Consider a finite population \( U = (U_1, U_2, \ldots, U_N) \) of \( N \) first stage units (psus) such that the sum of the weights of psu \( U_i = (1, 2, \ldots, M_i) \) consist of \( M_i \) second stage units (ssus) and \( M = \sum_{i=1}^{N} M_i \). Let \( y_{ij} \) and \( x_{ij} \) be the values of the study variable \( (y) \) and the auxiliary variable \( (x) \) respectively for the \( j \)-th ssu of the \( i \)-th psu \( U_i = (j = 1, 2, \ldots, M_i; i = 1, 2, \ldots, N) \). To estimate \( S_x^2 \) under two-stage sampling scheme, it is assumed that \( S_x^2 \) known.

Let we define the following notations and symbols:

\[
\begin{align*}
\gamma &= \frac{1-f}{n} ; \quad f = \frac{n}{N} ; \quad \gamma_{2u} = \left( \frac{M_i}{M} \right)^2 \frac{1}{n} \left( \frac{1}{m_i} - \frac{1}{M_i} \right) ; \\
p &= \frac{1}{n} \sum_{i=1}^{N} p_i , \quad \text{where} \quad p_i = \left( \frac{M_i}{M} \right) \sum_{j=1}^{M_i} p_{ij} ; \\
S_{ip}^2 &= \frac{1}{n-1} \sum_{i=1}^{N} \left( \frac{M_i}{M} p_i - \bar{p} \right)^2 ; \\
S_{1pq} &= \frac{1}{N-1} \sum_{i=1}^{N} \left( \frac{M_i}{M} \bar{p}_i - \bar{p} \right) \left( \frac{M_i}{M} \bar{q}_i - \bar{q} \right) ; \\
S_{2pl}^2 &= \frac{1}{M_i-1} \sum_{j=1}^{M_i} \left( p_{ij} - \bar{p}_j \right)^2 ; \\
S_{2pq}^2 &= \frac{1}{M_i-1} \sum_{j=1}^{M_i} \left( p_{ij} - \bar{p}_j \right) \left( q_{ij} - \bar{q}_j \right)
\end{align*}
\]

where, \( p, q = X, Y \); and \( P \neq Q \); \( S_{ip}^2 \) and \( S_{2pl}^2 \) are the variances among psus means and variances among ssus units for the \( i \)-th psu, while \( S_{1pq} \) and \( S_{2pq} \) are their corresponding covariances.

Assume that a sample of \( n \) psus is drawn from \( U \) and then a sample of \( m_i \) from \( M_i \) ssus units i.e., from the \( i \)-th selected psu using simple random sampling without replacement at both stages. We define the following relative error terms.
Let \( e_0 = \frac{s_{y(2s)}^2}{s_y^2} \) and \( e_1 = \frac{s_{y(2s)}^2 - s_x^2}{s_y^2} \) such that \( E(e_0) = E(e_1) = 0 \).

\[
E(e_0^2) = \gamma \beta_{2(y)} + \sum_{i=1}^N y_{2i} \beta_{2y(2y)}(y_{2i})'\; E(e_1^2) = \gamma \beta_{2(y)} + \sum_{i=1}^N y_{2i} \beta_{2y(2y)}(y_{2i})' \]

where,

\[
\beta_{2(y)} = \frac{\mu_{40(1)}}{\mu_{20(1)}} ; \beta_{2x(1x)} = \frac{\mu_{04(2)}}{\mu_{22(1)}} ; \beta_{2x(2x)} = \frac{\mu_{04(2)}}{\mu_{22(1)}} ; \\
\beta_{2y(2y)} = \beta_{2y(1y)} - 1 ; \beta_{2y(1x)} = \beta_{2y(1x)} - 1 ; y_{22(1)} = y_{22(1)} - 1 ; \\
\gamma_{rs(1)} = \frac{\mu_{rs(1)}}{\mu_{20(1)}\mu_{02(1)}} ; \mu_{rs(1)} = \frac{\Sigma_{i=1}^n (y_{ij} - \bar{y}_1)(x_{ij} - \bar{x}_1)^2}{N} ; \\
\gamma_{rs(2i)} = \frac{\mu_{rs(2i)}}{\mu_{20(2i)}\mu_{02(2i)}} ; \mu_{rs(2i)} = \frac{\Sigma_{i=1}^n (y_{ij} - \bar{y}_1)(x_{ij} - \bar{x}_1)^2}{N} \\
\] 

The usual variance estimator for population variance in two-stage sampling is given by:

\[
S_{y(2s)}^2 = s_y^2 
\]

The MSE of \( S_{y(2s)}^2 \) is given by:

\[
MSE(S_{y(2s)}^2) \equiv S_y^4\{\gamma \beta_{2(y)} + \sum_{i=1}^N y_{2i} \beta_{2y(2y)}(y_{2i})' \}
\]

The traditional regression estimator for population variance under two-stage sampling is given by:

\[
S_{reg(2s)}^2 = s_y^2 + b\left( s_x^2 - s_{x(2s)}^2 \right) 
\]

where, \( b \) is the sample regression coefficient in two stage sampling.

The MSE of \( S_{reg(2s)}^2 \) is given by:

\[
MSE(S_{reg(2s)}^2) \equiv S_y^4\{\gamma \beta_{2(y)}(1 - \rho^2) + \sum_{i=1}^N y_{2i} \beta_{2y(2y)}(1 - \rho^2) \}
\]

where, \( \rho_i^2 = \frac{\gamma_{2i(2i)}^2}{\beta_{2y(2y)}(1x) \beta_{2y(2y)}(y_{2i})} \) and \( p^2_{2i} = \frac{\gamma_{2i(2i)}^2}{\beta_{2y(2y)}(y_{2i}) \beta_{2y(2y)}(y_{2i})} \).

**PROPOSED ESTIMATOR**

On the lines of Gupta and Shabbir (2008), we propose the following an improved exponential ratio type estimator for population variance \( S_{y(2s)}^2 \) in two-stage sampling, given by:

\[
S_{p(2s)}^2 = \left\{ k_1 s_y^2 + k_2 (s_x^2 - s_{x(2s)}^2) \right\} \exp\left( \frac{s_x^2 - s_{x(2s)}^2}{s_y^2} \right) 
\]

where, \( k_1 \) and \( k_2 \) are suitably chosen constants. We discuss the two cases:

- When \( k_1 + k_2 \neq 1 \)
- When \( k_1 + k_2 = 1 \)

To find the properties of our proposed estimator \( S_{p(2s)}^2 \), we consider the following two cases.

**Case 1:** When \( k_1 + k_2 \neq 1 \).

Using notations from above section, we have:
Using (6), the bias of $S^2_P(2s)$ to first order of approximation, is given by:

$$\text{Bias}(S^2_P(2s)) \equiv (k_1 - 1)S^2_Y + k_1 S^2_Y \left( e_0 - \frac{e_1}{2} + \frac{3}{8} e_1^2 - \frac{1}{2} e_0 e_1 \right) - k_2 S^2_Y (e_1 - \frac{1}{2} e_1^2)$$  \hspace{1cm} (6)

Using (6), the MSE of $S^2_P(2s)$ to first order of approximation, is given by:

$$\text{MSE}(S^2_P(2s)) \equiv S^2_Y \left[ 1 + k_1^2 \left( 1 + \varphi (\beta_{2(1,y)}^* + \beta_{2(1,x)}^*) - 2 \gamma^*(22) \right) + \sum_{i=1}^{N} y_{2i} \left( \beta_{2(2,x,i)}^* + \gamma^*(22) \right) + k_2^2 \varphi^2 \left( \gamma^*(21,x) + \sum_{i=1}^{N} y_{2i} \beta_{2(2,x,i)}^* \right) - 2 k_1 \left( 1 + \frac{1}{2} \left( \gamma^*(21,x) - \gamma^*(22) \right) + \sum_{i=1}^{N} y_{2i} \beta_{2(2,x,i)}^* \right) \right]$$  \hspace{1cm} (7)

where, $\varphi = \frac{s^2_Y}{S^2_Y}$
or:

$$\text{MSE}(S^2_P(2s)) \equiv S^2_Y \left( 1 + k_1^2 A_1^* + k_2^2 B_1^* + 2 k_1 k_2 C_1^* - 2 k_1 D_1^* - 2 k_2 E_1^* \right)$$  \hspace{1cm} (8)

where,

$$A_1^* = 1 + \gamma (\beta_{2(1,y)}^* + \beta_{2(1,x)}^*) - 2 \gamma^*(22) + \sum_{i=1}^{N} y_{2i} \left( \beta_{2(2,y,i)}^* + \beta_{2(2,x,i)}^* \right) - 2 \gamma^*(22)$$

$$B_1^* = \varphi^2 \left( \gamma^*(21,x) + \sum_{i=1}^{N} y_{2i} \beta_{2(2,x,i)}^* \right)$$

$$C_1^* = \varphi^2 \left( \gamma^*(22) - \gamma^*(22) \right) + \sum_{i=1}^{N} y_{2i} \left( \beta_{2(2,x,i)}^* - \gamma^*(22) \right)$$

$$D_1^* = 1 + \frac{1}{2} \left( \gamma^*(21,x) - \gamma^*(22) \right) + \sum_{i=1}^{N} y_{2i} \left( \beta_{2(2,x,i)}^* - \gamma^*(22) \right)$$

$$E_1^* = \frac{1}{2} \varphi^2 \left( \gamma^*(21,x) + \sum_{i=1}^{N} y_{2i} \beta_{2(2,x,i)}^* \right)$$

Now differentiating (8) with respect to $k_1$ and $k_2$, we get:

$$k_{1opt} = \frac{B_1^* D_1^* - C_1^* E_1^*}{A_1^* B_1^* - C_1^* E_1^*}$$

$$k_{2opt} = \frac{A_1^* E_1^* - C_1^* D_1^*}{A_1^* B_1^* - C_1^* E_1^*}$$

Substituting the optimum values of $k_1$ and $k_2$ in (8), we get the minimum MSE of $S^2_P(2s)$, given by:

$$\text{MSE}(S^2_P(2s))_{min} \equiv S^2_Y \left( 1 - \frac{(A_1^* E_1^* - C_1^* D_1^*)}{A_1^* B_1^* - C_1^* E_1^*} \right)$$  \hspace{1cm} (9)

**Case 2:** When $k_1 + k_2 = 1$

Under Case 2, the proposed estimator becomes:

$$\hat{S}^2_P(2s) = \{k_1 S^2_Y + (1 - k_1) (S^2_X - s^2_{x(2s)}) \} \exp \left( \frac{S^2_Y - s^2_{y(2s)}}{S^2_Y + s^2_{x(2s)}} \right)$$  \hspace{1cm} (10)

Putting $k_2 = 1 - k_1$ in (7) and (8), we get the bias and $\text{MSE}$ of $\hat{S}^2_P(2s)$ respectively as given by:

$$\text{Bias}(\hat{S}^2_P(2s)) \equiv \left( k_1 - 1 \right) S^2_Y + k_1 S^2_Y \left\{ \gamma^*(21,x) - \gamma^*(22) + \sum_{i=1}^{N} y_{2i} \left( \beta_{2(2,x,i)}^* - \gamma^*(22) \right) \right\}$$

$$+ \frac{1}{2} \varphi \left( \gamma^*(21,x) + \sum_{i=1}^{N} y_{2i} \beta_{2(2,x,i)}^* \right)$$  \hspace{1cm} (11)
And
\[
MSE(\hat{S}_{P(2s)}^2) \equiv S_y^4 \{1 + k_1^*(A_1^* + B_1^* - 2C_1^*) - 2k_1(B_1^* - C_1^* + D_1^* - E_1^* + B_1^* - 2E_1^*) \}
\] (12)

Now differentiating (12) with respect to \( k_1 \), we get:
\[
k_{1opt}^* = \frac{(B_1^* - C_1^* + D_1^* - E_1^*)}{(A_1^* + B_1^* - 2C_1^*)}
\]

Substituting the optimum value of \( k_1 \) i.e., \( k_{1opt}^* \) in (12), we get the minimum \( MSE \) of \( \hat{S}_{P(2s)}^2 \), given by:
\[
MSE(\hat{S}_{P(2s)}^2)_{min} \equiv S_y^4 \left\{1 + B_1^* - 2E_1^* - \frac{(B_1^* - C_1^* + D_1^* - E_1^*)^2}{(A_1^* + B_1^* - 2C_1^*)} \right\}
\] (13)

**EFFICIENCY COMPARISONS**

We compare the proposed estimator with usual sample variance estimator and regression estimator in two-stage sampling as follows:

**Condition (1):** By (2) and (9):
\[
MSE(\hat{S}_{P(2s)}^2)_{min} < MSE(\hat{S}_{0(2s)}^2) \quad \text{if} \quad \gamma \beta_{2(\gamma y)}^* + \sum_{i=1}^{2} \gamma_{2i} \beta_{2(\gamma y)}^* (1 - \rho_{2i}) + \frac{(A_1^* + B_1^* - 2C_1^*)^2}{A_1^* + B_1^* - 2C_1^*} - 1 > 0
\]

**Condition (2):** By (4) and (9):
\[
MSE(\hat{S}_{P(2s)}^2)_{min} < MSE(\hat{S}_{reg(2s)}^2) \quad \text{if} \quad \gamma \beta_{2(\gamma y)}^* (1 - \rho_{2i}^2) + \sum_{i=1}^{N} \gamma_{2i} \beta_{2(\gamma y)}^* (1 - \rho_{2i}^2) + \frac{(A_1^* + B_1^* - 2C_1^*)^2}{A_1^* + B_1^* - 2C_1^*} - 1 > 0
\]

**Condition (3):** By (2) and (13):
\[
MSE(\hat{S}_{P(2s)}^2)_{min} < MSE(\hat{S}_{reg(2s)}^2) \quad \text{if} \quad \gamma \beta_{2(\gamma y)}^* + \sum_{i=1}^{N} \gamma_{2i} \beta_{2(\gamma y)}^* \left\{1 + B_1^* - 2E_1^* - \frac{(B_1^* - C_1^* + D_1^* - E_1^*)^2}{(A_1^* + B_1^* - 2C_1^*)} \right\} > 0
\]

**Condition (4):** By (4) and (13):
\[
MSE(\hat{S}_{P(2s)}^2)_{min} - MSE(\hat{S}_{reg(2s)}^2) \quad \text{if} \quad \gamma \beta_{2(\gamma y)}^* (1 - \rho_{2i}^2) + \sum_{i=1}^{N} \gamma_{2i} \beta_{2(\gamma y)}^* (1 - \rho_{2i}^2) - \left\{1 + B_1^* - 2E_1^* - B_1^* - C_1^* + D_1^* + E_1^* - 2C_1^* \right\} > 0
\]

**Note:** The proposed estimator will be more efficient than the usual variance and regression estimators in two-stage sampling under two cases, when above Conditions (1)-(4) are satisfied.

**Numerical illustration:** We use the following real data sets to observe the performances of estimators.

**Population:** (Sarndal *et al.*, 1992)

\( y \) : Revenues from the 1985 municipal taxation
\( x \) : 1975 population for \( M = 284 \) municipalities (\( ssus \)) divided into \( N = 50 \) clusters (\( psus \))

We use the following expression to obtain the Percent Relative Efficiency (\( PRE \)):
\[
PRE = \frac{MSE(\hat{S}_{0(2s)}^2)}{(\cdot)} \times 100
\]

where, (\( \cdot \)) denote the \( MSE(\hat{S}_{0(2s)}^2), \text{MSE}(\hat{S}_{reg(2s)}^2), \text{MSE}(\hat{S}_{P(2s)}^2)_{min} \text{ and } \text{MSE}(\hat{S}_{P(2s)}^2)_{min} \)
From every selected population, a second-stage sample of different sizes, ssus was again selected. Thus, we had 10,000 independent samples each of different sizes. For each sample from 1 to 10,000, values of the estimators were computed and then on the basis of these values calculated. Percentage Relative Efficiency (PRE) of different estimators with respect to \( s^2_{\text{psus}} \) for different sample sizes are given in Table 1.

### Table 1: Percent relative efficiency of different estimators with respect to \( s^2_{\text{psus}} \)

<table>
<thead>
<tr>
<th>Population</th>
<th>( m_i )</th>
<th>( n )</th>
<th>( s^2_{\text{psus}} )</th>
<th>( s^2_{\text{g2(2s)}} )</th>
<th>( s^2_{\text{reg(2s)}} )</th>
<th>( s^2_{\text{p2(2s)}} )</th>
<th>( s^2_{\text{P(2s)}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical study</td>
<td>3</td>
<td>100</td>
<td>724.86</td>
<td>1,734.11</td>
<td>409.15</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>100</td>
<td>777.87</td>
<td>957.11</td>
<td>352.62</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>100</td>
<td>877.59</td>
<td>1,324.65</td>
<td>396.15</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>641.99</td>
<td>841.55</td>
<td>353.40</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>100</td>
<td>671.00</td>
<td>785.75</td>
<td>342.38</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>100</td>
<td>727.40</td>
<td>895.86</td>
<td>220.65</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td>Simulated study</td>
<td>3</td>
<td>10</td>
<td>560.66</td>
<td>2,918.10</td>
<td>224.94</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>100</td>
<td>309.53</td>
<td>1,232.71</td>
<td>215.86</td>
<td>1,004.95</td>
<td>363.55</td>
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<tr>
<td></td>
<td>30</td>
<td>100</td>
<td>261.57</td>
<td>1,034.35</td>
<td>221.29</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
<tr>
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<td>4</td>
<td>10</td>
<td>460.29</td>
<td>2,403.23</td>
<td>226.64</td>
<td>1,004.95</td>
<td>363.55</td>
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<td>100</td>
<td>309.53</td>
<td>1,232.71</td>
<td>215.86</td>
<td>1,004.95</td>
<td>363.55</td>
</tr>
</tbody>
</table>

For simulation study, we selected 10,000 independent first-stage samples of different sizes from a population. From every selected psus, a second-stage sample of different sizes, ssus was again selected. Thus, we had 10,000 independent samples each of different sizes. For each sample from 1 to 10,000, values of the estimators were computed and then on the basis of these values simulated. Percentage Relative Efficiency (PRE) of different estimators with respect to \( s^2_{\text{psus}} \) for different sample sizes are given in Table 1.

### CONCLUSION

We proposed an improved exponential ratio type estimator for population variance in two stage sample under two cases:

- When sum of the weights cannot equal to one
- When sum of the weights are equal to one

Percentage Relative Efficiency (PRE) of different estimators for different sample sizes are given in Table 1. Both numerical and simulation studies show the same behavior of results.

From Table 1, we observed that the proposed estimator \( \hat{s}^2_{\text{P(2s)}} \) under Case 1 performs better than the usual sample variance estimator \( s^2_{\text{psus}} \) and regression estimator \( \hat{s}^2_{\text{reg(2s)}} \). The proposed estimator \( \hat{s}^2_{\text{P(2s)}} \) under Case 2 is better than the usual sample variance estimator \( s^2_{\text{psus}} \) but show the weaker performance than the traditional regression estimator \( s^2_{\text{reg(2s)}} \). So it is preferable to use the estimator \( \hat{s}^2_{\text{P(2s)}} \) under Case 1 for future study.

### REFERENCES


