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On the Fundamental Theorems of Γ- Regular Ring

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Abstract: In this paper, we have defined Γ - Regular ring and characterization of Γ- Regular ring have been established analogous to Barnes Γ - ring and Von – Neumann regular ring. Further we have defined ideal in Γ - Regular ring and fundamental theorems have been determined analogous to the fundamental theorem on rings found in Herstein.

Key words: Homomorphism, ideals, isomorphism, Γ - ring, regular ring, Γ- Regular ring.

INTRODUCTION

An element a of a ring R is said to be regular if and only if there exist an element x of R such that $axa = a$. The ring R is regular if and only if each element of R is regular. The concept of regular ring was introduced by Von Newmann (1936). According to Barnes (1966) Γ - ring have been defined in the following way. Let R and Γ be two additive abelian groups. If for all $x, y, z \in R$ and for all, $\alpha,\beta \in \Gamma$ the conditions

(1) $x\alpha y \in R$
(2) $(x+y)\alpha z = x\alpha z + y\alpha z; x(\alpha+\beta)z = x\alpha z + x\beta z$
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, and then R is called a Γ - ring. That is R is a ring with respect to Γ. Authors like Nobusawa (1964), Ravishankar *et al,* (1979) and Barnes (1966) himself have extended many fundamental results in ring theory to Γ - ring. Further Hatice Kandamar (2000) has studied K-derivation of a Gamma Ring. Here we have developed regular ring, Γ - ring to Γ - regular ring and fundamental theorems on Γ - regular ring have been established.

Definition: 1.1: Let R and Γ be two additive abelian groups. An element a ∈ R is said to be Γ - regular if and only if there exists an element x∈ Γ such that $axa = a$. A Γ - ring is said to be a Γ - regular if and only if each element of R is
Γ - regular
.

Definition: 1.2: If R is a Γ - ring and Γ is a R – ring, then (R, Γ) is called a ring pair. Throughout, we consider (R, Γ) to be a ring pair.

Lemma: 1.1: Let (R, Γ) be a ring pair. For a∈ Γ and y∈ Γ such that $a - aya$ is Γ - regular, then a is Γ - regular.

Proof: If $a - aya$ is Γ - regular, then there exists y,z∈ Γ such that $(a - aya)\, z\, (a - aya) = (a - aya)$ shows that $axa = a$. Thus, a is Γ - regular.

Definition: 1.3: Let (R, Γ) be a ring pair. Let $R_{m\times n}$ be the set of all mxn matrices over the Γ - ring R. Let $\Gamma_{n\times m}$ be the set of all nxm matrices over the R – ring Γ . A ring $\Gamma_{n\times n}$ is said to be Γ - regular ring if for $A \in R_{n\times m}$, there exists $X \in \Gamma_{n\times m}$ such that AXA = A.
Let R_n denotes the square matrix of order n over the Γ - ring R.

Theorem: 1.1: Let R, Γ be a ring pair. R is a Γ - regular ring if and only if R_n is a Γ - regular ring.

Proof: If R is a Γ - regular, then for $x \in R$, there is an element $\alpha \in \Gamma$ such that $xax = x$. Now, let

A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_2$ and $X = \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix}$ $Y = \begin{pmatrix} g' & 0 \\ 0 & i' \end{pmatrix}$

$Z = \begin{pmatrix} 0 & k' \\ 0 & 0 \end{pmatrix} \in \Gamma_2$ and denote $A - AXA = B$, $BYB - B = C$ and $C - CZC = 0$. Thus, C is a Γ - regular in R_2. Hence by lemma 1.1, B is a Γ - regular in R_2. Again applying by lemma 1.1, A is a Γ - regular in R_2.

If A ∈R_n let A_1 be the matrix of R_{2^k} with A in the upper left hand corner and zero's elsewhere. Now, as element of R_{2^k} A_1 is Γ - regular, then there exists a element X = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{2^k}$ Such that $A_1 X A_1 = A$.

A - AXA =0 and hence A is Γ - regular.
Conversely, if R_n is Γ - regular, then it is clear that R is Γ - regular.

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Theorem: 1.2: Let (R, Γ) be a ring pair. R is a Γ - regular if and only if is a Γ - regular ring.

Proof: By theorem 1.1 $R_{n\times n}$, is Γ - regular ring. Since $R_{m\times m}$ is a Γ - regular ring for all $B \in R_{m\times m}$, there exists $X \in \Gamma_{m\times m}$ such that $B\ X\ B = B$. For $A \in R_{m\times n}$ there exists $Y \in \Gamma_{n\times m}$ such that $A\ Y\ A = A$ by defining $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $A = \begin{pmatrix} A & 0 \end{pmatrix}$. Since A is arbitrary in $R_{m\times n}$, it follows that $R_{m\times n}$ is Γ - regular ring. Similarly, we have to prove the converse part also by using theorem 1.1.

Ideals in - regular ring:

Definition: 2.1: A subset S of the Γ - regular ring R is a right ideal (left ideal) of R, if S is a subgroup of R and $S\ \Gamma\ R = \{a\alpha c : a \in S, \alpha \in \Gamma, c \in R\}$ $\left[R\Gamma S = \{c\alpha a : a \in S, \alpha \in \Gamma, c \in R\}\right]$ is contained in S.

A subset S in a Γ - regular ring R is a two sided ideal or an ideal in Γ - regular ring R if it contain both left and right ideals.

Definition: 2.2: If A and B are ideals in Γ - regular ring R, then the sum of A and B is also an ideal of R. i.e., $A + B = \{a + b : a \in A, b \in B\}$.

Definition: 2.3: Let A be an ideal in Γ - regular ring R, then the set R/A is defined by R/A = {x + a αc: $x, a, c \in R, \alpha \in \Gamma$ } under the operation (+, •) is defined as follows

i) (x + a α c) + (y + a α c) = (x + y) + a α c
ii) (x + a α c) α (y + a α c) = (x α y) + a α c.

Then, (R/A , +, •) form a Γ - regular ring R.

Definition: 2.4: Let R and R' be Γ - regular rings. Let φ be a mapping from R → R' , then if (aαc) = bάc.

If φ is a homomorphism of $R \to$ Ŕ , then the kernel of homomorphism of φ is defined by Ker φ = {$r \in R$: φ(r) = ó} where 0 is the zero ideal of R and ó is the zero ideal of Ŕ.

Fundamental Theorem on homomorphism in Γ- regular rings

Theorem: 2.1: Every homomorphic image of a Γ - regular ring R is isomorphic to some residue class ring analogous to Herstein (1975).

Proof: Let Ŕ be the homomorphic image of a Γ - regular ring R and be the corresponding homomorphism. Then, f is a homomorphism of R onto Ŕ. Let K be the kernel of this homomorphism. Then K is an ideal of R. Then, R/K is a Γ - regular ring. We shall prove R/K .

Consider the mapping: $R/K \cong$ Ŕ

Consider the mapping $\varphi: R/K \to$ Ŕ such that

$\varphi(k + a\alpha c) = f(a\alpha c)\ \forall\ a, c \in R, \alpha \in \Gamma$.

To prove:

(i) φ is well defined and one to one.
(ii) φ[(k + aαc) + (k + bαc)] = φ (k + aαc) + (k + bαc)
(iii) φ(x + aαc) α(y + aαc) = φ(x + aαc)φ (α)φ(y +aαc),

it will be proved by usual way of Fundamental theorem on homomorphism of groups. Hence φ is isomorphic from R/K ≅ Ŕ .

Theorem: 2.2: Let R and R' be Γ - regular rings. and φ be a homomorphism from R onto Ŕ with kernel k. Then, R/$K \cong R'$. Moreover there exist one to one correspondences between the set of ideals of and the set of ideals R' which contains the same K. This correspondence can be achieved by associating with an ideal with W' in R' to an ideal W defined by W = { $x \in R: \varphi(x) \in W'$ } in R. Then, R/$W \cong R'/W'$ (or). $R'/W' \approx \dfrac{R/K}{W/K}$

Proof: Consider the mapping $\psi: R \to R'/W'$ is defined by $\psi(a\alpha c) = W' + \varphi(a\alpha c)$

To prove:

(i) ψ is well defined and one to one.
(ii) ψ(a_1αc +a_2αc) = ψ(a_1αc) + ψ(a_2αc) and
(iii) ψ[(a_1αc) α (a_2αc)] = ψ(a_1αc) ψ(α) ψ(a_2αc)

it will be proved by usual way of Fundamental theorem on homomorphism of groups. The kernel of homomorphism is defined by

Ker ψ = { $x \in R: \varphi(x) \in W'$ }

Now $x \in Ker\,\psi \Leftrightarrow \psi(a\alpha c) =$

$\Leftrightarrow \psi(a\alpha c) + W' = W'$

$\Leftrightarrow \varphi(a\alpha c) \in W'$

$\Leftrightarrow x \in W'$

Thus, $\psi: R \to \dfrac{R'}{W}$ is a homomorphism with kernel K.

By using Theorem 2.1, we get $\dfrac{R}{K} \approx R'$.

By restricting the homomorphism between the ideals W of R and W' of R' of having the same kernel K and again by applying Theorem 2.1, We obtain $\dfrac{W}{K} \approx W'$ and hence $R'/W' \approx \dfrac{R/K}{W/K}$.

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