Research Journal of Applied Sciences, Engineering and Technology 6(9): 1582-1587, 2013
ISSN: 2040-7459; e-ISSN: 2040-7467

Submitted: November 09, 2012 Accepted: January 07, 2013 Published: July 15, 2013

Relationship between Maxwell's Equations and Einstein Field Equation Base on EEG Source Localization in the Brain

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Abstract: Mathematics can assist to find EEG source localization, in particular during brainstorm. In this study, the Einstein field Equation is shown to lead to Maxwell's equations on route to model the brainstorm**.**

Keywords: Clifford bundle, Einstein field equation, lorentzian space-time, Maxwell's equations

INTRODUCTION

Clerc *et al*. (2012) and her colleagues, presented a new method for EEG source localization, based on rational approximation. The model was initially developed from Maxwell's equation under quasi-static assumptions. Then the equation led to a formulation of the electric potential as a solution to Laplace's equation. They considered the inverse EEG problem in their study.

The head $\Omega = U^2_{i=0}\,\Omega_i \subset R^3$ is modelled as a set of nested regions $\Omega_i \subset R^3$, i = 0, 1, 2 (brain, skull, scalp), separated by either spherical or ellipsoidal interfaces S_i (with $S_i = \partial\Omega$) and with piecewise constant conductivity, σ, $\sigma_{|\Omega i} = \sigma_i > 0 >$.

Considering a macroscopic physical model of brain activity and using a quasi-static approximation of the Maxwell equations, the spatial behaviour of the electric potential *u* in is related to the distribution of *N* dipolar sources located at $C_k \varepsilon \Omega_0$ with moments $P_k \in R^3$ by the following equation:

$$\begin{cases} \nabla.(\sigma\vec{\nabla}u) = \sum_{k=1}^{N} p_k.\nabla\delta_{C_k} & in \quad \Omega \subset R^3 \\ u = g \quad (Current\ flux)\ \partial_n u = \phi & on \quad S_0 \end{cases}$$

where, g, ϕ denote the given potential and current flux on the scalp respectively. The best application of this model is for epilepsy patients who suffer from an abnormality in the brain to localize epileptic foci (H¨am¨al¨ainen *et al*., 1993)

Maxwell's equations: Using Maxwell's equations under quasi-static assumptions, lead to a formulation of the magnetic potential $\vec{B}$, as a solution of a partial differential equation. The Maxwell's equations in vacuum are Jackson (1984):

$$\nabla.\vec{E} = \frac{\rho_c}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$$

where $\varepsilon_0 \backsimeq 8.85\times10^{-12}$ Fm^{-1} is the electrical permittivity of the vacuum and, p_c is the total charge density:

$$\nabla.\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

and $\mu_0 = 4\pi 10^{-7} Hm^{-1}$ is the magnetic permeability where by:

$\vec{r}$ = The position of the point r in
ρc ($\vec{r}$, t) = The (volumic) charge density at location $\vec{r}$ and time t
$\vec{E}$ ($\vec{r}$, t) = The electric field vector
$\vec{B}$ ($\vec{r}$, t) = The magnetic field vector
$\vec{j}$ ($\vec{r}$, t) = The current density vector

Because the frequencies of the electromagnetic signals of the brain are extremely low, the quasi-static approximation of Maxwell's equations allows ignoring the derivatives with respect to time. With this quasi-static assumption, the fourth Maxwell's equation is presented as:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{1}$$

Because $\mu_0 = 0$ at the exterior of the head, the curl of the magnetic field is zero:

$$\vec{\nabla} \times \vec{B} = 0$$

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Since the curl of the electric field is zero, this field is the gradient of electric potential: n:

$$\vec{E} = -\vec{\nabla} u \tag{2}$$

Inside the head, the current density $\vec{J}$ can be decomposed into:

- A volumic density $\vec{J}^{v} = \sigma \vec{E}$ where σ is the head conductivity
- A primary current $\vec{J}^{p}$:
- $\vec{J} = \vec{J}^{p} + \sigma \vec{E}$.

Using (2), we have:

$$\vec{J} = \vec{J}^{p} - \sigma \vec{\nabla} u \tag{3}$$

Using (1) and that the curl of the divergence is zero is transformed to:

$$\nabla . \vec{J} = 0$$

Thus, using (3):

$$\nabla . (\sigma \vec{\nabla} u) = \nabla . \vec{J}^{p}$$

The primary current generated by *N* dipolar sources C_k of moments P_k is:

$$\vec{J}^{p} = \sum_{k=1}^{N} p_k . \delta_{c_k}$$

where, δ_{c_k} is the Dirac distribution. Then,

$$\nabla . \vec{J}^{p} = \sum_{k=1}^{N} p_k . \nabla \delta_{c_k}$$

And

$$\nabla . (\sigma \vec{\nabla} u) = \sum_{k=1}^{N} p_k . \nabla \delta_{c_k} \tag{4}$$

Because the sources are localized inside the interior layer (brain), (4) is elaborated as follows:

$$\begin{cases} \nabla . (\sigma \vec{\nabla} u) = \sum_{k=1}^{N} p_k . \nabla \delta_{c_k} & in \quad \Omega \subset R^3 \\ u = g \quad (Current\ flux) \ \partial_n u = \phi & on \quad S_0 \\ \sigma \partial_n u = 0 & on \quad S_2 \end{cases}$$

where, here σ is the brain conductivity. The outer medium (air) is non-conductive in the homogeneous Neumann boundary condition. The current flowing through the neck is neglected.

EEG SOURCE LOCALIZATION METHOD BASED ON EINSTEIN FIELD EQUATION

Big Bang cosmology is a model to explain observations due to the evolution of the universe. It is remarkably similar to the initial sporadic electrons burst of epileptic brainstorm. Big Bang postulates about 15 billion years ago a tremendous explosion occurred from one hot, massive but tiny point of singularities which threw all matter outwards, initiated the birth of space and time into existence and started the expansion of the universe. This was not a conventional explosion but rather an explosion of space with all particles of the matter rushing away from each other. At the point of this event, all of the matter and energy of space was contained at one point. Since the Big Bang, the universe has been continuously expanding and, hence; there has been more and more distance between clusters of galaxies. Indeed, the Big Bang event is extremely closely similar to the event of sporadic burst of electron from epileptic foci as shown in Fig. 1.

The basic analogies between Creation of the universe event and Brainstorm epilepsy is presented in Table 1, (Ismail, 2012).

The Einstein Field Equation (EFE) in general relativity: The framework for the Big Bang model relies on Albert Einstein's general relativity. General relativity explains the fundamental force of gravitation can be described as a curved space-time caused by the presence of matter and energy (Einstein, 1916).

The Einstein Field Equation (EFE) in general relativity is an equation of tensor which relates the curvature of space-time due to the existence of energy and momentum within the space-time. The equation describes the expansion of space of the universe. It shows that the amount of mass in the universe influence the expansion that the universe is currently experience. Since the universe is mostly empty even though it is currently expanding in accelerated rate, it is postulate that dark mass is the one that providing the momentum and energy for the event. (Kroupa *et al*., 2010).

In geometrical form, the Einstein Field equation is written by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where, $R_{\mu\nu}$ is the Ricci curvature tensor, R the scalar curvature, $g_{\mu\nu}$ the metric tensor, $T_{\mu\nu}$ the stress-energy tensor and G is the Newton's gravitational constant.

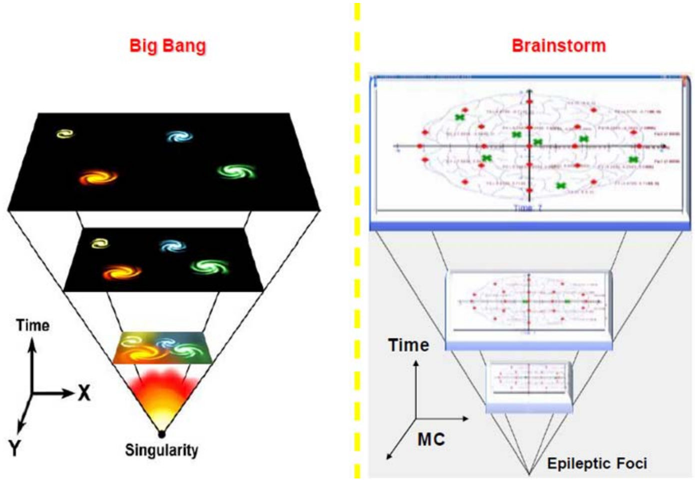

Fig. 1: Formations of a galaxy are the same as the journey of cluster of the electrons to the scalp area of the brain

Table 1: Similarities between creation of the universe event and brainstorm epilepsy event

Creation of the universe event	Brainstorm epilepsy event
Burst of space, stars and galaxies.	Burst of ionic charges.
Gravitational repulsion force.	Electrostatic repulsion force.
High density cloud of gas.	High accumulations of ionic charges.
Stars and galaxies move radially outward from the point of singularity. Gravitational force is radial.	Electrons repulse each other and move radially outward from seizure's foci. Coulomb's electrostatic force is radial.
Dark energy. Repulsive potential energy.	Electrostatic potential energy.
Generations of stars were created by secondary explosions of galaxies.	Secondary burst of electrons do occur at other secondary foci.
Stars have no solid boundary. It is in a state of hydrostatic equilibrium of hot plasma ionic gases, which takes the form of neutral gas-like clouds.	Ions have no sharp boundaries. The ionic radius is not a fixed property of a given ion, but varies with coordination number, spin state and other parameters.

Table 2: Four solution of Einstein field equation

	Non-rotating ($J = 0$)	Rotating ($J \neq 0$)
Uncharged $Q = 0$)	Schwarzschild	Kerr
Charged ($Q \neq 0$)	Reissner-nordström	Kerr–Newman

There exist several types of Einstein Field Equation solutions. They are summarized in the following Table 2. where, Q represents the body's electric charge and J represents its spin angular momentum.

In the case of epileptic seizures, the body is accumulations of electrons in a small volume. Therefore, this leads to the electro vacuum solutions of Reissner-Nordström metric in empty space, which describes the electromagnetic field of a charged, non-rotating, spherically symmetric body of mass M. (Nordstrom, 1918).

The solution based on Reissner–Nordström metric can be written as following:

$$c^2 d\tau^2 = \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)c^2 dt^2 - \frac{dr^2}{1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

where,

c = The speed of light in meters per second
t = The time coordinates in seconds
r = The radial coordinate in meters
θ = The colatitudes (angle from North) in radians
φ = The longitude in radians
r_s = The Schwarzschild radius (in meters) of the massive body, which is related to its mass M by

$$r_s = \frac{2GM}{c^2}$$

where,

G = The gravitational constant
r_Q = A length-scale corresponding to the electric charge Q of the mass $r^2_Q = Q2G/4\varepsilon 0c4$

where, $1/4\pi\varepsilon_0$ is Coulomb's force constant (Landau and Lifshitz, 1975).

Considering the values of the radial domains of the movements, or tracks of sources (epileptic foci) in given time within the brain based on Reissner-Nordstron solution of Einstein Field Equation.

COUPLING OF ELECTROMAGNETISM AND GENERAL RELATIVITY

In this section, we are concerned to reveal that any space-time structure(M, g, D, t_g, ↑) which is a model of a gravitational field in General Relativity, generated by an energy-momentum tensor T and which contains at least one Killing vector field A is such that the 2-form field $F = dA$ (where A = g (A,)), satisfies a Maxwell equation with a well determined current.

Two equations are shown to be equivalent in a precise mathematical viewpoint. We need some preliminaries before starting.

Preliminaries and notation: In this study, a space-time structure is a pentuple $\mathfrak{M}$ = (M, g, D, t_g, ↑) where (M,g,t_g) is a Lorentzian manifold, D is the Levi-Civita connection of g and ↑ is a similarity relation between timelike vector fields defining the time orientation (Rodrigues and Capelas de Oliveira, 2007). Also, g ε sec $T^2{}_0M$ denotes the metric of the cotangent bundle, $\wedge T^*M = \oplus_{r=0}^{r=4} \wedge T^r M \to \mathcal{C}\ell$(M,g) denotes the Cartan (Exterior) bundle of differential forms ,where $\mathcal{C}\ell$(M,g) denotes the Clifford bundle of differential forms (Blaine-Lawson and Michelson, 1989). Let $\{e_\mu\}$be an arbitrary coordinate basis for TU ⊂ TM and $\{\vartheta^v\}$ the corresponding dual basis of T*U⊂T*M =$\wedge^1$T*M then g = $g_{\mu v}\,\vartheta^\mu \otimes \vartheta^v = g^{\mu v}\,\vartheta_\mu \otimes \vartheta_v$, g = $g^{\mu v}\, e_\mu \otimes e_v = g_{\mu v}\, e^\mu \otimes e^v$ and $g^{\mu\alpha}.\ g_{\alpha v} = \delta^\mu{}_v$.

We recall that the set $\{e^\mu\}$ such that g(e_μ, e^v) = $\delta^\mu{}_v$ is called the reciprocal basis of $\{e_\mu\}$and the set $\{\vartheta_v\}$ such that g(ϑ^μ, ϑ_v) = $\delta^\mu{}_v$ is called the reciprocal basis of $\{\vartheta^v\}$.We denote g (ϑ^μ, ϑ_v) = $\vartheta^\mu \cdot \vartheta_v$, where, denotes the scalar product in $\mathcal{C}\ell$(M, g). Finally, δ = $v^u D_{e\mu}$ denotes the Dirac operator acting on sections of $\mathcal{C}\ell$ (M, g) and $\Box = \partial.\partial$ indicate the covariant D'Alembertian and δ^δ denotes the Ricci operators on sec$\wedge^1T^*M \to$sec $\mathcal{C}\ell$(M, g). The operator ♦ = σ^2 is called Hodge D'Alembertian ultimately, $\mathcal{R}^\mu \in$ sec$\wedge^1T^*M \to$ sec $\mathcal{C}\ell$(M, g) are the Ricci 1-form fields, with $\mathcal{R}^\mu = R^\mu{}_v\, \vartheta^v$, where $R^\mu{}_v$ are the components of the Ricci tensor.

Definition 1: Let $\mathfrak{M}$ = be a (M, g, D, t_g,↑) Lorentzian space-time and let $\mathcal{C}\ell$ (M, g) be the Clifford bundle of differential forms. The action of the Dirac operator ∂ on any $P \in \sec \wedge T^* M \to$ $\mathcal{C}\ell$(M, g) is $\partial P = (d - \delta)P$. Suppose electromagnetic field $F = dA \in \sec \wedge^2 T^* M \to$ $\mathcal{C}\ell$(M, g) generated by a current $J_e \in \sec \wedge^1 T^* M \to$ $\mathcal{C}\ell$(M, g), develops its dynamics in $\mathfrak{M}$, then F satisfies Maxwell equations (Rodrigues, 2010):

$$dF = 0,\ \delta F = -J_e$$

The Maxwell equation equivalent to Einstein equation: In this Section, we verify a two propositions and a corollary which establish, for any Lorentzian Space-time Structure (M, g, D, τ_g, ↑) representing a given gravitational field and containing an arbitrary Killing vector field A, the existence of Maxwell equations for the electromagnetic field F = dA (where A = g (A,)). Subsequently, it is shown to be equivalent to Einstein equation.

Proposition 1: If A vector field, A ∈ sec TM**,** with M part of the structure (M, g, D, τ_g,↑) is a killing vector field then δ_g A = 0, where A = g (A,) = $A_\mu\, \vartheta^\mu = A^\mu\, \vartheta_\mu$ (Rodrigues, 2010).

Proof: To verify the preposition1.it is essential to recall that:

$$\mathcal{L}_A g = 0 \Leftrightarrow D_\mu A_v + D_v A_\mu = 0 \tag{5}$$

and in the Clifford bundle formalism $\delta_g = -\partial \lrcorner A$. Then we have $\delta_g A = -\vartheta^\mu \lrcorner D_{e_\mu} A = -\vartheta^\mu\, [(D_\mu A_v)\, \vartheta^v] = g^{\mu v} D_\mu A_v$ $= 1/2 g^{\mu v}(D_\mu A_v + D_v A_\mu) = 0$ and the proposition is proved.

Proposition 2: If A ∈ sec TM (where M is part of the structure (M, g, D, τ_g,↑) is a Killing vector field then we have:

$$\partial \wedge \partial A = \Box A = \mathcal{R}^\mu A_\mu \tag{6}$$

where, δ = $v^u D_{e\mu}$ is the Dirac operator acting on the sections of the Clifford bundle $\mathcal{C}\ell$(M, g) and $\partial\wedge\partial$ is the Ricci operator acting on $\mathcal{C}\ell$(M, g) Finally $\mathcal{R}^\mu \in$ sec$\wedge^1T^*M \to$sec $\mathcal{C}\ell$(M, g) are the Ricci 1-form fields, with $\mathcal{R}^\mu = R^\mu{}_v\, \vartheta^v$, where $R^\mu{}_v$ are the components of the Ricci tensor.

Proof: To prove that $\partial\wedge\partial A = \mathcal{R}^\mu A_\mu$, it is well known that the Ricci operator is an extensorial entity (Rodrigues and Capelas de Oliveira, 2007). Specifically it satisfies $\partial\wedge\partial A = A_\mu\, \partial\wedge\partial\vartheta^\mu$ and since $\partial\wedge\partial\vartheta^\mu = \mathcal{R}^\mu$ it reads $\partial\wedge\partial A = \mathcal{R}^\mu A_\mu$

In order to determine that A = $\mathcal{R}^\mu A_\mu$ the definition of the covariant D'Alembertian is used (Rodrigues and Capelas de Oliveira, 2007) and it follows that: $\partial.\partial A = g^{\sigma v} D_\sigma D_v A_\mu \vartheta^\mu$ Now, the term $D_\sigma D_v A_\mu$ is calculated. Since A is a Killing vector field, satisfying Eq. (5) it is possible to write:

$$D_\sigma (D_v A_\mu + D_\mu A_v) = [D_\sigma, D_v]\, A_\mu + D_v D_\sigma A_\mu + [D_\sigma, D_\mu]\, A_v + D_\mu D_\sigma A_v = 0 \tag{7}$$

Taking into account that: $g^{\sigma v}\, [D_\sigma, D_v]\, A_\mu = 0$, $g^{\sigma v} D_\mu D_\sigma A_v = 1/2 g^{\sigma v} D_\mu (D_\sigma A_v + D_v A_\sigma) = 0$ and in addition that:

$$g^{\sigma v}\, [D_\sigma, D_v]\, A_\mu = -g^{\sigma v} A_\rho = -g^{\sigma v} A^\rho = -g^{\sigma v} A^\rho = -A^\rho \tag{8}$$

Multiplying Eq. (7) by $g^{\sigma v}$ it follows that: $g^{\sigma v} D_v D_\sigma A_\mu = R_{\rho u} A$ and thus $\partial.\partial A = g^{\sigma v} D_\sigma D_v A_\mu \vartheta^\mu = R_{\rho u} A\, \vartheta^\mu = A^\rho\, \mathcal{R}_\rho$, which proves the proposition. Now in a position that prove a theorem which demonstrate similarity between two equations as a follows Rodrigues *et al*. (2012).

Theorem: The field A satisfies the wave equation:

$$\square A - R/2A = - T(A) \quad (9)$$

where is the covariant D'Alembertian1, R is the scalar curvature, $T(A) := T^{\mu}A_{\mu} \in \sec\wedge^{1}T^{*}M$ where the $T^{\mu} = T^{\mu}{}_{v}v^{v} \in \sec\wedge^{1}T^{*}M$ are the energy-momentum 1-form fields, with $T = T_{\mu v}\,\vartheta^{\mu} \otimes \vartheta^{v}$ and Einstein equation is equivalent to Eq.(9).

Proof: Under the proposition above, Einstein equation (in geometrical units) is written as $Ricci - 1/2Rg = - T$ ($Ricci = R_{\mu v}\vartheta^{\mu} \otimes \vartheta^{v}$) and can be rewritten in the equivalent form:

$$\mathcal{R}^{\mu} - \frac{1}{2} R\vartheta^{\mu} = - T^{\mu} \quad (10)$$

Now, we use the fact that the Ricci operator satisfies $\partial\wedge\partial\vartheta^{\mu} = \mathcal{R}^{\mu}$ and moreover that it is an extensorial operator, i.e., $A_{\mu}\,(\partial\wedge\partial\vartheta^{\mu}) = \partial\wedge\partial A$, after multiplying Eq. (10) by A_{μ} it follows that:

$$\partial\wedge\partial A - 1/2RA = -T(A) \quad (11)$$

where $T(A) = T\mu\, A\mu \in \sec\wedge^{1}T^{*}M \rightarrow \sec\, \mathcal{C}\ell(M, g)$. Now using Eq. (6) which state that $\partial\wedge\partial A = \square A$ it reads: A-**1/2**RA= - T (A) which proves the theorem. Moreover, denoting F = dA and by δ_{g} the Hodge coderivative operator, we have the

Corollary: $dF = 0$, $\delta_{g} F = - RA + 2T(A)$

Proof: To prove the corollary, we sum $\partial.\,\partial A =$ A to both members of Eq. (11) and take into account that for any $\mathcal{C} \in \sec\, C\ell\,(M, g)$ the following Rodrigues and Capelas de Oliveira (2007). $\partial^{2}\mathcal{C} = \partial\wedge\partial\mathcal{C} + \partial.\partial\mathcal{C}$ holds. Then:

$$\partial^{2}A = 1/2RA - T(A) + \quad A$$

Now, since $\partial^{2}A = - \delta_{g}\, dA - d\, \delta_{g} A$ and proposition 1 implies that $\delta_{g} A = 0$, it follows that $\partial^{2}A = - \delta_{g} F$. Finally, taking into account Eq. (6) it follows that $\delta_{g} F = -J$ with $J = RA - 2T(A)$ and the corollary is proved.

CONCLUSION

We demonstrated that for each Lorentzian space-time representing a gravitation field in General Relativity, which contains, an arbitrary Killing vector field A, the field F = dA (where A = g (A,)) satisfies Maxwell equations with well determined current 1-form field. It is deduced to Einstein equation and this equivalence holds within a 4-dimensional space-time. The present relationship between two equations encourages further research into practical applications toward source localization in the brain. Now, we understood that their exist analytically link between EEG's source localization based on Rational approximation, which is under, the quasi-static assumption of Maxwell equation and radial domains of the tracks of cluster (epileptic foci) in given time within the brain based on Reissner-Nordstron solution of Einstein Field Equation is established.

ACKNOWLEDGMENT

The authorswould like to thank to Universiti Teknologi Malaysia (UTM) and Ministry of Higher Education (MOHE) for International Doctroal Fellowship (IDF) for this research.

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