

## The Efficiency of Convective-radiative Fin with Temperature-dependent Thermal Conductivity by the Differential Transformation Method

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**Abstract:** In this study, a simple and accurate solution for the temperature distribution of convective-radiative straight rectangular fins with temperature-dependent thermal conductivity is presented using an analytical method called the Differential Transformation Method (DTM). The governing differential equation for the temperature distribution in present problem contains two nonlinear terms, one due to temperature-dependent thermal conductivity and the other due to surface radiation. Here, the concept of differential transformation method is briefly introduced and then it is used to derive solutions of highly nonlinear equation. The obtained results from DTM are compared with those from the numerical solution to verify the accuracy of the proposed method. The investigations reveal that the differential transformation method can achieve suitable results in predicting the solution of such strong nonlinear problem. After this verification, the effects of some physical applicable parameters in this problem such as convection-conduction parameter, thermal conductivity parameter and the radiation-conduction parameter on efficiency of the fin are presented and discussed.

**Keywords:** Convection-conduction parameter, differential transformation method, fin efficiency, nonlinear heat transfer equation, radiation-conduction parameter, temperature-dependent thermal conductivity

### INTRODUCTION

The fin assembly is commonly used to increase the rate of heat transfer from a hot primary surface. An array of rectangular fins has widely used in industrial and engineering applications to enhance heat dissipation and better cooling of devices. The arrangement is most effective in a natural convection environment where the convection heat transfer coefficient is low. In this circumstance, the radiative component of heat loss from the fins is comparable to the natural convection heat loss. The fin heat transfer model must therefore include simultaneous surface convection and radiation. Furthermore, if the temperature change from the base to the tip of the fin is large, then for an accurate prediction of the performance of the fin, the model must also incorporate the variation of thermal conductivity with temperature. The differential equation for the temperature distribution in a convective-radiative fin with temperature-dependent thermal conductivity contains two nonlinear terms, one due to temperature-dependent thermal conductivity and the other due to surface radiation. Even with one nonlinear term, the equation does not admit an exact analytical solution. Consequently, the fin equation has been solved either

numerically or using a variety of approximate analytical methods. Aziz and Enamul-Huq (1973) considered a pure convection fin with temperature-dependent thermal conductivity and developed a three term regular perturbation expansion in terms of the thermal conductivity parameter. The analysis was extended by Aziz (1977) to include a uniform internal heat generation in the fin. Arslanturk (2005) used the Adomian Decomposition Method (ADM) to obtain the temperature distribution in a pure convection fin with thermal conductivity varying linearly with temperature. The same problem was solved by Ganji (2006) and Ganji and Rajabi (2006) using the homotopy perturbation method originally proposed by He (1999). Coskun and Atay (2008) and Miansari *et al.* (2008) used the Variation Iteration Method (VIM) to analyze some nonlinear fin problems. Khani and Aziz (2010) considered a trapezoidal fin with both the thermal conductivity and the convection heat transfer coefficient varying as functions of temperature and reported an analytic solution generated using the Homotopy Analysis Method (HAM). Chowdhury and Hashim (2008) and Chowdhury *et al.* (2009) investigated a rectangular fin with power law surface heat flux and made a comparative assessment of HAM, HPM and ADM. Other methods proposed for solving

nonlinear fin problems include the Optimal Homotopy Asymptotic Method (OHAM) used by Marinca and Herisanu (2008) and the Generalized Approximation Method (GAM) advocated by Khan (2009). The problem of a convective-radiative fin with temperature-dependent thermal conductivity, which is the focus of present work, was solved by Aziz and Benzies (1976) using a double series in two perturbation parameters, a thermal conductivity parameter  $\epsilon_c$  and a radiation-conduction parameter  $\epsilon_r$ . They obtained the first six terms of the perturbation series. The same problem was solved by Chiu and Chen (2003) using ADM, as well as by Malekzadeh *et al.* (2006) utilizing the Differential Quadrature Element Method (DQEM). Recognizing the complexities and limitations of the approximate analytical procedures, Bouaziz *et al.* (2001) and Aziz and Beers-Green (2009) have opted in favor of a numerical approach to study nonlinear fin problems. Recently Bouaziz and Aziz (2010) have introduced a new concept called the Double Optimal Linearization (DOLM) to derive simple and accurate expressions for predicting the thermal performance of a convective-radiative fin with temperature constant and dependent thermal conductivity.

An accurate review of the most approximate methods used for solving present problem, like HPM, ADM and VIM cited in the foregoing paragraphs reveals that these methods give accurate predictions only when the nonlinearities are weak. The methods such as HPM, HAM, ADM and VIM, if routinely implemented, can sometimes lead to erroneous results. Furthermore, the methods like DOLM and HAM often involve a complex mathematical analysis. Finally, the approximate analytical methods do not lend themselves to easy use by designers and practicing engineers who prefer simpler yet accurate expressions to evaluate the temperature distribution in the fin and the fin efficiency.

Our motivation in the present study is to investigate the temperature field of a convective-radiative fin with temperature dependent thermal conductivity using the DTM. We also intend to compare the results of simulation using the DTM with the results of simulation using the numerical method (shooting method, coupled with fourth-order Runge-Kutta).

### PROBLEM STATEMENT AND MATHEMATICAL FORMULATION

According to Fig. 1, consider a straight fin of rectangular profile area  $A$ , length  $L$ , constant thermal conductivity  $k_0$  and surface emissivity  $\epsilon$ . The fin is attached to a primary surface at fixed temperature  $T_b$  and loses heat by simultaneous convection and radiation to the surrounding medium. The sink temperatures for convection and radiation are  $T_\infty$

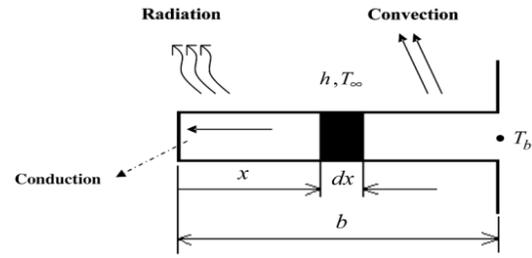


Fig. 1: Geometry of a straight fin

and  $T_s$ , respectively. The convective heat transfer coefficient  $h$  is assumed to be a constant. The heat loss from the tip of the fin compared with the top and bottom surfaces of the fin is taken to be negligible. Since the transverse Biot number should be small for the fin to be truly effective, the temperature variation in the transverse direction can be neglected. Thus heat conduction occurs only in the longitudinal direction.

**Governing equation:** For the problem just described, the appropriate differential equation and the boundary conditions may be written as (Aziz and Benzies, 1976):

$$\frac{d^2T}{dx^2} - \frac{hP}{k_0A}(T - T_\infty) - \frac{\epsilon\sigma P}{k_0A}(T^4 - T_s^4) = 0, \quad (1)$$

$$\begin{aligned} x = 0, & \quad \frac{dT}{dx} = 0 \\ x = L, & \quad T = T_0, \end{aligned} \quad (2)$$

where  $x$  is measured from the tip of the fin. For simplicity, the case of  $T_\infty = T_s = 0$  is treated. With the introduction of following dimensionless quantities:

$$\theta = \frac{T}{T_b}, X = \frac{x}{L}, N^2 = \frac{hPL^2}{k_0A}, \epsilon_r = \epsilon\sigma PL^2 T_b^3 / k_0A \quad (3)$$

Applying Eq. (3) to (1) and (2) gives:

$$\frac{d^2\theta}{dX^2} - N^2\theta - \epsilon_r\theta^4 = 0 \quad (4)$$

$$\begin{aligned} X = 0, & \quad \frac{d\theta}{dX} = 0 \\ X = 1, & \quad \theta = 1. \end{aligned} \quad (5)$$

Assuming that the thermal conductivity of the fin varies linearly with temperature, we have:

$$k = k_0(1 + \beta T) \quad (6)$$

where, the constant  $\beta$  is a measure of the thermal conductivity variation with temperature. Equation (6) may be written in dimensionless form as:

$$k/k_0 = 1 + \epsilon_c\theta \quad (7)$$

where,  $\varepsilon_c = \beta T_b$ . The governing differential equation in this case becomes (Aziz and Benzie, 1976):

$$\frac{d}{dX} [(1 + \varepsilon_c \theta) \frac{d\theta}{dX}] - N^2 \theta - \varepsilon_r \theta^4 = 0 \tag{8}$$

With the boundary conditions given by Eq. (5) for convective-radiative fin with temperature-dependent thermal conductivity.

**Fin efficiency:** The heat transfer rate from the fin is found by using Newton's law of cooling:

$$Q = \int_a^b P(T - T_\infty) dx \tag{9}$$

The ratio of actual heat transfer from the fin surface to the other side while whole fin surface is at the same temperature is called the fin efficiency:

$$\eta = \frac{Q}{Q_{ideal}} = \frac{\int_0^b P(T - T_\infty) dx}{Pb(T_b - T_\infty)} = \int_{x=0}^1 \theta(x) dx \tag{10}$$

### ANALYTICAL APPROXIMATIONS BY MEANS OF DTM

The differential transformation method is an analytical method for a vast variety of differential equations including ODEs and PDEs (Zhou, 1986). This method uses polynomials form to approximate the exact solutions. We now take a brief review to the DTM. The differential transform of the  $k^{th}$  derivative of function  $f(t)$  is defined as follows (Zhou, 1986):

$$F(k) = \frac{1}{k!} \left[ \frac{d^k f(t)}{dt^k} \right]_{t=t_0} \tag{11}$$

where,

$f(t)$  : The base function

$F(k)$  : The transformed function

The differential inverse transform of  $F(k)$  is defined as:

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^k \tag{12}$$

Equation (11) and (12) give the following:

$$f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \left[ \frac{d^k f(t)}{dt^k} \right]_{t=t_0} \tag{13}$$

This shows that differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which is described by the transformed equations of the base function. We approximate  $f(t)$  by a finite series and Eq. (12) can be written as:

$$f(t) \approx \sum_{k=0}^N F(k)(t-t_0)^k \tag{14}$$

The main steps of the DTM are the following. First, we apply the differential transform (11) to the given differential equation or a system of differential equations to obtain a recursive relation. Second, solving the recursive relation and then using the differential inverse transform (12) we obtain the solution of the problem.

Using Eq. (11) and (12) the following theorems can be deduced as follow:

- **Theorem 1:** If  $u(t) = x(t) \pm y(t)$  then  $U(k) = X(k) \pm Y(k)$
- **Theorem 2:** If  $u(t) = \alpha x(t)$ , then  $U(k) = \alpha X(k)$ , where  $\alpha$  is a constant
- **Theorem 3:** If  $u(t) = (d^m x(t))/dt^m$ , then  $U(k) = ((m+k)!/k!) X(k+m)$
- **Theorem 4:** If  $u(t) = x(t) y(t)$ , then  $U(k) = \sum_r^k X(r) Y(k-r)$
- **Theorem 5:** If  $u(t) = t^n$ , then  $U(k) = \delta(k-n)$ ;  $\delta(k-n) = \begin{cases} 1, & k = n \\ 0 & k \neq n \end{cases}$

Taking differential transform from Eq. (8) for convective-radiative fin with temperature dependent thermal conductivity:

$$\begin{aligned} & (k+1)(k+2)F(k+3) \\ & + \varepsilon_c \left\{ - \sum_{r=0}^k ((r+1)(k-r+1)F(r)F(k-r+1)) \right. \\ & \left. - \varepsilon_r \sum_{r=0}^k \left( \sum_{l=0}^r F(r-l) \times \sum_{l=0}^r F(k-l)F(k-r+l) \right) \right\} = 0 \end{aligned}$$

The boundary condition, Eq. (5), is transformed into:

$$F(1) = 0, \sum_{k=0}^{\infty} F(k) = 0$$

For example for  $\varepsilon_r = 0$  and  $\varepsilon_c = 0$ ,  $f(t)$  can be expressed as:

$$f(t) = 0.886818883970074 + 0.110852360496259 t^2 + 0.002309424177005 t^4 + 0.000019245201475 t^6 + \dots$$

for  $N = 0.5$ ,  $\varepsilon_r = 0$  and  $\varepsilon_c = 0.4$ :

$$f(t) = 0.9159604482103925 + 0.08379419036t^2 + 0.000249870451t^4 - 0.000004605456649 t^6 + \dots$$

and for  $N = 0.5$ ,  $\varepsilon_r = 0.4$  and  $\varepsilon_c = 0.4$ :

$$f(t) = 0.83861390032131 + 0.1526001296t^2 + 0.009058011765t^4 + 0.0000001028649t^6 + \dots$$

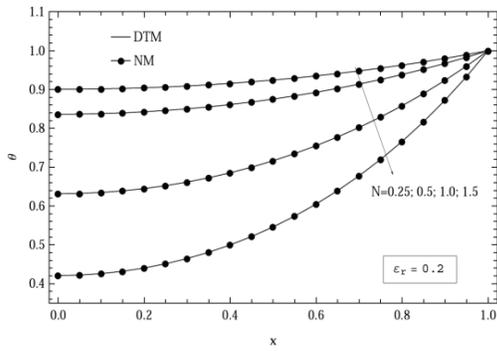


Fig. 2: The influence of convection-conduction parameter  $N$  on the temperature distribution in a convective-radiative fin of constant thermal conductivity, when  $\epsilon_r = 0.2$

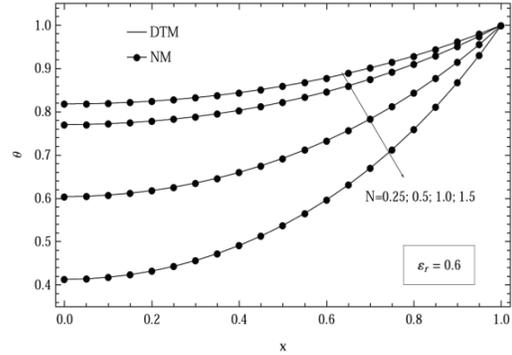


Fig. 4: The influence of convection-conduction parameter  $N$  on the temperature distribution in a convective-radiative fin of constant thermal conductivity, when  $\epsilon_r = 0.6$

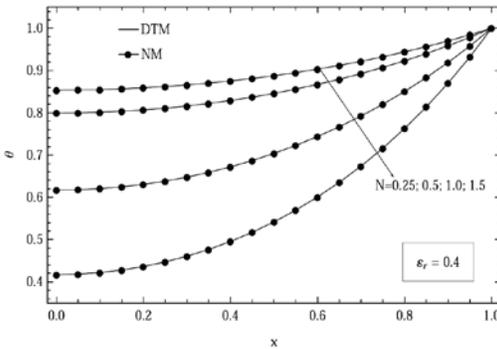


Fig. 3: The influence of convection-conduction parameter  $N$  on the temperature distribution in a convective-radiative fin of constant thermal conductivity, when  $\epsilon_r = 0.4$

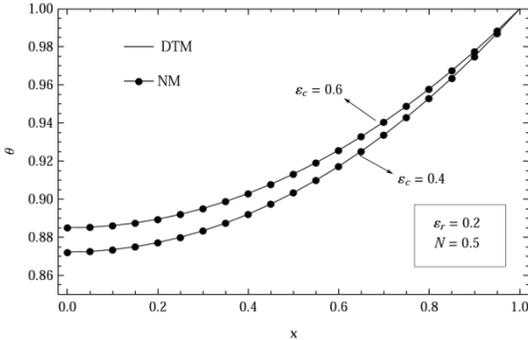


Fig. 5: The influence of temperature-dependent thermal conductivity  $\epsilon_c$  on the temperature distribution, when  $N = 0.5$  and  $\epsilon_r = 0.2$

**Numerical simulation:** The shooting method works by considering the boundary conditions as a multivariate function of initial conditions at some point, reducing the boundary value problem to finding the initial conditions that give a root. The advantage of the shooting method is that it takes advantage of the speed and adaptively of methods for initial value problems. The basic concept of the shooting method can be obtained from Roberts and Shipman (1972). In this study the shooting method, coupled with the fourth-order Runge-Kutta scheme is used for solving the temperature field of convective-radiative fin.

### RESULTS AND DISCUSSION

In this section the effects of physical parameters including thermal conductivity parameter ( $\epsilon_c$ ), radiation-conduction parameter ( $\epsilon_r$ ) and convection-conduction parameter ( $N$ ) on temperature distribution and fin efficiency will be presented. Figure 2 shows the temperature distributions in the fin obtained for the radiation-conduction parameter  $\epsilon_r = 0.2$ . The curves for

different values of convection-conduction parameter  $N$  show that DTM and the numerical results are completely coincident. The same accuracy can be observed for  $\epsilon_r = 0.4$  and  $\epsilon_r = 0.6$  in Fig. 3 and 4, respectively. It can be seen from Fig. 2 to 4 that for a certain value of convection-conduction parameter ( $N$ ), global behavior of curves are repeated, in other words general behavior of curves for different values of  $\epsilon_r$  are the same. According to Fig. 2 to 4 by increasing the convective heat transfer coefficient  $h$ , proportional to the conductive heat transfer coefficient  $k_0$ , temperature field become more uniform and its value become closer to the base temperature  $T_b$  of the fin. Within the same procedure as Fig. 2 to 4 repeated for Fig. 5 to 7 where in which the effects of physical parameters on temperature field for the convective-radiative fin with temperature dependent thermal conductivity are shown. As seen in these figures, although the nonlinearity of the problem increases, still accuracy of the results obtained by DTM is noticeable. Figure 8 shows the fin efficiency as a function of the convection-conduction parameter ( $N$ ), for different values of the thermal conductivity parameter ( $\epsilon_c$ ) in certain value of radiation conduction parameter ( $\epsilon_r = 0.4$ ). As expected

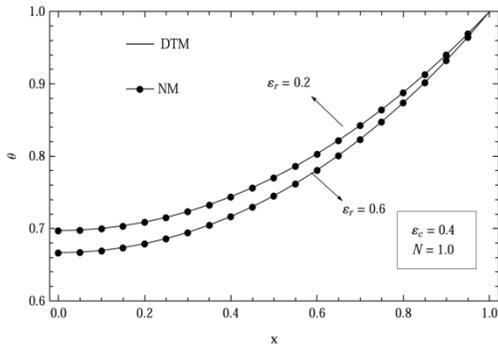


Fig. 6: The influence of radiation-conduction parameter  $\epsilon_r$  on the temperature distribution in a convective-radiative fin of temperature-dependent thermal conductivity, when  $N = 1$  and  $\epsilon_c = 0.4$

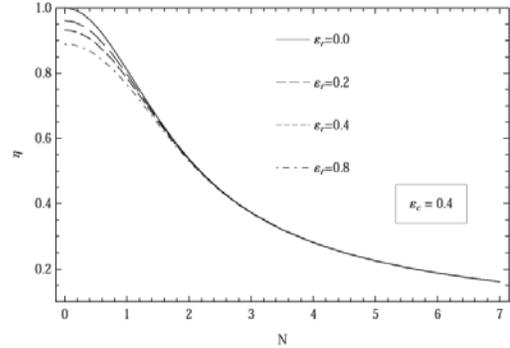


Fig. 9: Variation of the fin efficiency with the convection-conduction parameter ( $N$ ) for different values of the radiation-conduction parameter  $\epsilon_r$ , when  $\epsilon_c = 0.4$

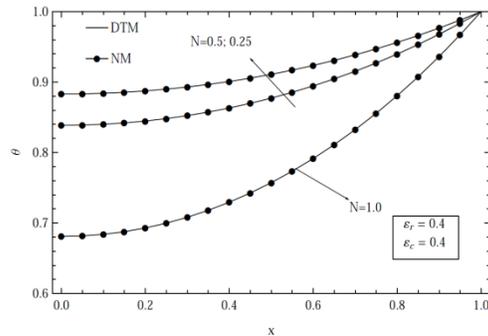


Fig. 7: The influence of convection-conduction parameter  $N$  on the temperature distribution in a convective-radiative fin of temperature-dependent thermal conductivity, when  $\epsilon_r = 0.4$  and  $\epsilon_c = 0.4$

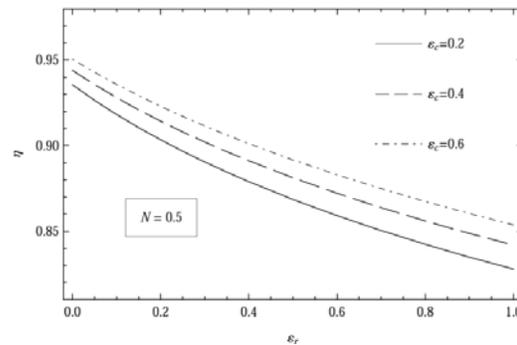


Fig. 10: Variation of the fin efficiency with the radiation-conduction parameter  $\epsilon_r$  for different values of the thermal conductivity parameter  $\epsilon_c$ , when  $N = 0.5$

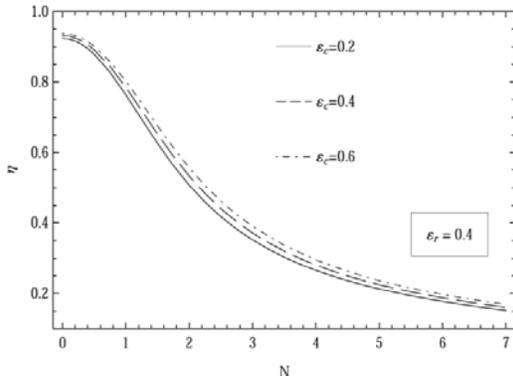


Fig. 8: Variation of the fin efficiency with the convection-conduction parameter ( $N$ ) for different values of the thermal conductivity parameter  $\epsilon_c$ , when  $\epsilon_r = 0.4$

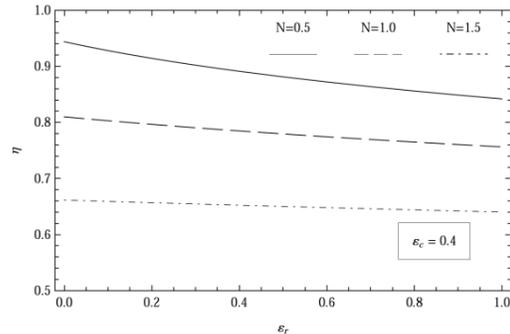


Fig. 11: Variation of the fin efficiency with the radiation-conduction parameter  $\epsilon_r$  for different values of the convection-conduction parameter  $N$ , when  $\epsilon_c = 0.4$

physically, increasing the value of thermal conductivity makes the convective-radiative fin more efficient. Furthermore by increasing the ratio of the convective heat transfer coefficient  $h$ , to the conductive heat transfer coefficient  $k_0$ , temperature distribution of the fin become farther apart from the fin base temperature

$T_b$ . Figure 9 presents the effects of the convection-conduction parameter ( $N$ ), on the fin efficiency for different values of  $\epsilon_r$  for a certain value of  $\epsilon_c$ . One of the more interesting points that can be seen from this figure is that for values greater than 1.5 of the convection-conduction parameter ( $N$ ), the fin efficiency is independent of the radiation-conduction fin parameter ( $\epsilon_r$ ). It means that for values greater than a certain value of  $N$ , the effects of radiation heat transfer mechanism for the fin can be negligible. Figure 10 shows the effects of radiation-conduction fin parameter

( $\varepsilon_r$ ) and thermal conductivity fin parameter ( $\varepsilon_c$ ) on the fin efficiency. Figure 11 presents some interesting physical properties of heat transfer for a fin. In this figure variation of the fin efficiency with ( $\varepsilon_r$ ) and  $N$  plotted. It can be seen that for  $N = 1.5$ , fin efficiency is approximately independent of ( $\varepsilon_r$ ). But by decreasing  $N$ , the rate of decreasing of the fin efficiency becomes more apparent.

## CONCLUSION

In this study, the Differential Transformation Method (DTM) has been applied to solve nonlinear differential equation arising in radiative-convective straight fins with temperature-dependent thermal conductivity problem. Comparison results obtained by DTM with those of numerical solution by shooting method, coupled with the fourth-order Runge-Kutta scheme, showed efficiency of this method to solve strong nonlinear equations. Finally, the effects of physical parameter for a radiative-convective straight fin were discussed. The results showed that for certain value of thermal conductivity parameter, decreasing the convection-conduction parameter, lead to increase the effects of radiation heat transfer mechanism on fin efficiency. Furthermore for fixed value of convection-conduction parameter, uniform variation of the fin efficiently could be observed with the radiation-conduction parameter.

## REFERENCES

- Arslanturk, C., 2005. A decomposition method for fin efficiency of convective straight fin with temperature dependent thermal conductivity. *Int. Commun. Heat Mass Transfer*, 32: 831-841.
- Aziz, A., 1977. Perturbation solution for convective fin with internal heat generation and temperature dependent thermal conductivity. *Int. J. Heat Mass Transfer*, 20: 1253-5.
- Aziz, A. and S.M. Enamul-Huq, 1973. Perturbation solution for convecting fin with temperature dependent thermal conductivity. *J. Heat Transfer*, 97: 300-301.
- Aziz, A. and J.Y. Benzie, 1976. Application of perturbation techniques to heat transfer problems with variable thermal properties. *Int. J. Heat Mass Transfer*, 19: 271-6.
- Aziz, A. and A. Beers-Green, 2009. Performance and optimum design of convective-radiative rectangular fin with convective base heating, wall conduction resistance and contact resistance between the wall and the fin base. *Energ. Convers. Manage.*, 50: 2622-2631.
- Bouaziz, M.N. and A. Aziz, 2010. Simple and accurate solution for convective-radiative fin with temperature dependent thermal conductivity using double optimal linearization. *Energ. Convers. Manage.*, 51: 2776-2782.
- Bouaziz, M.N., S. Rechak, S. Hanini, Y. Bal and K. Bal, 2001. Numerical study of nonlinear heat transfer in longitudinal fins. *Int. J. Therm. Sci.*, 40: 843-857.
- Chiu, CH. and C.K. Chen, 2003. Application of adomian decomposition procedure to analysis of convective-radiative fins. *J. Heat Transfer*, 125: 312-6.
- Chowdhury, M.S.H. and I. Hashim, 2008. Analytical solutions to heat transfer equations by homotopy-perturbation method revisited. *Phys. Lett. A*, 372: 1240-1243.
- Chowdhury, M.S.H., I. Hashim and O. Abdulaziz, 2009. Comparison of homotopy analysis method and homotopy-perturbation method for purely nonlinear fin-type problems. *Commun. Nonlinear Sci. Numer. Simult.*, 14: 371-378.
- Coskun, S.B. and M.T. Atay, 2008. Fin efficiency analysis of convective straight fin with temperature dependent thermal conductivity using variational iteration method. *Appl. Therm. Eng.*, 28: 2345-2352.
- Ganji, D.D., 2006. The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. *Phys. Lett. A*, 355: 337-341.
- Ganji, D.D. and A. Rajabi, 2006. Assessment of homotopy-perturbation and perturbation methods in heat radiation equations. *Int. Commun. Heat Mass Transfer*, 33: 391-400.
- He, J.H., 1999. Homotopy perturbation method. *Comp. Methods Appl. Mech. Eng.*, 178: 257-262.
- Khan, A.R., 2009. The generalized approximation method and nonlinear heat transfer equations. *Elect. J. Qual. Theory Diff. Eq.*, 2: 1-15.
- Khani, F. and A. Aziz, 2010. Thermal analysis of a longitudinal trapezoidal fin with temperature dependent thermal conductivity and heat transfer coefficient. *Commun. Nonlinear Sci. Numer. Sci. Simul.*, 15: 590-601.
- Malekzadeh, P., P. Rahideh and G. Karami, 2006. Optimization of convective-radiative fins using differential quadrature element method. *Energ. Convers. Manage.*, 47: 1505-14.
- Marinca, V. and N. Herisanu, 2008. Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. *Int. Commun. Heat Mass Transfer*, 35: 710-705.
- Miansari, M.O., D.D. Ganji and M.E. Miansar, 2008. Application of He's variational iteration method to nonlinear heat transfer equations. *Phys. Lett. A*, 372: 770-785.
- Roberts, S.M. and J.S. Shipman, 1972. *Two Point Boundary Value Problems: Shooting Methods*. American Elsevier Pub. Co., New York.
- Zhou, J.K., 1986. *Differential Transformation and Its Applications for Electrical Circuits*. Huazhong University Press, Wuhan, China.