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On Fuzzy-Γ-ideals of Γ-Abel-Grassmann's Groupoids

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Abstract: In this study, we have introduced the notion of Γ-fuzzification in Γ-AG-groupoids which is in fact the generalization of fuzzy AG-groupoids. We have studied several properties of an intra-regular Γ-AG**-groupoids in terms of fuzzy Γ-left (right, two-sided, quasi, interior, generalized bi-, bi-) ideals. We have proved that all fuzzy Γ-ideals coincide in intra-regular Γ-AG**-groupoids. We have also shown that the set of fuzzy Γ-two-sided ideals of an intra-regular Γ-AG**-groupoid forms a semilattice structure.

Keywords: Γ-AG-subgroupoid, Γ-AG**-groupoid, Γ-AG**-groupoid and intra-regular, fuzzy subset, fuzzy Γ-ideals, fuzzy Γ-AG-groupoid

INTRODUCTION

The real world has a lot of different aspects which are not usually been specified. Models for problems in almost all fields of knowledge like engineering, medical science, mathematics, physics, computer science and artificial intelligence, can be constructed. Some times the models are very difficult to handle and the possible solutions may be impossible. Therefore, the classical set theory, which is precise and exact, is not appropriate for such problems of uncertainty.

In today's world, many theories have been developed to deal with such uncertainties for instance fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionistic fuzzy sets and theory of rough sets. The theory of soft sets has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc. The basic concept of fuzzy set theory was first given by Zadeh (1965). Zadeh (1965) discussed the relationships between fuzzy set theory and probability theory. Rosenfeld (1971) initiated the fuzzy groups in fuzzy set theory. Mordeson *et al*. (2003) have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines.

Abel-Grassmann's groupoid (AG-groupoid) is the generalization of semigroup theory with wide range of usages in theory of flocks (Naseeruddin, 1970). The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid (Mushtaq and Yousuf, 1978).

The concept of a Γ-semigroup has been introduced by Sen (1981) as follows: A non-empty set S is called a Γ-semigroup if $x\alpha y \in S$ and $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup is the generalization of semigroup.

In this study we characterize Γ-AG**-groupoids by the properties of their fuzzy Γ-ideals and generalize some results. A Γ-AG-groupoid is the generalization of AG-groupoid. Let S and Γ be any non-empty sets. If there exists a mapping $S \times \Gamma \times S \to S$ written as (x, α, y) by $x\alpha y$, then S is called a Γ-AG-groupoid if $x\alpha y \in S$ such that the following Γ-left invertive law holds for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ $(x\alpha y)\beta z = (z\alpha y)\beta x$.

A Γ-AG-groupoid also satisfies the Γ-medial law for all $w, x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (w\alpha y)\beta(x\gamma z).$$

Note that if a Γ-AG-groupoid contains a left identity, then it becomes an AG-groupoid with left identity.

A Γ-AG-groupoid is called a Γ-AG**-groupoid if it satisfies the following law for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ $x\alpha(y\beta z) = y\alpha(x\beta z)$.

A Γ-AG**-groupoid also satisfies the Γ-paramedial law for all $w, x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$ $(w\alpha x)\beta(y\gamma z) = (z\alpha y)\beta(x\gamma w)$.

PRELIMINARIES

The following definitions are available in Shah and Rehman (2010).

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Let S be a Γ -AG-groupoid, a non-empty subset A of S is called a Γ -AG-subgroupoid if $a\gamma b \in A$ for all $a,b \in A$ and $\gamma \in \Gamma$ or if $A\Gamma A \subseteq A$.

A subset A of a Γ -AG-groupoid S is called a Γ -left (right) ideal of S if $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$ and A is called a Γ -two-sided-ideal of S if it is both a Γ -left ideal and a Γ -right ideal.

A subset A of a Γ -AG-groupoid S is called a Γ -generalized bi-ideal of S if $(A\Gamma S)\Gamma A \subseteq A$.

A sub Γ -AG-groupoid A of a Γ -AG-groupoid S is called a Γ-bi-ideal of S if $(A\Gamma S)\Gamma A \subseteq A$.

A subset A of a Γ-AG-groupoid S is called a Γ -interior ideal of S if $(S\Gamma A)\Gamma S \subseteq A$.

A subset A of a Γ-AG-groupoid S is called a Γ -quasi-ideal of S if $S\Gamma A \cap A\Gamma S \subseteq A$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \to [0,1]$, where $[0,1]$ the usual is closed interval of real numbers.
Now we introduce the following definitions.

Let f and g be any fuzzy subsets of a Γ -AG-groupoid S , then the Γ -product $f \circ_\Gamma g$ is defined by

$$(f \circ_\Gamma g)(a) = \begin{cases} \bigvee_{a=b\alpha c} \{f(b) \wedge g(c)\}, \text{if } \exists\, b,c \in S \ni a = b\alpha c \text{ where } \alpha \in \Gamma. \\ 0, \qquad \text{otherwise.} \end{cases}$$

A fuzzy subset f of a Γ -AG-groupoid S is called a fuzzy Γ -AG-subgroupoid if $f(x\alpha y) \geq f(x) \wedge f(y)$ for all x , $y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset f of a Γ -AG-groupoid S is called fuzzy Γ -left ideal of S if $f(x\alpha y) \geq f(y)$ for all x , $y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset f of a Γ -AG-groupoid S is called fuzzy Γ -right ideal of S if $f(x\alpha y) \geq f(x)$ for all x , $y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset f of a Γ -AG-groupoid S is called fuzzy Γ -two-sided ideal of S if it is both a fuzzy Γ -left ideal and a fuzzy Γ -right ideal of S.

A fuzzy subset f of a Γ -AG-groupoid S is called fuzzy Γ -generalized bi-ideal of S if $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$ and $\alpha, \beta \in \Gamma$.

A fuzzy Γ-AG-subgroupoid f of a Γ -AG-groupoid S is called fuzzy Γ -bi-ideal of S if $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$, for all x , y and $z \in S$ and $\alpha, \beta \in \Gamma$.

A fuzzy subset f of a Γ -AG-groupoid S is called fuzzy Γ -interior ideal of S if $f((x\alpha y)\beta z) \geq f(y)$,for all x , y and $z \in S$ and $\alpha, \beta \in \Gamma$.

A fuzzy subset f of a Γ -AG-groupoid S is called fuzzy Γ -interior ideal of S if $(f \circ_\Gamma S) \cap (S \circ_\Gamma f) \subseteq f$.

Γ-FUZZIFICATION IN Γ-AG-GROUPOIDS

Example 1: Let $S = \{1,2,3,4,5,6,7,8,9\}$. The following multiplication table shows that S is an AG-groupoid and also an AG-band:

.	1	2	3	4	5	6	7	8	9
1	1	4	7	3	6	8	2	9	5
2	9	2	5	7	1	4	8	6	3
3	6	8	3	5	9	2	4	1	7
4	5	9	2	4	7	1	6	3	8
5	3	6	8	2	5	9	1	7	4
6	7	1	4	8	3	6	9	5	2
7	8	3	6	9	2	5	7	4	1
8	2	5	9	1	4	7	3	8	6
9	4	7	1	6	8	3	5	2	9

Clearly S is non-commutative and non-associative because $23 \neq 32$ and $(42)3 \neq 4(23)$.

Let $\Gamma = \{\alpha, \beta\}$ and define a mapping $S \times \Gamma \times S \to S$ by $a\alpha b = a^2 b$ and $a\beta b = ab^2$ for all $a,b \in S$. Then it is easy to see that S is a Γ-AG-groupoid and also a Γ-AG-band. Note that S is non-commutative and non-associative because $9\alpha 1 \neq 1\alpha 9$ and $(6\alpha 7)\beta 8 \neq 6\alpha(7\beta 8)$.

Example 2: Let $\Gamma = \{1,2,3\}$ and define a mapping $\mathrm{Z} \times \Gamma \times \mathrm{Z} \to \mathrm{Z}$ by $a\beta b = b - \beta - a - \beta - z$ for all $a,b,z \in \mathrm{Z}$ and $\beta \in \Gamma$, where " $-$ " is a usual subtraction of integers. Then Z is a Γ-AG-groupoid. Indeed

$$\begin{aligned}(a\beta b)\gamma c &= (b - \beta - a - \beta - z)\gamma c = c - \gamma - (b - \beta - a - \beta - z) - \gamma - z \\ &= c - \gamma - b + \beta + a + \beta + z - \gamma - z = c - b + 2\beta + a - 2\gamma.\end{aligned}$$

And

$$\begin{aligned}(c\beta b)\gamma a &= (b - \beta - c - \beta - z)\gamma a = a - \gamma - (b - \beta - c - \beta - z) - \gamma - z \\ &= a - \gamma - b + \beta + c + \beta + z - \gamma - z = a - 2\gamma - b + 2\beta + c \\ &= c - b + 2\beta + a - 2\gamma.\end{aligned}$$

Which shows that $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a,b,c \in \mathrm{Z}$ and $\beta, \gamma \in \Gamma$. This example is the generalizaion of a Γ -AG-groupoid given by Shah and Rehman (2010).

Example 3: Assume that S is an *AG*-groupoid with left identity and let $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \to S$ by $x1y = xy$ for all $x, y \in S$, then S is a Γ -*AG*-groupoid. Thus we have seen that every *AG*-groupoid is *a* Γ-*AG*-groupoid for $\Gamma = \{1\}$, that is, Γ-*AG*-groupoid is the generalization of *AG*-groupoid. Also S is a Γ-AG^{**}-groupoid because $x1(y1z) = y1(x1z)$ for all $x, y, z \in S$.

Example 4: Let S be an *AG*-groupoid and $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \to S$ by $x1y = xy$ for all $x, y \in S$, then we know that S is *a* Γ -*AG*-groupoid. Let L be a left ideal of an AG-groupoid S, then $S\Gamma L = SL \subseteq L$. Thus L is Γ -left ideal of S. This shows that every Γ-left ideal of Γ-AG-groupoid is a generalization of a left ideal in an AG-groupoid (for suitable Γ). Similarly all the fuzzy Γ -ideals are the generalizations of fuzzy ideals.

By keeping the generalization, the proof of Lemma 1 and Theorem 1 are same as in (Nouman, 2010).

Lemma 1: Let f be a fuzzy subset of a Γ-*AG*-groupoid S, then $S \circ_\Gamma f = f$.

Theorem 1: Let S be a Γ -*AG*-groupoid, then the following properties hold in S .

- $(f \circ_\Gamma g) \circ_\Gamma h = (h \circ_\Gamma g) \circ_\Gamma f$ for all fuzzy subsets f, g and h of S .
- $(f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k) = (f \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma k)$ for all fuzzy subsets f, g, h and k of S .

Theorem 2: Let S be a Γ-AG^{**} -groupoid, then the following properties hold in S .

(i) $f \circ_\Gamma (g \circ_\Gamma h) = g \circ_\Gamma (f \circ_\Gamma h)$ for all fuzzy subsets f, g and h of S .

(ii) $(f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k) = (k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f)$ for all fuzzy subsets f, g, h and k of S .

Proof:

(i) Assume that x is an arbitrary element of a Γ-AG** -groupoid S and let $\alpha, \beta \in \Gamma$. If x is not expressible as a product of two elements in S , then $(f \circ_\Gamma (g \circ_\Gamma h))(x) = 0 = (g \circ_\Gamma (f \circ_\Gamma h))(x)$. Let there exists y and z in S such that $x = y\alpha z$, then by using (3), we have:

$$
\begin{aligned}
((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) &= \bigvee_{x=y\alpha z} \{(f \circ_\Gamma g)(y) \wedge (h \circ_\Gamma k)(z)\} \\
&= \bigvee_{x=y\alpha z} \left\{ \bigvee_{y=p\beta q} \{f(p) \wedge g(q)\} \wedge \bigvee_{z=u\gamma v} \{h(u) \wedge k(v)\} \right\} \\
&= \bigvee_{x=(p\beta q)\alpha(u\gamma v)} \{f(p) \wedge g(q) \wedge h(u) \wedge k(v)\} \\
&= \bigvee_{x=(v\beta u)\alpha(q\gamma p)} \{k(v) \wedge h(u) \wedge_\Gamma g(q) \wedge f(p)\} \\
&= \bigvee_{x=m\alpha n} \left\{ \bigvee_{m=v\beta u} \{k(v) \wedge h(u)\} \wedge \bigvee_{n=q\gamma p} \{g(q) \wedge f(p)\} \right\} \\
&= \bigvee_{x=m\alpha n} \{(k \circ_\Gamma h)(m) \wedge (g \circ_\Gamma f)(n)\} \\
&= ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x).
\end{aligned}
$$

If z is not expressible as a product of two elementsin S, then $(f \circ_\Gamma (g \circ_\Gamma h))(x) = 0 = (g \circ_\Gamma (f \circ_\Gamma h))(x)$. Hence, $(f \circ_\Gamma (g \circ_\Gamma h))(x) = (g \circ_\Gamma (f \circ_\Gamma h))(x)$ for all x in S .

(ii) If any element x of S is not expressible as product of two elements in S at any stage, then, $((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) = 0 = ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x)$. Assume that $\alpha, \beta, \gamma \in \Gamma$ and let there exists y, z in S such that $x = y\alpha z$, then by using (4), we have:

$$
\begin{aligned}
((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) &= \bigvee_{x=y\alpha z} \{(f \circ_\Gamma g)(y) \wedge (h \circ_\Gamma k)(z)\} \\
&= \bigvee_{x=y\alpha z} \left\{ \bigvee_{y=p\beta q} \{f(p) \wedge g(q)\} \wedge \bigvee_{z=u\gamma v} \{h(u) \wedge k(v)\} \right\} \\
&= \bigvee_{x=(p\beta q)\alpha(u\gamma v)} \{f(p) \wedge g(q) \wedge h(u) \wedge k(v)\} \\
&= \bigvee_{x=(v\beta u)\alpha(q\gamma p)} \{k(v) \wedge h(u) \wedge_\Gamma g(q) \wedge f(p)\} \\
&= \bigvee_{x=m\alpha n} \left\{ \bigvee_{m=v\beta u} \{k(v) \wedge h(u)\} \wedge \bigvee_{n=q\gamma p} \{g(q) \wedge f(p)\} \right\} \\
&= \bigvee_{x=m\alpha n} \{(k \circ_\Gamma h)(m) \wedge (g \circ_\Gamma f)(n)\} \\
&= ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x).
\end{aligned}
$$

By keeping the generalization, the proof of the following two lemmas is same as in Mordeson *et al*. (2003).

Lemma 2: Let f be a fuzzy subset of a Γ -*AG*-groupoid S , then the following properties hold:

- f is a fuzzy Γ -AG-subgroupoid of S if and only if $f \circ_\Gamma f \subseteq f$
- f is a fuzzy Γ -left (right) ideal of S if and only if $S \circ_\Gamma f \subseteq f$ $(f \circ_\Gamma S \subseteq f)$
- f is a fuzzy Γ -two-sided ideal of S if and only if $S \circ_\Gamma f \subseteq f$ and $f \circ_\Gamma S \subseteq f$

Lemma 3: Let f be a fuzzy Γ -*AG*-subgroupoid of a Γ -*AG*-groupoid S, then f is a fuzzy Γ -bi-ideal of S if and only if $(f \circ_\Gamma S) \circ_\Gamma f \subseteq f$.

Lemma 4: Let f be any fuzzy Γ -right ideal and g be any fuzzy Γ -left ideal of Γ -*AG*-groupoid S, then $f \cap g$ is a fuzzy Γ -quasi ideal of S .

Proof: It is easy to observe the following

$$((f \cap g) \circ_\Gamma S) \cap (S \circ_\Gamma (f \cap g)) \subseteq (f \circ_\Gamma S) \cap (S \circ_\Gamma g) \subseteq f \cap g.$$

Lemma 5: *Ev*ery fuzzy Γ -quasi ideal of a Γ -*AG*-groupoid S is a fuzzy Γ -*AG*-subgroupoid of S .

Proof: Let f be any fuzzy Γ -quasi ideal of S , then $f \circ_\Gamma f \subseteq f \circ_\Gamma S$ and $f \circ_\Gamma f \subseteq S \circ_\Gamma f$, therefore $f \circ_\Gamma f \subseteq f \circ_\Gamma S \cap S \circ_\Gamma f \subseteq f$.
Hence f is a fuzzy Γ -AG-subgroupoid of S .
A fuzzy subset f of a Γ -AG-groupoid S is called Γ -idempotent, if $f \circ_\Gamma f = f$.

Lemma 6: In a Γ -*AG*-groupoid S, every Γ-idempotent fuzzy Γ -quasi ideal is a fuzzy Γ -bi-ideal *of* S .

Proof: Let f be any fuzzy Γ -quasi ideal of S, then by lemma 5, f is a fuzzy Γ -AG-subgroupoid. Now by using (2), we have

$$(f \circ_\Gamma S) \circ_\Gamma f \subseteq (S \circ_\Gamma S) \circ_\Gamma f \subseteq S \circ_\Gamma f$$

And

$$\begin{aligned}(f \circ_\Gamma S) \circ_\Gamma f &= (f \circ_\Gamma S) \circ_\Gamma (f \circ_\Gamma f) = (f \circ_\Gamma f) \circ_\Gamma (S \circ_\Gamma f) \\ &\subseteq f \circ_\Gamma (S \circ_\Gamma S) \subseteq f \circ_\Gamma S.\end{aligned}$$

This implies that $(f \circ_\Gamma S) \circ_\Gamma f \subseteq (f \circ_\Gamma S) \cap (S \circ_\Gamma f) \subseteq f$. Hence by lemma bi-ideal, f is a fuzzy Γ -bi-ideal of S

Lemma 7: In a Γ -*AG*-groupoid S each one sided fuzzy Γ - (left, right) ideal is a fuzzy Γ -quasi ideal of S .

Proof: It is obvious.

Corollary 1: In a Γ -*AG*-groupoid S, every fuzzy Γ -two- sided ideal of S is a fuzzy Γ -quasi ideal of S .

Lemma 8: In a Γ -*AG*-groupoid S, each one sided fuzzy Γ -(left, right) ideal of S is a fuzzy Γ -generalized bi-ideal of *S*.

Proof: Assume that f be any fuzzy Γ -left ideal of *S*. Let $a,b,c \in S$ and let $\alpha,\beta \in \Gamma$. Now by using (1), we have $f((a\alpha b)\beta c) \geq f((c\alpha b)\beta a) \geq f(a)$ and $f((a\alpha b)\beta c) \geq f(c)$. Thus $f((a\alpha b)\beta c) \geq f(a) \wedge f(c)$. Similarly in the case of fuzzy Γ -right ideal.

Lemma 9: Let *f* or g is a Γ -idempotent fuzzy Γ -quasi ideal of a Γ -AG**-groupoid S, then $f \circ_\Gamma g$ or $g \circ_\Gamma f$ is a fuzzy Γ -bi-ideal of S

Proof: Clearly $f \circ g$ is a fuzzy Γ -AG-subgroupoid. Now using lemma bi-ideal, (1), (4) and (2), we have:

$$\begin{aligned}((f \circ_\Gamma g) \circ_\Gamma S) \circ_\Gamma (f \circ_\Gamma g) &= ((S \circ_\Gamma g) \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma g) \\ &\subseteq ((S \circ_\Gamma S) \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma g) \\ &= (S \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma g) \\ &= (g \circ_\Gamma f) \circ_\Gamma (f \circ_\Gamma S) \\ &= ((f \circ_\Gamma S) \circ_\Gamma f) \circ_\Gamma g \subseteq (f \circ_\Gamma g).\end{aligned}$$

Similarly we can show that $g \circ f$ is a fuzzy Γ -bi-ideal of S .

Lemma 10: The product of two fuzzy Γ -left (right) ideal of a Γ -*AG*** -groupoid S is a fuzzy Γ - left (right) ideal of S .

Proof: Let f and g be any two fuzzy Γ -left ideals of S , then by using (3), we have:

$$S \circ_\Gamma (f \circ_\Gamma g) = f \circ_\Gamma (S \circ_\Gamma g) \subseteq f \circ_\Gamma g.$$

Let f and g be any two fuzzy Γ -right ideals of S , then by using (2), we have:

$$(f \circ_\Gamma g) \circ_\Gamma S = (f \circ_\Gamma g) \circ_\Gamma (S \circ_\Gamma S) = (f \circ_\Gamma S) \circ_\Gamma (g \circ_\Gamma S) \subseteq f \circ_\Gamma g.$$

Γ -FUZZIFICATION IN INTRA-REGULAR Γ -AG** -GROUPOIDS

An element a of a Γ -AG-groupoid S is called an intra-regular if there exists $x, y \in S$ and $\beta,\gamma,\xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$ and S is called intra-regular if every element of S is intra-regular.

Note that in an intra-regular Γ -AG-groupoid S , we can write $S \circ_\Gamma S = S$.

Example 5: Let $S = \{1,2,3,4,5\}$ be an *AG*-groupoid with the following multiplication table:

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

Let $\Gamma = \{1\}$ and define a mapping $S \times \Gamma \times S \to S$ by $x1y = xy$ for all $x, y \in S$, then S is a Γ -AG** -groupoid because $(x1y)1z = (z1y)1x$ and $x1(y1z) = y1(x1z)$ for all $x, y, z \in S$. Also S is an intra-regular because $a = (b1(a1a))1a$, $b = (c1(b1b))1d$, $c = (c1(c1c))1d$, $d = (c1(d1d))1e$, $e = (c1(e1e))1c$.

It is easy to observe that $\{a, b\}$ is a Γ -two-sided ideal of an intra-regular Γ-AG** -groupoid S.

It is easy to observe that in an intra-regular Γ-AG-groupoid S, the following holds $S = S\Gamma S$.

Lemma 11: *A* fuzzy subset f of an intra-regular Γ-*AG*-groupoid S is a fuzzy Γ-right ideal if and only if it is a fuzzy Γ-left ideal.

Proof: Assume that f is a fuzzy Γ -right ideal of S . Since S is an intra-regular Γ-AG-groupoid, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \xi, \gamma \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (1), we have:

$$f(a\alpha b) = f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \geq f(b\gamma y) \geq f(b).$$

Conversely, assume that f is a fuzzy Γ -left ideal of S . Now by using (1), we have:

$$\begin{aligned} f(a\alpha b) &= f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &\geq f(x\beta(a\xi a)) \geq f(a\xi a) \geq f(a). \end{aligned}$$

Theorem 3: Every fuzzy Γ -left ideal of an intra-regular Γ -*AG***-groupoid S is Γ -idempotent.

Proof: Assume that f is a fuzzy Γ-left ideal of S, then clearly $f \circ_\Gamma f \subseteq S \circ_\Gamma f \subseteq f$. Since S is an intra-regular Γ -AG-groupoid, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (3) and (1) we have $a = (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a.$ Thus, we have:

$$\begin{aligned} (f \circ_\Gamma f)(a) &= \bigvee_{a=(y\beta(x\xi a))\gamma a} \{f(y\beta(x\xi a)) \wedge f(a)\} \geq f(y\beta(x\xi a)) \wedge f(a) \\ &\geq f(a) \wedge f(a) = f(a). \end{aligned}$$

Corollary 2: Every fuzzy Γ -two-sided ideal of an intra-regular Γ -*AG*** -*groupoid* S is Γ -idempotent.

Theorem 4: In an intra-regular Γ -*AG*** -groupoid S, $f \cap g = f \circ_\Gamma g$ for every fuzzy Γ -right ideal f and every fuzzy Γ -left ideal g of S.

Proof: Assume that S is intra-regular Γ -AG** -groupoid. Let f and g be any fuzzy Γ-right and fuzzy Γ-left ideal of S, then

$f \circ_\Gamma g \subseteq f \circ_\Gamma S \subseteq f$ and $f \circ_\Gamma g \subseteq S \circ_\Gamma g \subseteq g$ which implies that $f \circ_\Gamma g \subseteq f \cap g$.

Since S is an intra-regular, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \xi, \gamma \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\psi \in \Gamma$, then by using (3), (1), (5) and (4) we have:

$$\begin{aligned} a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a \\ &= ((u\psi v)\beta(x\xi a))\gamma a = ((a\psi x)\beta(v\xi u))\gamma a. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} (f \circ_\Gamma g)(a) &= \bigvee_{a=((a\psi x)\beta(v\xi u))\gamma a} \{f((a\psi x)\beta(v\xi u)) \wedge g(a)\} \\ &\geq f((a\psi x)\beta(v\xi u)) \wedge g(a) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a). \end{aligned}$$

Corollary 3: In an intra-regular Γ-*AG***-groupoid S $f \cap g = f \circ_\Gamma g$ for every fuzzy Γ-right ideal f and g *of* S.

Theorem 5: The set of fuzzy Γ-two-sided ideals of an intra-regular Γ-AG** -groupoid S forms a semilattice structure with identity S.

Proof: Let l_Γ be the set of fuzzy Γ -two-sided ideals of an intra-regular Γ-AG** -groupoid S and f, g and $h \in l_\Gamma$, then clearly l_Γ is closed and by corollary 2 and corollary 3, we have $f = f \circ_\Gamma f$ and $f \circ_\Gamma g = f \cap g$, where f and g are fuzzy Γ-two-sided ideals of S.

Clearly $f \circ_\Gamma g = g \circ_\Gamma f$ and now by using (1), we get $(f \circ_\Gamma g) \circ_\Gamma h = (h \circ_\Gamma g) \circ_\Gamma f = f \circ_\Gamma (g \circ_\Gamma h)$. Also by using (1) and lemma 1, we have:

$$f \circ_\Gamma S = (f \circ_\Gamma f) \circ_\Gamma S = (S \circ_\Gamma f) \circ_\Gamma f = f \circ_\Gamma f = f.$$

A fuzzy Γ-two-sided ideal f of a Γ-AG-groupoid S is said to be a Γ-strongly irreducible if and only if for fuzzy Γ-two-sided ideals g and h of S, $g \cap h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$.

The set of fuzzy Γ-two-sided ideals of a Γ-AG-groupoid S is called a Γ-totally ordered under inclusion if for any fuzzy Γ-two-sided ideals f and g of S either $f \subseteq g$ or $g \subseteq f$.

A fuzzy Γ-two-sided ideal h of a Γ-AG-groupoid S is called a Γ-fuzzy prime ideal of S, if for any fuzzy Γ-two-sided f and g of S, $f \circ_\Gamma g \subseteq h$, implies that $f \subseteq h$ or $g \subseteq h$.

Theorem 6: In an intra-regular Γ-AG^{**}-groupoid S, a fuzzy Γ-two-sided ideal is Γ-strongly irreducible if and only if it is Γ-fuzzy prime.

Proof: It follows from corollary 3.

Theorem 7: Every fuzzy Γ-two-sided ideal of an intra-regular Γ-AG^{**}-groupoid S *is* Γ-fuzzy prime if and only if the set of fuzzy Γ-two-sided ideals of S is Γ-totally ordered under inclusion.

Proof: It follows from corollary 3.

Theorem 8: For a fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid, the following statements are equivalent.

(i) f is a fuzzy Γ-two-sided ideal of S.
(ii) f is a fuzzy Γ-interior ideal of S.

Proof: $(i) \Rightarrow (ii)$: Let f be any fuzzy Γ-two- sided ideal of S, then obviously f is a fuzzy Γ-interior ideal of S.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-interior ideal of S and $a, b \in S$. Since S is an intra-regular Γ-AG-groupoid, so for each $a, b \in S$ there exist $x, y, u, v \in S$ and $\beta, \xi, \gamma, \delta, \psi, \eta \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$ and $b = (u\delta(b\psi b))\eta v$. Now let $\alpha, \in \Gamma$, thus by using (1), (3) and (2), we have:

$$\begin{aligned} f(a\alpha b) &= f((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &= f((b\gamma y)\alpha(a\beta(x\xi a))) = f((b\gamma a)\alpha(y\beta(x\xi a))) \geq f(a). \end{aligned}$$

Also by using (3), (4) and (2) we have:

$$\begin{aligned} f(a\alpha b) &= f(a\alpha((u\delta(b\psi b))\eta v)) = f((u\delta(b\psi b))\alpha(a\eta v)) = f((b\delta(u\psi b))\alpha(a\eta v)) \\ &= f((v\delta a)\alpha((u\psi b)\eta b)) = f((u\psi b)\alpha((v\delta a)\eta b)) \geq f(b). \end{aligned}$$

Hence f is a fuzzy Γ-two- sided ideal of S.

Theorem 9: A fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid is fuzzy Γ-two-sided ideal if and only if it is a fuzzy Γ-quasi ideal.

Proof: Let f be any fuzzy Γ-two-sided ideal of S, then obviously f is a fuzzy Γ-quasi ideal of S.

Conversely, assume that f is a fuzzy Γ-quasi ideal of S, then by using corollary 2 and (4), we have:

$$f \circ_\Gamma S = (f \circ_\Gamma f) \circ_\Gamma (S \circ_\Gamma S) = (S \circ_\Gamma S) \circ_\Gamma (f \circ_\Gamma f) = S \circ_\Gamma f.$$

Therefore, $f \circ_\Gamma S = (f \circ_\Gamma S) \cap (S \circ_\Gamma f) \subseteq f$. Thus f is a fuzzy Γ-right ideal of S and by lemma 11, f is a fuzzy Γ-left ideal of S.

Theorem 10: For a fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.

(i) f is a fuzzy Γ-bi-ideal of S.
(ii) f is a fuzzy Γ-generalized bi-ideal of S.

Proof: $(i) \Rightarrow (ii)$: Let f be any fuzzy Γ-bi-ideal of S, then obviously f is a fuzzy Γ-generalized bi-ideal of S.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-generalized bi-ideal of S and $a, b \in S$. Since S is an intra-regular Γ-AG-groupoid, so for each $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha, \delta, \in \Gamma$, then by using (5), (4), (2) and (3), we have:

$$\begin{aligned} f(a\alpha b) &= f(((x\beta(a\xi a))\gamma y)\alpha b) = f(((x\beta(a\xi a))\gamma(u\delta v))\alpha b) \\ &= f(((v\beta u)\gamma((a\xi a)\delta x))\alpha b) = f(((a\xi a)\gamma((v\beta u)\delta x))\alpha b) \\ &= f(((x\gamma(v\beta u))\delta(a\xi a))\alpha b) = f((a\delta((x\gamma(v\beta u))\xi a))\alpha b) \geq f(a) \wedge f(b). \end{aligned}$$

Therefore, f is a fuzzy bi-ideal of *S*.

Theorem 11: For a fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.

(i) f is a fuzzy Γ-two- sided ideal of S.

(ii) f is a fuzzy Γ-bi-ideal of S.

Proof: $(i) \Rightarrow (ii)$: Let f be any fuzzy Γ-two- sided ideal of S, then obviously f is a fuzzy Γ-bi-ideal of S.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-bi-ideal of S. Since S is an intra-regular Γ-AG-groupoid, so for each $a, b \in S$ there exist $x, y, u, v \in S$ and $\beta, \xi, \gamma, \delta, \psi, \eta \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$ and $b = (u\delta(b\psi b))\eta v$. Now let $\alpha \in \Gamma$, then by using (1), (4), (2) and (3) we have:

$$\begin{aligned}
f(a\alpha b) &= f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\
&= f(((a\xi a)\gamma x)\alpha(y\beta b)) = f(((y\beta b)\gamma x)\alpha(a\xi a)) \\
&= f((a\gamma a)\alpha(x\xi(y\beta b))) = f(((x\xi(y\beta b))\gamma a)\alpha a) \\
&= f(((x\xi(y\beta b))\gamma((x\beta(a\xi a))\gamma y))\alpha a) \\
&= f(((x\beta(a\xi a))\gamma((x\xi(y\beta b))\gamma y))\alpha a) \\
&= f(((y\beta(x\xi(y\beta b)))\gamma((a\xi a)\gamma x))\alpha a) \\
&= f(((a\xi a)\gamma((y\beta(x\xi(y\beta b)))\gamma x))\alpha a) \\
&= f(((x\xi(y\beta(x\xi(y\beta b))))\gamma(a\gamma a))\alpha a) \\
&= f((a\gamma((x\xi(y\beta(x\xi(y\beta b))))\gamma a))\alpha a) \\
&\geq f(a) \wedge f(a) = f(a).
\end{aligned}$$

Now by using (3), (4) and (1) we have:

$$\begin{aligned}
f(a\alpha b) &= f(a\alpha(u\delta(b\psi b))\eta v) = f((u\delta(b\psi b))\alpha(a\eta v)) = f((v\delta a)\alpha((b\psi b)\eta u)) \\
&= f((b\psi b)\alpha((v\delta a)\eta u)) = f(((((v\delta a)\eta u)\psi b)\alpha b) \\
&= f((((v\delta a)\eta u)\psi((u\delta(b\psi b))\eta v))\alpha b) \\
&= f(((u\delta(b\psi b))\psi(((v\delta a)\eta u)\eta v))\alpha b) \\
&= f(((v\delta((v\delta a)\eta u))\psi((b\psi b)\eta u))\alpha b) \\
&= f(((b\psi b)\psi((v\delta((v\delta a)\eta u))\eta u))\alpha b) \\
&= f(((u\psi(v\delta((v\delta a)\eta u)))\psi(b\eta b))\alpha b) \\
&= f((b\psi((u\psi(v\delta((v\delta a)\eta u)))\eta b))\alpha b) \\
&\geq f(b) \wedge f(b) = f(b).
\end{aligned}$$

Theorem 12: Let f be a subset of an intra-regular Γ-AG^{**}-groupoid S, then the following conditions are equivalent.

(i) f is a fuzzy Γ-bi-ideal of S.

(ii) $(f \circ_\Gamma S) \circ_\Gamma f = f$ and $f \circ_\Gamma f = f$.

Proof: $(i) \Rightarrow (ii)$: Let f be a fuzzy Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S. Let $a \in A$, then there exists $x, y \in S$ and $\beta, \gamma, \xi \in \Gamma$ such that $a = (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (3), (1), (5), (4) and (2) and we have:

$$\begin{aligned}
a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a = (y\beta(x\xi((x\beta(a\xi a))\gamma y)))\gamma a \\
&= ((u\alpha v)\beta(x\xi((a\beta(x\xi a))\gamma y)))\gamma a = ((((a\beta(x\xi a))\gamma y)\alpha x)\beta(v\gamma u))\gamma a \\
&= (((x\gamma y)\alpha(a\beta(x\xi a)))\beta(v\gamma u))\gamma a = ((a\alpha((x\gamma y)\beta(x\xi a)))\beta(v\gamma u))\gamma a \\
&= ((a\alpha((x\gamma x)\beta(y\xi a)))\beta(v\gamma u))\gamma a = (((v\alpha u)\beta((x\gamma x)\beta(y\xi a)))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((x\gamma x)\beta(y\xi((x\beta(a\xi a))\gamma(u\alpha v)))))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((x\gamma x)\beta(y\xi((v\beta u)\gamma((a\xi a)\alpha x)))))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((x\gamma x)\beta(y\xi((a\xi a)\gamma((v\beta u)\alpha x)))))\gamma a)\alpha a \\
&= (((v\alpha u)\beta((x\gamma x)\beta((a\xi a)\xi(y\gamma((v\beta u)\alpha x)))))\gamma a)\gamma a \\
&= (((v\alpha u)\beta((a\xi a)\beta((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))))\gamma a)\gamma a \\
&= (((a\xi a)\beta((v\alpha u)\beta((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))))\gamma a)\gamma a \\
&= (((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta(a\beta a))\gamma a)\gamma a \\
&= ((a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a)\gamma a = (p\gamma a)\gamma a
\end{aligned}$$

where,

$$p = a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a).$$

Therefore,

$$\begin{aligned}
((f \circ_\Gamma S) \circ_\Gamma f)(a) &= \bigvee_{a=(pa)a} \{(f \circ_\Gamma S)(pa) \wedge f(a)\} \geq \bigvee_{pa=pa} \{f(p) \circ_\Gamma S(a)\} \wedge f(a) \\
&\geq \{f(a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a)) \wedge S(a)\} \wedge f(a) \\
&\geq f(a) \wedge 1 \wedge f(a) = f(a).
\end{aligned}$$

Now by using (3), (1), (5) and (4) we have:

$$\begin{aligned}
a &= (x\beta(a\xi a))\gamma y = (a\beta(x\xi a))\gamma y = (y\beta(x\xi a))\gamma a \\
&= (y\beta(x\xi((x\beta(a\xi a))\gamma(u\alpha v))))\gamma a = (y\beta(x\xi((v\beta u)\gamma((a\xi a)\alpha x))))\gamma a \\
&= (y\beta(x\xi((a\xi a)\gamma((v\beta u)\alpha x))))\gamma a = (y\beta((a\xi a)\xi(x\gamma((v\beta u)\alpha x))))\gamma a \\
&= ((a\xi a)\beta(y\xi(x\gamma((v\beta u)\alpha x))))\gamma a = (((x\gamma((v\beta u)\alpha x))\xi y)\beta(a\xi a))\gamma a \\
&= (a\beta((x\gamma((v\beta u)\alpha x))\xi y))\gamma a = (a\beta p)\gamma a
\end{aligned}$$

where, $p = ((x\gamma((v\beta u)\alpha x))\xi y)$. Therefore:

$$\begin{aligned}
((f \circ_\Gamma S) \circ f)(a) &= \bigvee_{a=(a\beta p)\gamma a} \{(f \circ_\Gamma S)(a\beta p) \wedge f(a)\} \\
&= \bigvee_{a=(a\beta p)\gamma a} \left(\bigvee_{a\beta p = a\beta p} f(a) \wedge S(p) \right) \wedge f(a) \\
&= \bigvee_{a=(a\beta p)\gamma a} \{f(a) \wedge 1 \wedge f(a)\} = \bigvee_{a=(a\beta p)\gamma a} f(a) \wedge f(a) \\
&\leq \bigvee_{a=(a\beta p)\gamma a} f((a\beta((x\gamma((v\beta u)\alpha x))\xi y))\gamma a) = f(a).
\end{aligned}$$

Thus $(f \circ_\Gamma S) \circ_\Gamma f = f$. As we have shown that:

$$a = ((a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a)\gamma a.$$

Let $a = p\gamma a$ where

$$p = (a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a.$$

Therefore,

$$(f\circ_\Gamma f)(a)=\bigvee_{a=p\gamma a}\{f((a\beta((((x\gamma x)\xi(y\gamma((v\beta u)\alpha x)))\xi(v\alpha u))\beta a))\gamma a)\wedge f(a)\}$$
$$\geq f(a)\wedge f(a)\wedge f(a)=f(a).$$

Now by using lemma 2, we get $f\circ_\Gamma f=f$.

$(ii)\Rightarrow(i)$: Assume that f is a fuzzy subset of an intra-regular Γ-AG**-groupoid S and let $\beta,\gamma\in\Gamma$, then

$$f((x\beta a)\gamma y)=((f\circ_\Gamma S)\circ_\Gamma f)((x\beta a)\gamma y)=\bigvee_{(x\beta a)\gamma y=(x\beta a)\gamma y}\{(f\circ_\Gamma S)(xa)\wedge f(y)\}$$
$$\geq\bigvee_{x\beta a=x\beta a}\{f(x)\wedge S(a)\}\wedge f(y)\geq f(x)\wedge 1\wedge f(y)=f(x)\wedge f(z).$$

Now by using lemma 2, f is fuzzy Γ-bi-ideal of S.

Theorem 13: Let f be a subset of an intra-regular Γ-AG^{**}-groupoid S, then the following conditions are equivalent.

(i) f is a fuzzy Γ-interior ideal of S.
(ii) $(S\circ_\Gamma f)\circ_\Gamma S=f$.

Proof: $(i)\Rightarrow(ii)$: Let f be a fuzzy Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S. Let $a\in A$, then there exists $x,y\in S$ and $\beta,\gamma,\xi\in\Gamma$ such that $a=(x\beta(a\xi a))\gamma y$. Now let $\alpha\in\Gamma$, then by using (3), (5), (4) and (1) we have:

$$a=(x\beta(a\xi a))\gamma y=(a\beta(x\xi a))\gamma y=((u\alpha v)\beta(x\xi a))\gamma y$$
$$=((a\alpha x)\beta(v\xi u))\gamma y=(((v\xi u)\alpha x)\beta a)\gamma y.$$

Therefore,

$$((S\circ_\Gamma f)\circ_\Gamma S)(a)=\bigvee_{a=(((v\xi u)\alpha x)\beta a)\gamma y}\{(S\circ_\Gamma f)(((v\xi u)\alpha x)\beta a)\wedge S(y)\}$$
$$\geq\bigvee_{((v\xi u)\alpha x)\beta a=((v\xi u)\alpha x)\beta a}\{(S((v\xi u)\alpha x)\wedge f(a)\}\wedge 1$$
$$\geq 1\wedge f(a)\wedge 1=f(a).$$

Now again:

$$((S\circ_\Gamma f)\circ_\Gamma S)(a)=\bigvee_{a=(x\beta(a\xi a))\gamma y}\{(S\circ_\Gamma f)(x\beta(a\xi a))\wedge S(y)\}$$
$$=\bigvee_{a=(x\beta(a\xi a))\gamma y}\left(\bigvee_{(x\beta(a\xi a))=(x\beta(a\xi a))}S(x)\wedge f(a\xi a)\right)\wedge S(y)$$
$$=\bigvee_{a=(x\beta(a\xi a))\gamma y}\{1\wedge f(a\xi a)\wedge 1\}=\bigvee_{a=(x\beta(a\xi a))\gamma y}f(a\xi a)$$
$$\leq\bigvee_{a=(x\beta(a\xi a))\gamma y}f((x\beta(a\xi a))\gamma y)=f(a).$$

Hence it follows that $(S\circ_\Gamma f)\circ_\Gamma S=f$.

$(ii)\Rightarrow(i)$: Assume that f is a fuzzy subset of an intra-regular Γ-AG**-groupoid S and let $\beta,\gamma\in\Gamma$, then:

$$f((x\beta a)\gamma y)=((S\circ_\Gamma f)\circ_\Gamma S)((x\beta a)\gamma y)=\bigvee_{(x\beta a)\gamma y=(x\beta a)\gamma y}\{(S\circ_\Gamma f)(x\beta a)\wedge S(y)\}$$
$$\geq\bigvee_{x\beta a=x\beta a}\{(S(x)\wedge f(a)\}\wedge S(y)\geq f(a).$$

Lemma 12: Let f be a fuzzy subset of an intra-regular Γ-AG^{**}-groupoid S, then $S\circ_\Gamma f=f=f\circ_\Gamma S$.

Proof: Let f be a fuzzy Γ-left ideal of an intra-regular Γ-AG**-groupoid S and let $a\in S$, then there exists $x,y\in S$ and $\beta,\gamma,\xi\in\Gamma$ such that $a=(x\beta(a\xi a))\gamma y$

Let $\alpha\in\Gamma$, then by using (5), (4) and (3) we have:

$$a=(x\beta(a\xi a))\gamma y=(x\beta(a\xi a))\gamma(u\alpha v)=(v\beta u)\gamma((a\xi a)\alpha x)=(a\xi a)\gamma((v\beta u)\alpha x)$$
$$=(x\xi(v\beta u))\gamma(a\alpha a)=a\gamma((x\xi(v\beta u))\alpha a).$$

Therefore:

$$(f\circ_\Gamma S)(a)=\bigvee_{a=a\gamma((x\xi(v\beta u))\alpha a)}\{f(a)\wedge S((x\xi(v\beta u))\alpha a)\}$$
$$=\bigvee_{a=a\gamma((x\xi(v\beta u))\alpha a)}\{f(a)\wedge 1\}$$
$$=\bigvee_{a=a((x(ye))a)}f(a)=f(a).$$

The rest of the proof can be followed from lemma 1.

Theorem 14: For a fuzzy subset f of an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.

(i) f is a fuzzy Γ-left ideal of S.
(ii) f is a fuzzy Γ-right ideal of S.
(iii) f is a fuzzy Γ-two-sided ideal of S.
(iv) f is a fuzzy Γ-bi-ideal of S.
(v) f is a fuzzy Γ-generalized bi- ideal of S.
(vi) f is a fuzzy Γ-interior ideal of S.
(vii) f is a fuzzy Γ-quasi ideal of S.
$(viii)$ $S\circ_\Gamma f=f=f\circ_\Gamma S$.

Proof: $(i)\Rightarrow(viii)$: It follows from lemma 12.

$(viii)\Rightarrow(vii)$: It is obvious.

$(vii)\Rightarrow(vi)$: Let f be a fuzzy Γ-quasi ideal of an intra-regular Γ-AG**-groupoid S and let $a\in S$, then there exists $b,c\in S$ and $\beta,\gamma,\xi\in\Gamma$ such that $a=(b\beta(a\xi a))\gamma c$. Let $\delta,\eta\in\Gamma$, then by using (3), (4) and (1), we have:

$$(x\delta a)\eta y=(x\delta(b\beta(a\xi a))\gamma c)\eta y=((b\beta(a\xi a))\delta(x\gamma c))\eta y$$
$$=((c\beta x)\delta((a\xi a)\gamma b))\eta y=((a\xi a)\delta((c\beta x)\gamma b))\eta y$$
$$=(y\delta((c\beta x)\gamma b))\eta(a\xi a)=a\eta((y\delta((c\beta x)\gamma b))\xi a)$$

and from above:

$$(x\delta a)\eta y = (y\delta((c\beta x)\gamma b))\eta(a\xi a) = (a\delta a)\eta(((c\beta x)\gamma b)\xi y)$$
$$= ((((c\beta x)\gamma b)\xi y)\delta a)\eta a.$$

Now by using lemma 12, we have:

$$f((x\delta a)\eta y) = ((f \circ_\Gamma S) \cap (S \circ_\Gamma f))((x\delta a)\eta y)$$
$$= (f \circ_\Gamma S)((x\delta a)\eta y) \wedge (S \circ_\Gamma f)((x\delta a)\eta y).$$

Now:

$$(f \circ_\Gamma S)((x\delta a)\eta y) = \bigvee_{(x\delta a)\eta y = a\eta((y\delta((c\beta x)\gamma b))\xi a)} \{f(a) \wedge S_\delta((y\delta((c\beta x)\gamma b))\xi a)\} \geq f(a)$$

And

$$(S \circ_\Gamma f)((x\delta a)\eta y) = \bigvee_{(x\delta a)\eta y = ((((cx)b)y)a)a} \{S((((c\beta x)\gamma b)\xi y)\delta a) \wedge f(a)\} \geq f(a).$$

This implies that $f((x\delta a)\eta y) \geq f(a)$ and therefore f is a fuzzy Γ-interior ideal of S.

$(vi) \Rightarrow (v)$: It follows from theorems 8, 11 and 10.

$(v) \Rightarrow (iv)$: It follows from theorem 10.

$(iv) \Rightarrow (iii)$: It follows from theorem 11.

$(iii) \Rightarrow (ii)$: It is obvious and $(ii) \Rightarrow (i)$ can be followed from lemma 11.

CONCLUSION

In this study we introduced Γ-ideals in Γ- AG-groupouds. We showed that Γ-AG-groupoids satisfy all the laws that is, gamma left invertive law, gamma medial and gamma paramedical laws. Moreover we introduced gamma intra-regular AG-groupoids and characterized gamma ideals. In our future we will focus on some new characterizations of gamma AG-groupoids.

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