conduct structural modification or further research on the near defective systems, reanalysis should be performed on modified systems. In choosing a suitable reanalysis method, the following three factors are considered: the accuracy of the calculations, the computational effort involved and the ease of implementation. High accuracy, however, is often achieved at the expense of more computational effort. The CA approach is most suitable for efficient-accurate evaluation of the structural response at various modified designs (Kirsch, 2000; Kirsch and Bogomolni, 2007).

This paper aims to solve structural modification reanalysis problems and give a solution to the problem of near defective vibration systems based on the advantage of shift-relaxation combined approximations approach. First, there is a brief review of the near defective systems. Then the generalized modal theory of defective systems is discussed, the introduction of relaxation factor is a kind of equivalent technique which makes it possible and effective to deal with the problems of the modified defective systems. The proposed method is feasible after overcoming the difficulties of irreversible condition. Moreover, by frequency shift, the reanalysis problems of near

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defective systems with close eigenvalues can be transformed into one of the defective systems with repeated ones, which is equal to the average of the close ones. Finally, the results obtained by numerical examples prove that the proposed method is effective and promising to solve these kinds of problems briefly.

PROBLEM FORMULATION

The natural vibration equation of the general linear system is:

$$\mathbf{M}\ddot{\mathbf{r}}(t)+\mathbf{C}\dot{\mathbf{r}}(t)+\mathbf{K}\mathbf{r}(t)=\mathbf{0} \quad (1)$$

where, M, C and Kare the mass, damping and stiffness matrices, respectively; r(t), $\dot{r}$(t) and $\ddot{r}$(t) are vectors of displacement, speed and acceleration, respectively, t denotes time.

We assume λ are the eigenvalue of the system(1). The eigenvector corresponding to each eigenvalue is modal vector u.

Assume that:

$$\mathbf{A}=\begin{pmatrix}-\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K}\\ \mathbf{I}_n & \mathbf{0}\end{pmatrix}_{2n\times 2n} \quad (2)$$

in Eq. (2), I_n is the identity matrix and 0 is the zero matrix of the same order. Using the state vector:

$$\boldsymbol{\psi}=\begin{pmatrix}\lambda\mathbf{u}\\ \mathbf{u}\end{pmatrix} \quad (3)$$

We have the eigenproblem

$$\mathbf{A}\boldsymbol{\psi}=\lambda\boldsymbol{\psi} \quad (4)$$

where, Ψ is the state vector. Denote 2n = N briefly.

In the following, we give the definitions for classifying the non-defective system and the defective system. The system must be non-defective if λ is distinct eigenvalue or the algebra multiplicity of the eigenvalue λ is equal to geometric multiplicity. The system must be defective if the algebra multiplicity of the eigenvalue λ is greater than the geometric multiplicity, so the defective system has an incomplete set of eigenvectors to span the state space.

The defective system with repeated eigenvalues is ill-conditioned because the dynamic characteristic is very sensitive to the parameters changes of the defective system and it can be changed into a near defective system with close eigenvalues. Therefore, an outline of development of reanalysis theory for near defective systems with close eigenvalues is necessary.

SOLUTION FOR DEFECTIVE SYSTEMS BY CA METHOD

As we mentioned before, there will not be a set of complete eigenvectors to span the corresponding space if A is a defective matrix. It means that the modal expansion theory in the non-defective system is invalid for the defective system. In this case, generalized modal theory is applied for solving the eigenproblems of defective systems.

Generalized modal theory of the defective systems

For the characteristic Eq. (4), from the algebra theory, it can be shown that there exists non-singular matrix Ψ, such that:

$$\mathbf{A}\boldsymbol{\Psi}=\boldsymbol{\Psi}\mathbf{J} \quad (5)$$

where. Ψ is the generalized modal matrix of the state matrix A, given by:

$$\mathbf{J}=\begin{pmatrix}\mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_t\end{pmatrix},(1\le t<N),\ \mathbf{J}_i=\begin{pmatrix}\lambda_i & 1 & & \\ & \lambda_i & & \\ & & \ddots & 1\\ & & & \lambda_i\end{pmatrix}_{m_i\times m_i},\ \left(\sum_{i=1}^{t}m_i=N\right). \quad (6)$$

Matrix J is a Jordan canonical form of A and m_i is the multiplicities of eigenvalues λ_i.

The conjugate and transpose of the state matrix A is called adjoined system, for A^H the generalized modes satisfy the following equation:

$$\mathbf{A}^H\boldsymbol{\Phi}=\boldsymbol{\Phi}\mathbf{J}^H \quad (7)$$

where $(.)^H$ denotes the conjugate transpose of the matrix and ϕ is the generalized modal matrix of the adjoined system.

The right generalized modal matrix Ψ and the left generalized modal matrix ϕ satisfy the following orthogonal condition:

$$\boldsymbol{\Phi}^H\boldsymbol{\Psi}=\mathbf{I} \quad (8)$$

It should be noted that for the linear vibration system with distinct or repeated eigenvalues, the system matrix A can be diagonalized by the similarity transformation:

$$\mathbf{A}=\boldsymbol{\Psi}\mathbf{J}\boldsymbol{\Psi}^{-1} \quad (9)$$

then the system must be non-defective. In this case the right generalized modal matrix Ψ and the left ones ϕ are the complete eigenvectors and can be used to span the eigenspace. The defective matrix, however, can not be put into a diagonal matrix, there is a sub matrix j_i of order (m_i x m_i), (m_i > 1) in J, at least.

The eigenspace can be obtained by using normal methods for solving the linear equations if the system is non-defective, however, if the system is defective or near defective, that is, the eigenspace is incomplete or near incomplete, fatal mistakes may occur while computing the generalized modes. Therefore, it is important to give a reliable method for computing the generalized modes corresponding to defective and near defective systems.

CA method for defective systems: For the purpose of improving the accuracy of the calculation and eliminating the numerical errors, the approximate modes and basis vectors for the defective systems are obtained with a suitable relaxation factor embedded in the CA method.

For simplicity of presentation, suppose that the state matrix A has N repeated defective eigenvalues λ, and the Jordan canonical form J corresponding to A is:

$$\mathbf{J}=\begin{pmatrix}\lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda\end{pmatrix}_{N\times N}. \tag{10}$$

Then Eq.(5) can be rewritten as:

$$(\mathbf{A}-\lambda\mathbf{I})\boldsymbol{\Psi}=\boldsymbol{\Psi}\mathbf{B}, \tag{11}$$

where, B = (0, e_1, …, e_{N-1}). e_i ($1 \le i \le N-1$) is the unit vector.

The initial stiffness matrix K_0 = A- λI needs to be reversible in the CA method. In order to apply this method, we introduce a relaxation factor $\omega \neq 0$ and then Eq. (11) is equivalent to:

$$(\mathbf{A}-\lambda\mathbf{I}+\omega\mathbf{I})\boldsymbol{\Psi}=\boldsymbol{\Psi}\mathbf{B}+\omega\boldsymbol{\Psi}, \tag{12}$$

Using the form $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_N)$ the characteristic equations are:

$$\begin{cases}(\mathbf{A}-\lambda\mathbf{I}+\omega\mathbf{I})\boldsymbol{\psi}_1=\omega\boldsymbol{\psi}_1, \\ (\mathbf{A}-\lambda\mathbf{I}+\omega\mathbf{I})\boldsymbol{\psi}_i=\boldsymbol{\psi}_{i-1}+\omega\boldsymbol{\psi}_i, \quad i=2,3,\cdots,N.\end{cases} \tag{13}$$

If the structural parameters have small changes, such that the state matrix A has a change ΔA, for the modified state matrix A + ΔA, there exists a reversible matrix $\tilde{\Psi}$, such that:

$$(\mathbf{A}+\Delta\mathbf{A})\tilde{\boldsymbol{\Psi}}=\tilde{\boldsymbol{\Psi}}\tilde{\mathbf{J}}, \tag{14}$$

where, $\tilde{A} = A + \Delta A, \tilde{J} = J + \Delta J, \tilde{J}$ is the new Jordan canonical form of the state matrix $\tilde{A}$.

Introduce the notation:

$$\mathbf{L}_0=\mathbf{A}-\lambda\mathbf{I}+\omega\mathbf{I},\ \Delta\mathbf{L}=\Delta\mathbf{A}-\Delta\mathbf{J}-\omega\mathbf{I},\ \mathbf{L}=\mathbf{L}_0+\Delta\mathbf{L}, \tag{15}$$

(14) and (15) lead to:

$$(\mathbf{L}_0+\Delta\mathbf{L})\tilde{\boldsymbol{\Psi}}=\tilde{\boldsymbol{\Psi}}\mathbf{B}. \tag{16}$$

Expand Eq. (16) and write it to the form of characteristic equations:

$$\begin{cases}(\mathbf{L}_0+\Delta\mathbf{L})\tilde{\boldsymbol{\psi}}_1=\mathbf{0}, \\ (\mathbf{L}_0+\Delta\mathbf{L})\tilde{\boldsymbol{\psi}}_i=\tilde{\boldsymbol{\psi}}_{i-1}, \quad i=2,3,\cdots,N.\end{cases} \tag{17}$$

The selection of relaxation factor should guarantee Lo (= A-$\lambda I + \omega I$) is reversible. It is an equivalent technology and the value of $\omega \neq 0$ does not affect the results.

For i =1in Eq. (17), we can get the generalized eigenvector. $\tilde{\Psi}_1$

Based on the CA approach, we obtain:

$$\tilde{\boldsymbol{\psi}}_i^k=-\mathbf{L}_0^{-1}\Delta\mathbf{L}\,\tilde{\boldsymbol{\psi}}_i^{k-1},\quad k=1,2,\cdots,s,\quad i=2,3,\cdots,N, \tag{18}$$

from (18), the basis vectors can be given by:

$$\tilde{\boldsymbol{\Psi}}_i^B=\left(\tilde{\boldsymbol{\psi}}_i^1,\tilde{\boldsymbol{\psi}}_i^2,\cdots,\tilde{\boldsymbol{\psi}}_i^s\right),\quad i=2,3,\cdots,N. \tag{19}$$

The vector $\tilde{\Psi}_i$ is a linear combination of the basis vectors $\tilde{\Psi}_i^B$ and the coefficient vectors y_i, it follows that:

$$\tilde{\boldsymbol{\psi}}_i=\tilde{\boldsymbol{\Psi}}_i^B\mathbf{y}_i=y_i^1\tilde{\boldsymbol{\psi}}_i^1+\cdots+y_i^s\tilde{\boldsymbol{\psi}}_i^s,\quad i=2,3,\cdots,N, \tag{20}$$

where, the vectors of coefficient are to be determined:

$$\mathbf{y}_i=\left(y_i^1,y_i^2,\cdots,y_i^s\right)^T \quad i=2,3,\cdots,N. \tag{21}$$

Let

$$\mathbf{L}_i^R=\left(\tilde{\boldsymbol{\Psi}}_i^B\right)^T\mathbf{L}\tilde{\boldsymbol{\Psi}}_i^B,\quad \mathbf{R}_i^R=\left(\tilde{\boldsymbol{\Psi}}_i^B\right)^T\tilde{\boldsymbol{\psi}}_{i-1},\quad s,\ i=2,3,\cdots,N.$$

Therefore, only to solve the smaller s × s system:

$$\mathbf{L}_i^R\mathbf{y}_i=\mathbf{R}_i^R,\ i=2,3,\cdots,N \tag{23}$$

Can we get the vectors of coefficients y_i and the computation is much less than the original Eq.(14). Put the vectors of coefficient to (20) and repeat the above

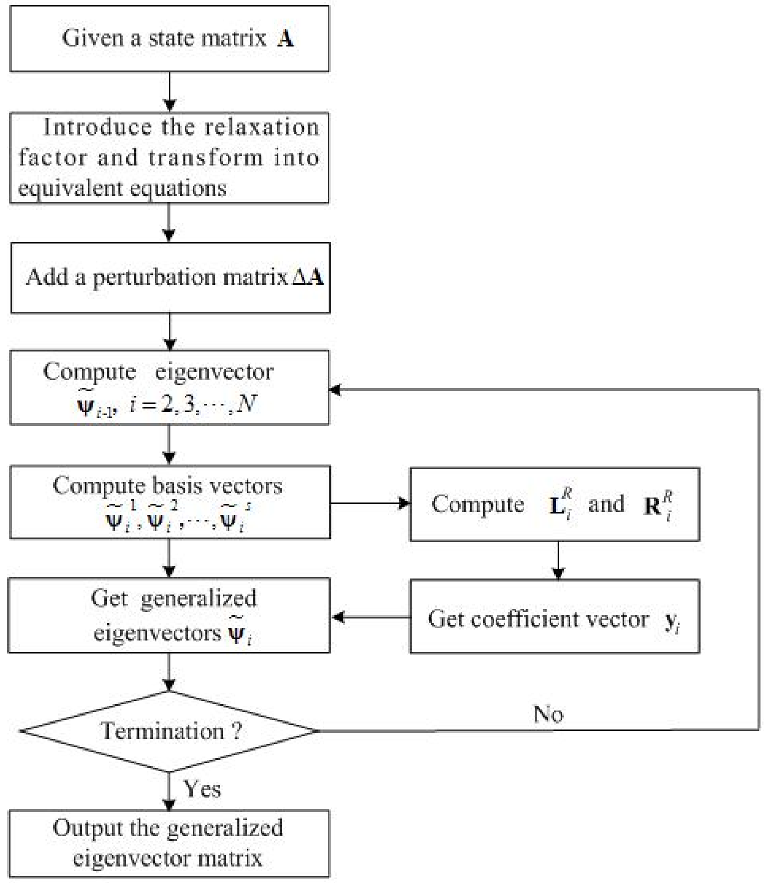

Fig. 1: Flowchart of solving defective systems

iterative scheme for i = 2, 3, …, N, we get the generalized eigenvectors $\tilde{\Psi}_i$. Summing up the above ideas, the modified generalized eigenvector matrix $\tilde{\Psi}$ can be obtained.

By the analysis and discussion of the method, the whole idea for solving the problems with N repeated defective eigenvalues λ can be summarized into the flowchart in Fig. 1.

The state matrix A contains N repeated defective eigenvalue λ is a special case, it is easy to implement compared with the general situation. That is, λ is the t − multiply ($2 \le t \le N$) defective eigenvalues, the rest eigenvalues are distinct, i.e., $\lambda, \ldots, \lambda, \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_N$. Detailed analysis and calculation processes are given which can be used as an extension of this algorithm in other cases.

SOLUTION FOR NEAR DEFECTIVE SYSTEMS BY SHIFT-RELAXATION CA METHOD

The difficulties arise when the eigenvectors corresponding to the close eigenvalues are near linearly independent which cannot be obtained by the common computational methods. For this reason, we need to seek a method to deal with the reanalysis problem of the near defective systems. A new approximate algorithm, which is developed based on the CA approach with a relaxation factor and frequency-shift, is presented in this section as an efficient reanalysis method for the near defective systems.

Assume that N eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_N$ of the state matrix A are close at first. The generalized modal matrices Ψ and ϕ satisfies the Eq. (5), (7) and (8). According to the closed property of the eigenvalues, taking the algebra average of $\lambda_1, \lambda_2 \ldots \lambda_N$:

$$\bar{\lambda} = \frac{1}{N}\sum_{i=1}^{N}\lambda_i \tag{24}$$

For simplicity, let

$$\bar{\mathbf{J}} = \begin{pmatrix} \bar{\lambda} & 1 & & 0 \\ & \bar{\lambda} & \ddots & \\ & & \ddots & 1 \\ 0 & & & \bar{\lambda} \end{pmatrix}, \ \delta\mathbf{J} = \begin{pmatrix} \lambda_1 - \bar{\lambda} & -1 & & 0 \\ & \lambda_2 - \bar{\lambda} & \ddots & \\ & & \ddots & -1 \\ 0 & & & \lambda_N - \bar{\lambda} \end{pmatrix}, \tag{25}$$

the Jordan canonical form J can be expressed as:

$$\mathbf{J} = \bar{\mathbf{J}} + \delta\mathbf{J} \tag{26}$$

The state matrix A can be written in the following form:

$$\mathbf{A} = \mathbf{\Psi J \Phi}^H = \mathbf{\Psi}\bar{\mathbf{J}}\mathbf{\Phi}^H + \mathbf{\Psi}\delta\mathbf{J}\mathbf{\Phi}^H = \bar{\mathbf{A}} + \delta\mathbf{A} \tag{27}$$

where,

$$\bar{\mathbf{A}} = \mathbf{\Psi}\bar{\mathbf{J}}\mathbf{\Phi}^H, \quad \delta\mathbf{A} = \mathbf{\Psi}\delta\mathbf{J}\mathbf{\Phi}^H. \tag{28}$$

Since the orthogonal transform cannot change the eigenvalues of matrix, the eigenvalues and the corresponding eigenvectors of $\bar{A}$ are the same as those of $\bar{J}$. If $\lambda_1, \lambda_2 \ldots \lambda_N$ are closed eigenvalues and $\delta = \max|\lambda_i - \bar{\lambda}|$, it can be proved that δA is an error matrix.

Through the above analysis, it is observed that in the reanalysis problem of the near defective system, the matrix A can be expressed in terms of the defective matrix $\bar{A}$ with N repeated eigenvalues and an error matrix δA .

Assume that the parameter changes are introduced to the structure as $\tilde{M}$ = M+εM, $\tilde{K}$= K+εK, εM and εK represent changes due to the behavior of the structure. For the perturbed structure, there is a reversible matrix $\tilde{\Psi}$, such that:

$$(\mathbf{A} + \varepsilon\mathbf{A})\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{J}}, \tag{29}$$

where, $\tilde{J}$ is the new Jordan canonical form of state matrix A + εA.

Let:

$$\Delta\mathbf{A} = \varepsilon\mathbf{A} + \delta\mathbf{A} \tag{30}$$

Substituting Eq. (27) into (29) yields:

$$(\bar{\mathbf{A}} + \Delta\mathbf{A})\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{J}} \tag{31}$$

Eq. (31) shows that by the skill of frequency shift, the analysis of the near defective systems with N close eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_N$ has been transformed into ones of the defective systems with repeated eigenvalues, which is equal to the average of the close ones. Hence, the method discussed in section 3.2 for the defective systems can be used to deal with the reanalysis problem for the near defective systems.

In generally, suppose that the t repeated eigenvalues of A are close and the rest are distinct, i.e. $\lambda_1 = \lambda_2 = \ldots = \lambda_t = \lambda$ $(2 \le t < N)$, $\lambda_{t+1}, \lambda_{t+2} \ldots \lambda_N$. In this case, the modal matrices Ψ and ϕ in Eq. (7) and (8) can be partitioned by column as:

$$\mathbf{\Psi} = (\mathbf{\Psi}_t, \mathbf{\Psi}_{N-t}) \tag{32}$$

$$\mathbf{\Phi} = (\mathbf{\Phi}_t, \mathbf{\Phi}_{N-t}), \tag{33}$$

where,

$$\mathbf{\Psi}_t = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \cdots, \boldsymbol{\psi}_t),$$
$$\mathbf{\Psi}_{N-t} = (\boldsymbol{\psi}_{t+1}, \boldsymbol{\psi}_{t+2}, \cdots, \boldsymbol{\psi}_N)$$
$$\mathbf{\Phi}_t = (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_t),$$
$$\mathbf{\Phi}_{N-t} = (\boldsymbol{\varphi}_{t+1}, \boldsymbol{\varphi}_{t+2}, \cdots, \boldsymbol{\varphi}_N)$$

The Jordan canonical matrix of A in Eq. (6) can be written in the following matrix block form:

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{N-t} \end{pmatrix}, \tag{34}$$

where,

$$\mathbf{J}_t = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_2 & & \\ & & \ddots & 1 \\ & & & \lambda_t \end{pmatrix}_{t\times t}, \quad \mathbf{J}_{N-t} = \begin{pmatrix} \lambda_{t+1} & & 0 \\ & \lambda_{t+2} & \\ 0 & & \lambda_N \end{pmatrix}_{(N-t)\times(N-t)}. \tag{35}$$

Similar to (25), J_t can be expressed as:

$$\mathbf{J}_t = \bar{\mathbf{J}}_t + \delta\mathbf{J}_t$$
$$= \begin{pmatrix} \bar{\lambda} & 1 & & 0 \\ & \bar{\lambda} & \ddots & \\ & & \ddots & 1 \\ 0 & & & \bar{\lambda} \end{pmatrix}_{t\times t} + \begin{pmatrix} \lambda_1 - \bar{\lambda} & -1 & & 0 \\ & \lambda_2 - \bar{\lambda} & \ddots & \\ & & \ddots & -1 \\ 0 & & & \lambda_t - \bar{\lambda} \end{pmatrix}_{t\times t} \tag{36}$$

As noted before, $\bar{\lambda}$ represents the algebra average of $\lambda_1, \lambda_2 \ldots \lambda_t$.

As a consequence:

$$\begin{aligned} \mathbf{A} &= \mathbf{\Psi J \Phi}^H \\ &= \mathbf{\Psi}_t\bar{\mathbf{J}}_t\mathbf{\Phi}_t^H + \mathbf{\Psi}_{N-t}\mathbf{J}_{N-t}\mathbf{\Phi}_{N-t}^H + \mathbf{\Psi}_t\delta\mathbf{J}_t\mathbf{\Phi}_t^H \\ &= \bar{\mathbf{A}} + \delta\mathbf{A}. \end{aligned} \tag{37}$$

Similarly, the system has been transformed into one of the defective systems with t repeated eigenvalues and N-t distinct eigenvalues by the close eigenvalues shift.

More generally, if the eigenvalues of the state matrix A are consisted of t close eigenvalues, s multiple defective eigenvalues and N–t–s distinct eigenvalues, partitioning the modal matrices Ψ and Φ by columns according to the property of the eigenvalues, then the problem can be solved similarly to the above steps.

The true percentage error in the approximate generalized eigenvectors $\tilde{\Psi}_1$, relative to the exact ones $\tilde{\Psi}_1$, is defined as:

$$\mu_i = \frac{\left|(\tilde{\boldsymbol{\psi}}_i)^H \cdot (\bar{\boldsymbol{\psi}}_i)\right|}{\|\tilde{\boldsymbol{\psi}}_i\| \cdot \|\bar{\boldsymbol{\psi}}_i\|} \tag{38}$$

where $\|.\|$ denotes the L^2 norm. Obviously, if $\mu_i \to 1$, this indicates that $\tilde{\Psi}_i$ is very accurate.

The quality of the approximation and the efficiency of the computation are usually two conflictin factors in selecting an approximate reanalysis method. The number of algebraic operations (multiplication and division) needed to solve an N × N set of equations is $N^3/3$. The number of operations needed by the CA method is $3N^2 s + Ns^2 + s^3/3$, where s the number of basis vector is. The total CPU effort involved in solution by the CA approach is usually much smaller than those needed to carry out complete analysis of modified designs. The proposed extended method not only makes it possible for the relatively complex near defective problems, but also obtains the higher efficiency of reanalysis.

NUMERICAL EXAMPLES

We give two examples to check the validity of the method. The first one with step-by-step illustrations of the algorithm will verify the correctness of the approach for defective systems. The second one validates further the proposed shift-relaxation CA method is efficient for near defective systems.

Example 1: Assume the state matrix A and the perturbation matrix ΔA are, respectively:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & 3 \end{pmatrix},$$

$$\Delta\mathbf{A} = \begin{pmatrix} -0.5 & -1.5 & 0 & 2.5 & 0.5 \\ 0 & 0 & 0 & -4 & -2 \\ -1.5 & 1.5 & 2 & -7.5 & -5.5 \\ -1 & 1 & 2 & -6 & -5 \\ 0.5 & -1.5 & 0 & 2.5 & -0.5 \end{pmatrix}$$

Then the modified state matrix can be written as $\tilde{A} = A + \Delta A$ A is a defective matrix, by the method for the repeated defective system, we can get the modified generalized eigenvector matrix $\tilde{\Psi}$.

The following diagram (Fig. 2) shows the relationship between the relaxation factor and the error with four basis vectors when solving for generalized eigenvectors form $\tilde{\Psi}_2$ to $\tilde{\Psi}_5$.

The example demonstrates the method does not depend on the value of the relaxation factor ω at all. To simplify the calculation, we usually take ω as a positive real number. The excellent results of the generalized eigenvectors can be easily obtained and are shown in Table 1. Here, we take the relaxation factors as ω = (5, 5, 6, 7) and choose the number of basis vectors as four.

From Table 1 it can be seen that the vibration reanalysis for defective systems discussed in Section 3 is seen to yield good results and the stability can be guaranteed.

Example 2: As an illustrative example in case of the near defective system with close eigenvalues, the 6 Degree Of Freedom (DOF) spring-mass mechanical system shown in Fig. 3 is considered. It is assumed that only vibrations in the vertical plane are possible.

The components of the mass matrix m of the system are:

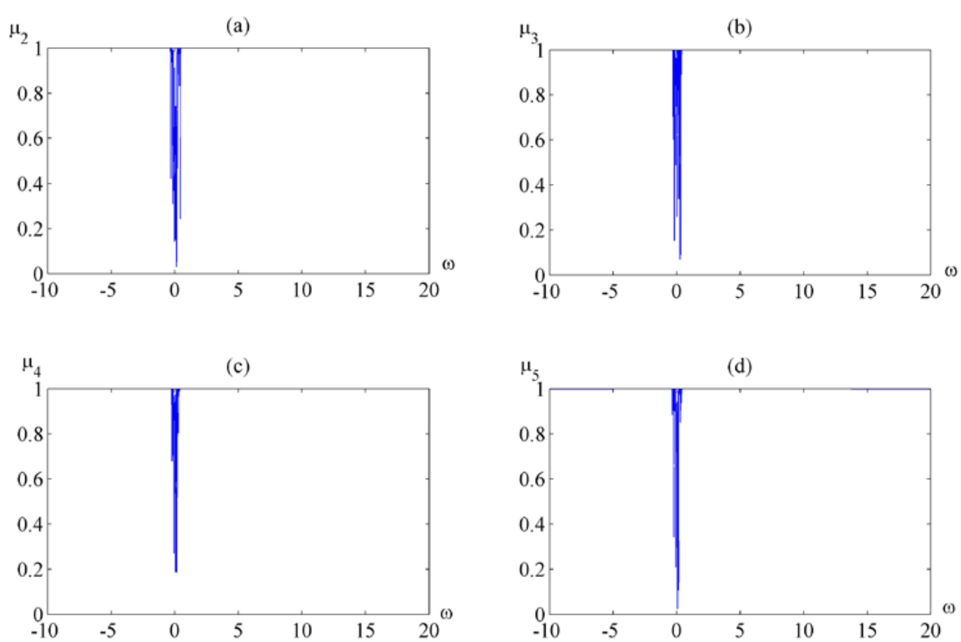

Fig. 2: Relationship between the relaxation factor and the error of generalized eigenvectors

Table 1: Error of generalized eigenvectors by the method

	1		2		3		4		5	
Mode case	RCA*	Exact	RCA	Exact	RCA	Exact	RCA	Exact	RCA	Exact
1	1.0000	1.0000	2.8000	2.8000	6.4400	6.4400	3.8222	3.8222	5.2176	5.2176
2	1.0000	1.0000	0.8000	0.8000	3.8400	3.8400	2.3268	2.3268	2.7870	2.7870
3	0.0000	0.0000	1.0000	1.0000	3.8000	3.8000	1.8480	1.8480	2.5552	2.5552
4	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	0.5601	0.5600	0.5439	0.5439
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.2000	-0.2000	-0.0802	-0.0802

Table 2: Mode shapes of spring-mass system using shift-relaxation CA method

	1		2		3	
Mode case	SRCA*	Exact	SRCA	Exact	SRCA	Exact
1	0.0001	0.0001	0.9954	0.9954	0.0038	0.0038
2	-0.0069	-0.0069	-0.0923	-0.0923	0.0545	0.0545
3	0.5575	0.5575	0.0095	0.0095	0.5769	0.5769
4	-0.4588	-0.4588	0.0145	0.0145	0.4687	0.4687
5	-0.4588	-0.4588	0.0145	0.0145	0.4687	0.4687
6	-0.5178	-0.5178	0.0141	0.0141	0.4742	0.4742
	4		5		6	
Mode case	SRCA	Exact	SRCA	Exact	SRCA	Exact
1	-0.1232	-0.1232	0.0000	0.0000	0.0000	0.0000
2	-0.8736	-0.8736	-0.0005	-0.0005	0.0000	0.0000
3	0.2254	0.2254	0.0191	0.0191	0.0000	0.0000
4	0.2389	0.2389	-0.5305	-0.5305	-0.7071	-0.7071
5	0.2389	0.2389	-0.5305	-0.5305	0.7071	0.7071
6	0.2380	0.2380	0.6609	0.6609	0.0000	0.0000

*SRCA represents the relaxation combined approximations method

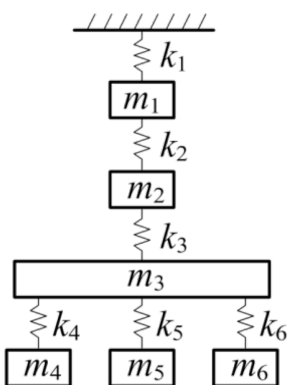

Fig. 3: 6-DOF spring-mass system

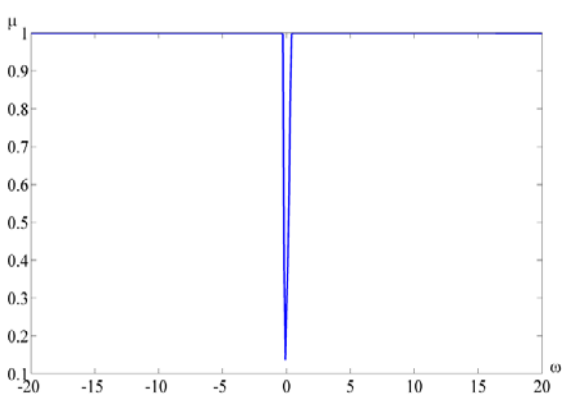

Fig. 4: Relationship between the relaxation factor and the corresponding error of generalized eigenvectors

m_1 = 20kg, m_2 = 20kg, m_3 = 20kg, m_4 = 20kg, m_5 = 20kg, m_6 = 20kg. The elements of the stiffness matrix are given as: k_1 = 200 N/m, k_2 = 1700 N/m, k_3 = 1900 N/m, k_4 = 200 N/m, k_5 = 850 N/m, k_6 = 850 N/m. The system has two close eigenvalues i.e., λ_1 = 42.4805, λ_2 = 42.5000. The algebra of λ_1, λ_2 is:

$$\overline{\lambda}=\frac{1}{2}\sum_{i=1}^{2}\lambda_i = 42.4903$$

by eigenvalues shifting, the vibration reanalysis of the near defective system with close eigenvalues can be transformed into a problem with repeated ones which is equal to the average value of the close ones.

In the following, we give the perturbation analysis of the system. The changed structure can be described as: m_1 = 200 kg, m_2 = 300 kg, m_3 = 50 kg, m_4 = 20 kg, m_5 = 20 kg, m_6 = 30 kg m, k_1 = 1500 N/m, k_2 = 200 N/m, k_3 = 200 N/m, k_4 = 500 N/m, k_5 = 500 N/m , k_6 = 800 N/m. The solutions of the modified structure are shown in Table 2, which are obtained by shift-relaxation CA method.

The diagram (Fig. 4) gives the relationship between the relaxation factor and the error with four basis vectors when solving for generalized eigenvectors near defective systems. The state vectors can be expressed by a combination of the basis vectors and the order of solving equations is reduced. Numerical examples show that the proposed method is effective and stable in reanalysis problems of the near defective systems.

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REFERENCES

Chen, S.H., 2007. Matrix Perturbation Theory in Structural Dynamics. Science Press, Beijing, China.

Chen, Y.D. and S.H. Chen, 2003. Design procedure for modal controllers for defective and nearly defective systems. Struct. Eng. Mech., 15(5): 551-562.

Chen, Y.D., S.H. Chen and Z.S. Liu, 2001. Modal optimal control procedure for near defective systems. J. Sound Vib., 245(1): 113-132.

Kirsch, U., 2000. Combined approximations-a general reanalysis approach for structural optimization. Struct. Multidisc. Optim., 20: 97-106.

Kirsch, U. and M. Bogomolni, 2007. Nonlinear and dynamic structural analysis using combined approximations. Comput. Struct., 85(10): 566-578.

Luongo, A., 1995. Free vibration and sensitivity analysis of a defective two degree-of-freedom system. AIAA J., 33: 120-127.