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Ternary 4-Point Interpolating Scheme for Curve Sketching

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Abstract: Subdivision schemes as an alternative approach in computer graphics are very much appreciated, a new ternary 4-point interpolating subdivision scheme has been proposed that generates the smooth limiting curves of C^2 continuity using a global control tension parameter. A comparison of the proposed subdivision scheme along with examples has been demonstrated with the existing schemes. Comparison established that the proposed subdivision scheme behaves better than the existing schemes for a wider range of tension parameter. The limiting curve of the scheme has been observed to be close to the control polygon.

Keywords: Convergence and smoothness, interpolating subdivision scheme, Laurent polynomial, mask, ternary

INTRODUCTION

Subdivision schemes have significance due to their applications in computer aided geometric design, computer graphics, geometric modeling, medical surgery simulations, engineering and reverse engineering. Curve subdivision schemes can be classified into two important branches, interpolating and approximating. Interpolating subdivision schemes preserve the data obtained at the former stage and refines new data by inserting values corresponding to intermediate points.

Binary 4-point interpolating subdivision schemes have been studied and investigated their properties by Deslauriers and Dubuc (1989), Dyn *et al.* (1987), Tang *et al.* (2005), Beccari *et al.* (2009) and Jena *et al.* (2003). Weissman (1989) introduced a binary 6-point interpolating subdivision scheme that generates the limiting curve of C^2 continuity. Siddiqi and Ahmad (2006) investigate the smoothness of the binary 6-point interpolating subdivision scheme, introduced by Weissman (1989), using Laurent polynomial method. Siddiqi and Rehan (2012a) introduced a new corner cutting piecewise approximating method that generates the limiting curve of C^1 continuity. Hassan and Dodgson (2003) presented a ternary 3-point interpolating subdivision scheme that generates a family of C^1 limiting curve. Zheng *et al.* (2005) also developed a ternary 3-point interpolating subdivision scheme that generates C^1 limiting curves. Zheng *et al.* (2007a) analyzed the fractal property of ternary 3-point interpolating subdivision scheme with two parameters introduced by Hassan and Dodgson (2003). Hassan *et al.* (2002) proposed a ternary 4-point interpolating subdivision scheme that generates C^2 limiting curve for a certain range of tension parameter. Siddiqi and Rehan (2012b) proposed a symmetric ternary 4-point interpolating subdivision scheme that generates a family of C^1 limiting curves for certain range of tension parameter. Beccari *et al.* (2007) introduced a non-stationary ternary 4-point interpolating subdivision scheme that generates C^2 limiting curves using tension parameter. Zheng *et al.* (2007b) investigated the differentiable properties of the ternary 4-point interpolating subdivision scheme, introduced by Hassan *et al.* (2002).

A subdivision algorithm recursively refines the initial polygon to produce a sequence of finer polygons that converge to a smooth limiting curve. Each subdivision scheme is associated with a mask a = $\{a_i\}$, $i \in Z$. The ternary subdivision scheme is the process which recursively define a sequence of control points $f^k = f_i^k, i \in Z$, by the rule of the form with mask a = $\{a_i\}, \in Z$:

$$f_i^{k+1} = \sum a_{i-3j} f_j^{\,k}, i \in Z$$

Which is formally denoted by $f^{k+1} = Sf^k = S^k f^0$ A subdivision scheme is said to be uniformly convergent if for every initial data $f^0 = \{f_i\}, i \in Z$, there is a continuous function f such that for any closed interval [a, b]:

$$\lim_{k\to\infty} \sup_{i \in Z \cap 3^k[a,b]} \left| f_i^k - f(3^{-k} i) \right| = 0$$

Obviously $f = S^\infty f^0$ is considered to be a limit function of subdivision scheme S.

A new ternary 4-point interpolating subdivision scheme is introduced using a tension parameter. The

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ternary 4-point interpolating subdivision scheme is defined as

$$\left.\begin{array}{l} f_{3i}^{k+1} = f_i^k, \\ f_{3i+1}^{k+1} = \left(-\frac{7}{108}-\frac{\mu}{12}\right)f_{i-1}^k + \left(\frac{3}{4}+\frac{\mu}{4}\right)f_i^k + \left(\frac{13}{36}-\frac{\mu}{4}\right)f_{i+1}^k + \left(\frac{-5}{108}+\frac{\mu}{12}\right)f_{i+2}^k, \\ f_{3i+2}^{k+1} = \left(\frac{-5}{108}+\frac{\mu}{12}\right)f_{i-1}^k + \left(\frac{13}{36}-\frac{\mu}{4}\right)f_i^k + \left(\frac{3}{4}+\frac{\mu}{4}\right)f_{i+1}^k + \left(-\frac{7}{108}-\frac{\mu}{12}\right)f_{i+2}^k, \end{array}\right\} \quad (1)$$

where, f_i^0 is a set of initial control points. The proposed subdivision scheme generates a family of C^2 limiting curves for the range of tension parameter $\mu \epsilon \left]\frac{-1}{3},\frac{1}{9}\right[$. For geometric point of view, the purpose of ternary 4-point interpolating subdivision scheme is to gives more flexibility with in the discrete set of data points as compare to existing subdivision schemes.

ANALYSIS OF TERNARY SUBDIVISION SCHEME

For the convergent subdivision scheme S, the corresponding mask $\{a_i\}$, $i \epsilon Z$ necessarily satisfies:

$$\sum_{j\in Z} a_{3j} = \sum_{j\in Z} a_{3j+1} = \sum_{j\in Z} a_{3j+2} = 1. \quad (2)$$

Introducing a symbol called the Laurent polynomial $a(z) = \sum_{i\epsilon Z} a_i Z^i$ of a mask $\{a_i\}$, $i \epsilon Z$ with finite support. The corresponding symbols play an efficient role to analyze the convergence and smoothness of subdivision scheme.

With the symbol, Hassan *et al.* (2002) provided a sufficient and necessary condition for a uniform convergent subdivision scheme. A subdivision scheme S is uniform convergent if and only if there is an integer L≥1, such that $\left\|\left(\frac{1}{3}S_1\right)^L\right\|_\infty < 1$.

The subdivision scheme S_1 with symbol $a_1(z)$ is related to subdivision scheme S with symbol a(Z), where $a_1(Z) = \frac{3z^2}{1+z+z^2}a(z)$. The subdivision scheme S with symbol a(z) satisfies Eq. (2) then there exists a subdivision scheme S_1 with the property:

$$df^k = S_1 df^{k-1}, \quad k = 1, 2, \ldots,$$

where, $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 3^k(f_{i+1}^k - f_i^k) : i \in Z\}$. The norm $\|S\|_\infty$ of a subdivision scheme S with a mask $\{a_i\}$, $i \epsilon Z$ is defined by

$$\|S\|_\infty = \max\{\sum_{i\in Z}|a_{3i}|, \sum_{i\in Z}|a_{3i+1}|, \sum_{i\in Z}|a_{3i+2}|\}$$

Theorem 1: Ternary 4-point interpolating subdivision scheme defined in equation (1.1) converges and has smoothness C^2 for the range of tension parameter $\mu \epsilon \left]\frac{-1}{3},\frac{1}{9}\right[$.

Proof: Ternary 4-point interpolating subdivision scheme is:

$$\begin{array}{l} f_{3i}^{k+1} = f_i^k, \\ f_{3i+1}^{k+1} = \left(-\frac{7}{108}-\frac{\mu}{12}\right)f_{i-1}^k + \left(\frac{3}{4}+\frac{\mu}{4}\right)f_i^k + \left(\frac{13}{36}-\frac{\mu}{4}\right)f_{i+1}^k + \left(\frac{-5}{108}+\frac{\mu}{12}\right)f_{i+2}^k, \\ f_{3i+2}^{k+1} = \left(\frac{-5}{108}+\frac{\mu}{12}\right)f_{i-1}^k + \left(\frac{13}{36}-\frac{\mu}{4}\right)f_i^k + \left(\frac{3}{4}+\frac{\mu}{4}\right)f_{i+1}^k + \left(-\frac{7}{108}-\frac{\mu}{12}\right)f_{i+2}^k. \end{array}$$

The Laurent polynomial a (z) for the mask of the subdivision scheme can be written as:

$$\begin{aligned} a(z) = {} & \left(\frac{-5}{108}+\frac{\mu}{12}\right)z^{-5} + \left(\frac{-7}{108}-\frac{\mu}{12}\right)z^{-4} + \left(\frac{13}{36}-\frac{\mu}{4}\right)z^{-2} + \left(\frac{3}{4}+\frac{\mu}{4}\right)z^{-1} + 1.z^0 \\ & + \left(\frac{3}{4}+\frac{\mu}{4}\right)z^1 + \left(\frac{13}{36}-\frac{\mu}{4}\right)z^2 + \left(\frac{-7}{108}-\frac{\mu}{12}\right)z^4 + \left(\frac{-5}{108}+\frac{\mu}{12}\right)z^5. \end{aligned}$$

Laurent polynomial method is used to prove the smoothness of the ternary 4-point interpolating subdivision scheme to be C^2. Taking:

$$b^{[m,L]}(z) = \frac{1}{3^L}a_m^{[L]}(z), \; L = 1, 2, \ldots, m$$

where,

$$a_m(z) = \left(\frac{3z^2}{1+z+z^2}\right)a_{m-1}(z) = \left(\frac{3z^2}{1+z+z^2}\right)^m a(z)$$

and

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{3^j})$$

With a choice of m = 1 and L = 1, it can be written as:

$$\begin{aligned} b^{[1,1]}(z) = \frac{1}{3^1}a_1(z) = {} & \left(\frac{-5}{108}+\frac{\mu}{12}\right)z^{-3} + \left(\frac{-1}{54}-\frac{\mu}{6}\right)z^{-2} + \left(\frac{7}{108}+\frac{\mu}{12}\right)z^{-1} + \left(\frac{17}{54}-\frac{\mu}{6}\right)z^0 \\ & + \left(\frac{10}{27}+\frac{\mu}{3}\right)z^1 + \left(\frac{17}{54}-\frac{\mu}{6}\right)z^2 + \left(\frac{7}{108}+\frac{\mu}{12}\right)z^3 + \left(\frac{-1}{54}-\frac{\mu}{6}\right)z^4 + \left(\frac{-5}{108}+\frac{\mu}{12}\right)z^5. \end{aligned}$$

To determine the convergence of subdivision scheme S, consider:

$$\begin{aligned} \left\|\frac{1}{3}S_1\right\|_\infty &= \max\left\{\sum_\beta \left|b_{\gamma+3\beta}^{[1,1]}\right| : \gamma = 0,1,2\right\} \\ &= \max\left\{\left|\frac{-5}{108}+\frac{\mu}{12}\right| + \left|\frac{17}{54}-\frac{\mu}{6}\right| + \left|\frac{7}{108}+\frac{\mu}{12}\right|, \left|\frac{10}{27}+\frac{\mu}{3}\right| + 2\left|\frac{-1}{54}-\frac{\mu}{6}\right|\right\} < 1, \end{aligned}$$

Therefore, the subdivision scheme S is convergent.

In order to prove the ternary 4-point interpolating subdivision scheme to be C^1. Consider m = 2 and L = 1, the Laurent polynomial gives:

$$b^{[2,1]}(z)=\frac{1}{3^1}a_2(z)=\left(\frac{-5}{36}+\frac{\mu}{4}\right)z^{-1}+\left(\frac{1}{12}-\frac{3\mu}{4}\right)z^0+\left(\frac{1}{4}+\frac{3\mu}{4}\right)z^1+\left(\frac{11}{18}-\frac{\mu}{2}\right)z^2$$
$$+\left(\frac{1}{4}+\frac{3\mu}{4}\right)z^3+\left(\frac{1}{12}-\frac{3\mu}{4}\right)z^4+\left(\frac{-5}{36}+\frac{\mu}{4}\right)z^5.$$

To determine the convergence of subdivision scheme S_1, consider:

$$\left\|\frac{1}{3}S_2\right\|_\infty=\max\left\{\sum_\beta\left|b_{\gamma+3\beta}{}^{[2,1]}\right|:\gamma=0,1,2\right\}$$
$$=\max\left\{\left|\frac{1}{12}-\frac{3\mu}{4}\right|+\left|\frac{1}{4}+\frac{3\mu}{4}\right|,\left|\frac{11}{18}-\frac{\mu}{2}\right|+2\left|\frac{-5}{36}+\frac{\mu}{4}\right|\right\}<1,$$

Therefore, the subdivision scheme S_1 is convergent and the subdivision scheme $S\epsilon C^1$.

In order to prove the ternary 4-point interpolating subdivision scheme to be C^2. Consider m = 3 and L = 1; the Laurent polynomial gives:

$$b^{[3,1]}(z)=\frac{1}{3^1}a_3(z)=\left(\frac{-5}{12}+\frac{3\mu}{4}\right)z^1+\left(\frac{2}{3}-3\mu\right)z^2+\left(\frac{1}{2}+\frac{9\mu}{2}\right)z^3+\left(\frac{2}{3}-3\mu\right)z^4+\left(\frac{-5}{12}+\frac{3\mu}{4}\right)z^5.$$

To determine the convergence of subdivision scheme S_2, consider:

$$\left\|\frac{1}{3}S_3\right\|_\infty=\max\left\{\sum_\beta\left|b_{\gamma+3\beta}{}^{[3,1]}\right|:\gamma=0,1,2\right\}$$
$$=\max\left\{\left|\frac{1}{2}+\frac{9\mu}{2}\right|,\left|\frac{-5}{12}+\frac{3\mu}{4}\right|+\left|\frac{2}{3}-3\mu\right|\right\}<1,$$

Therefore, the subdivision scheme S_2 is convergent and the subdivision scheme $S\epsilon C^2$ for the range of tension parameter $\mu\epsilon\left]\frac{-1}{3},\frac{1}{9}\right[$.

COMPARISON OF C^2 INTERPOLATING SCHEMES

Weissman (1989) proposed the binary 6-point interpolating subdivision scheme by taking the convex combination of the two (Deslauriers and Dubuc, 1989) subdivision schemes. The subdivision scheme is defined as:

$$f_{2i}^{k+1}=f_i^k,$$
$$f_{2i+1}^{k+1}=\left(\frac{9}{16}+2\mu\right)\left(f_i^k+f_{i+1}^k\right)-\left(\frac{1}{16}+3\mu\right)\left(f_{i-1}^k+f_{i+2}^k\right)+(\mu)\left(f_{i-2}^k+f_{i+3}^k\right),$$

The scheme generates a family of C^2 limiting curves for the range of the tension parameter $\mu\epsilon\left]0,\frac{1}{50}\right[$.

Hassan *et al*. (2002) presented a ternary 4-point interpolating subdivision scheme. The subdivision scheme is defined as:

$$f_{3i}^{k+1}=f_i^k,$$
$$f_{3i+1}^{k+1}=a_0f_{i-1}^k+a_1f_i^k+a_2f_{i+1}^k+a_3f_{i+2}^k,$$
$$f_{3i+2}^{k+1}=a_3f_{i-1}^k+a_2f_i^k+a_1f_{i+1}^k+a_0f_{i+2}^k.$$

Table 1: Comparison of the interpolating C^2 subdivision schemes

Interpolating subdivision Scheme	Support (size)	Continuity	Range of parameter
Binary 6-point (1989)	10	C^2	$\mu\epsilon\left]0,\frac{1}{50}\right[$
Ternary 4-point (2002)	5	C^2	$\mu\epsilon\left]\frac{1}{15},\frac{1}{9}\right[$
Proposed ternary 4-point	5	C^2	$\mu\epsilon\left]\frac{-1}{3},\frac{1}{9}\right[$

where, $a_0=\left(\frac{-1}{18}-\frac{1}{6}\mu\right)$, $a_1\left(\frac{13}{18}+\frac{1}{2}\mu\right)$, $a_2\left(\frac{7}{18}-\frac{1}{2}\mu\right)$ and $a_3=\left(\frac{-1}{18}+\frac{1}{6}\mu\right)$. The Scheme generates a family of C^2 limiting curves for the range of the tension parameter $\mu\epsilon\left]\frac{1}{15},\frac{1}{9}\right[$.

Comparison Table 1 demonstrated that binary 6-point 1989 and ternary 4-point (2002) schemes are required to generate interpolating C^2 limiting curves. Each new point generated by binary 6-point scheme (1989) requires a total of 11 floating point operations (6 multiplies and 5 adds). On the other hand, the proposed subdivision scheme requires a total of 7 floating point operations (4 multiplies and 3 adds) to compute each new point, same as that of ternary 4-point 2002 scheme but more efficient in terms of computational cost as compared to binary 6-point 1989 scheme. Comparison Table 1 also illustrated that the proposed subdivision scheme generates family of C^2 limiting curves in the wider range of tension parameter as compare to the interpolating binary 6-point 1989 and ternary 4-point 2002 subdivision schemes. In other words, the range of the tension parameter $\mu\epsilon\left]0,\frac{1}{50}\right[$ of the binary 6-point interpolating scheme 1989 and the range of the tension parameter $\mu\epsilon\left]\frac{1}{15},\frac{1}{9}\right[$ of the ternary 4-point interpolating scheme are the proper subset of the range of the proposed subdivision scheme.

EXAMPLES

Three examples are considered to demonstrate the role of the tension parameter μ = 1/50 and μ = 0 as shown in Fig. 1, 2 and 3. Continuous curve belongs to the proposed subdivision scheme, dashed curve belongs to Hassan *et al.* (2002) and the dotted curve belongs to Weissman (1989). For the value of tension parameter μ = 1/50, comparison of the three subdivision schemes illustrated that the proposed scheme is close to the control polygon as compare to Weissman (1989) and maintains the smoothness better than Hassan *et al.* (2002) as shown in examples. For the value of tension parameter μ = 0, comparison of the three subdivision schemes exposed that Weissman (1989) and Hassan *et al.* (2002) have lost the C^2 continuity, where as the proposed subdivision scheme still maintains C^2 continuity. In fact, the limiting curves generated by the proposed scheme maintain C^2 smoothness for the wider range of the tension parameter as compare to Weissman (1989) and Hassan *et al.* (2002).

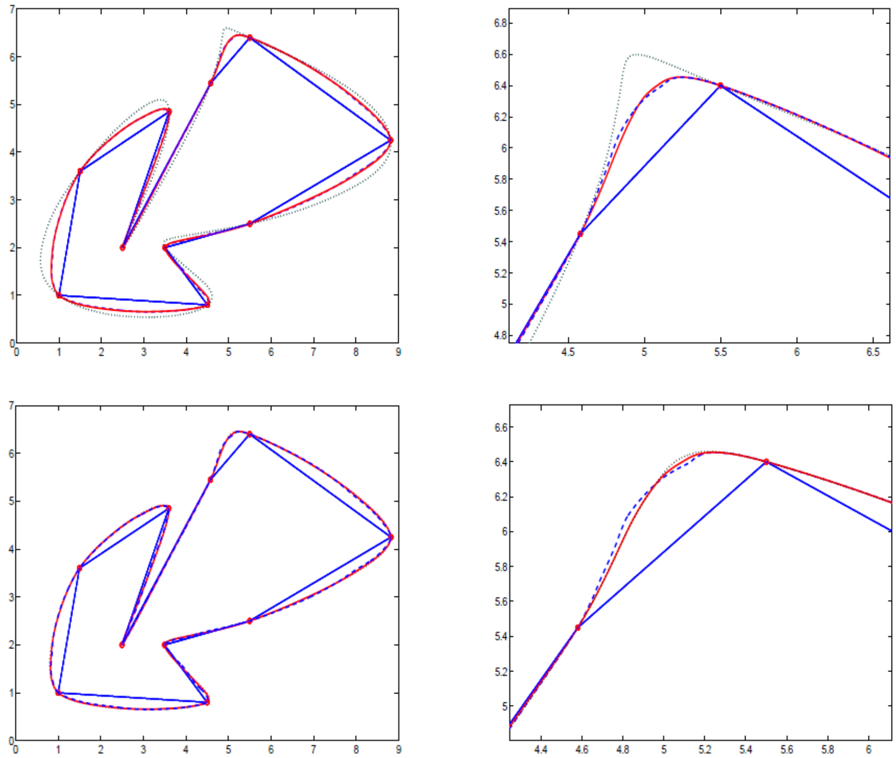

Fig. 1: Behavior of the interpolating subdivision schemes for the tension parameter μ = 1/50 (In first row) and μ = 0 (in second row); dotted curve generated by Weissman (1989), dashed curve generated by Hassan *et al*. (2002) and continuous curve generated by proposed subdivision scheme

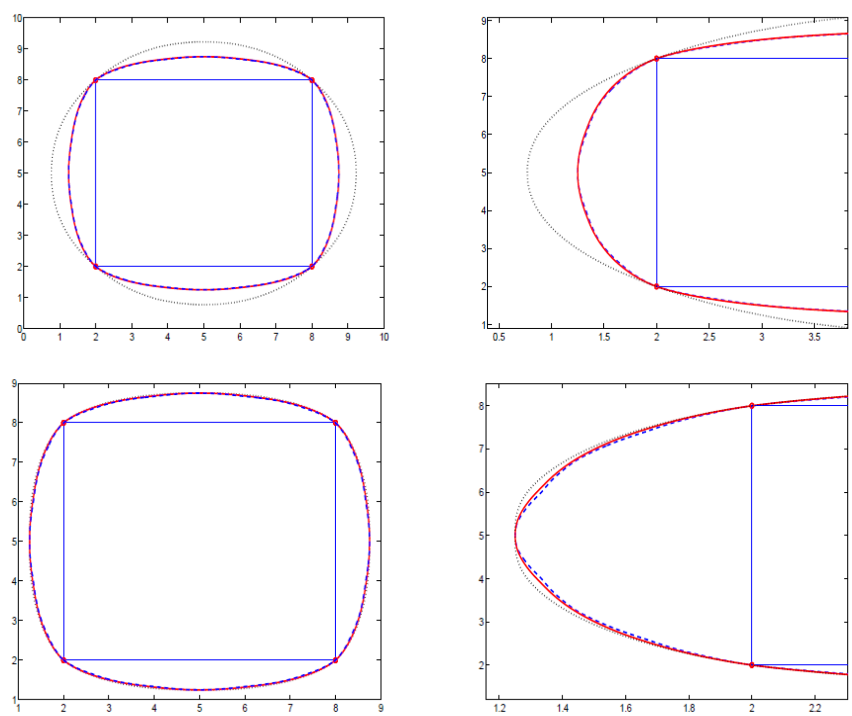

Fig. 2: Behavior of the interpolating subdivision schemes for the tension parameter μ = 1/50 (In first row) and μ = 0 (in second row); dotted curve generated by Weissman (1989), Dashed curve generated by Hassan *et al*. (2002) and continuous curve generated by proposed subdivision scheme

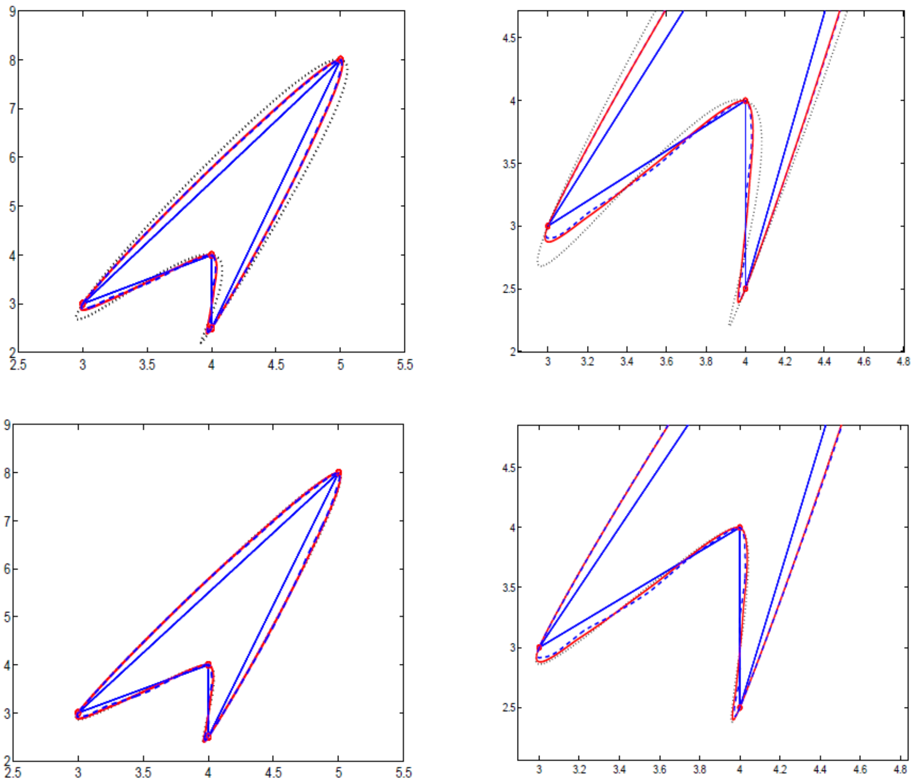

Fig. 3: Behavior of the interpolating subdivision schemes for the tension parameter μ = 1/50 (In first row) and μ = 0 (in second row); dotted curve generated by Weissman (1989), dashed curve generated by Hassan *et al.* (2002) and continuous curve generated by proposed subdivision scheme

CONCLUSION

A ternary 4-point interpolating subdivision scheme is introduced that generates the family of C^2 limiting curves for the range of tension parameter $\mu \epsilon \left]\frac{-1}{3}, \frac{1}{9}\right[$. Beauty of the proposed subdivision scheme is that it generates smooth limiting curves of C^2 continuity for the larger range of the parameter as compare to Weissman (1989) and Hassan *et al.* (2002). Laurent polynomial method is used to investigate the smoothness of the proposed subdivision scheme. Three examples demonstrate that the proposed scheme is close to the initial control polygon as compare to Weissman (1989) and it gives less distortion as compare to Hassan *et al.* (2002). For geometric designers, the proposed subdivision scheme gives more flexibility with in the discrete set of data points as compare to subsist subdivision schemes.

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