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Estimation of Reliability in Non-accumulating Damage Shock Model for Two-component(non I.i.d.) Parallel System

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Abstract: A parallel system of two components is subjected to a sequence of shocks causing damage to both the components of the system. The maximum likelihood estimator and Bayes estimator of reliability function are obtained under non-accumulating damage shock model setup. The computation of estimators and their comparison is also considered.

Key words: Bayes estimator, fatal shock, life testing experiment, maximum likelihood estimator

INTRODUCTION

Arrangement of components in parallel is one of the basic methods of increasing reliability. The interest in two-component parallel system currently is exhibited by health scientists, engineers, mathematicians, economists and administrators. We come across several two-component parallel systems in our day-today life such as pair of eyes, pair of kidneys, pair of hands and legs, pair of elevators, pair of engines in aircrafts, pair of directors and deputy directors to run a firm, etc.. So, it is necessary to develop mathematical methods to assess the reliability of the two- component parallel systems.

There have been several formulations of bivariate exponential and related distributions. These include the distributions of (Gumbel, 1960; Freund, 1961; Marshall and Olkin, 1967; Downton, 1970; Basu, 1971; Hawkes, 1972; Block and Basu, 1974; Weier, 1981 and Klein and Basu, 1985) models. Esary et al.(1973) studied some shock models and their properties. Barlow and Proschan (1975) have considered shock models yielding bivariate distributions. Kunchur and Munoli (1993) have considered the estimation of reliability in cumulative damage shock model.

Suppose the system of two-components each with thresholds u_1 and u_2 is subjected to a sequence of shocks from a single source causing damage to both the components. Shocks are occurring randomly in time as events of a Poisson process of intensity λ, $\lambda > 0$. Let X_j and Y_j denote the damages caused due to j-th shock to the two components respectively and X_j , Y_j have exponential distribution with means θ_1 and θ_2, θ_1 , $\theta_2 > 0$. The component fails whenever the damage due to a shock exceeds the threshold of the component. If the damage does not exceeds the threshold of the component, then the component works as good as new one. After failure of one of the two components the surviving component receives shocks with the same intensity. The system fails when both the components fail (parallel system).

The reliability function and the life testing experiment are considered in Section 2. Estimators of reliability function and their computations are considered in Section 3. Findings of the study are discussed in Section 4.

MATERIALS AND METHODS

Reliability Function: The life distribution of the model discussed in Section 1 is given by

$$H(t) = \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} P_n \tag{1}$$

where

$$P_n = \sum_{k=1}^{n} \left(1-e^{-\frac{u_1}{\theta_1}}\right)^{k-1} e^{-\frac{u_1}{\theta_1}} \left(1-e^{-\frac{u_2}{\theta_2}}\right)^{k-1} e^{-\frac{u_2}{\theta_2}}$$

$$+ \sum_{k<l}^{n} \left(1-e^{-\frac{u_1}{\theta_1}}\right)^{l-1} e^{-\frac{u_1}{\theta_1}} \left(1-e^{-\frac{u_2}{\theta_2}}\right)^{k-1} e^{-\frac{u_2}{\theta_2}}$$

$$+ \sum_{k>l}^{n} \left(1-e^{-\frac{u_1}{\theta_1}}\right)^{l-1} e^{-\frac{u_1}{\theta_1}} \left(1-e^{-\frac{u_2}{\theta_2}}\right)^{k-1} e^{-\frac{u_2}{\theta_2}}$$

$$= \sum_{k=l}^{n} \left(1-e^{-\frac{u_1}{\theta_1}}\right)^{k-1} e^{-\frac{u_1}{\theta_1}} \left(1-e^{-\frac{u_2}{\theta_2}}\right)^{k-1} e^{-\frac{u_2}{\theta_2}}$$

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$$+\sum_{k=l}^{n}\sum_{l=1}^{k-1}\left\{\left(1-e^{-\frac{u_1}{\theta_1}}\right)^{l-1}e^{-\frac{u_1}{\theta_1}}\left(1-e^{-\frac{u_2}{\theta_2}}\right)^{k-1}e^{-\frac{u_2}{\theta_2}}\right.$$

$$\left.+\left(1-e^{-\frac{u_1}{\theta_1}}\right)^{k-1}e^{-\frac{u_1}{\theta_1}}\left(1-e^{-\frac{u_2}{\theta_2}}\right)^{l-1}e^{-\frac{u_2}{\theta_2}}\right\} \tag{2}$$

Substituting this probability in (1) and interchanging the order of summations over n and k, we have

$$H(t)=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{(\lambda t)^n}{n!}\left\{e^{-\left(\frac{u_1}{\theta_1}+\frac{u_2}{\theta_2}\right)}\left(1-e^{-\frac{u_1}{\theta_1}}\right)^{k-1}\right.$$

$$\left(1-e^{\frac{u_2}{\theta_2}}\right)^{k-1}+\sum_{i=1}^{k-1}e^{-\left(\frac{u_1}{\theta_1}+\frac{u_2}{\theta_2}\right)}\left[\left(1-e^{-\frac{u_1}{\theta_1}}\right)^{k-1} \tag{3}\right.$$

$$\left.\left.\left(1-e^{-\frac{u_2}{\theta_2}}\right)^{l-1}+\left(1-e^{-\frac{u_1}{\theta_1}}\right)^{l-1}\left(1-e^{-\frac{u_2}{\theta 2}}\right)^{k-1}\right]\right\}$$

and the reliability of the system at mission time τ is given by

$$R(\tau) = 1 - H(\tau). \tag{4}$$

Life Testing Experiment: Suppose r systems each of two components (say A and B) having life distribution H(t) given in (3) are subjected to life testing experiment. The experiment is conducted until all the systems fail. The i-th system fails when both the components of the system fail. Let for i-th system, component A fails due to m_i-th shock and component B fails due to n_i-th shock. Let $X_{i1}, X_{i2}, \ldots, X_{im_i}$ and $Y_{i1}, Y_{i2}, \ldots, Y_{in_i}$ be damages due to shocks to the two components A and B respectively. Let X_{ij}'s and Y_{ij}'s be exponentially distributed random variables with means θ_1 and θ_2, θ_1, $\theta_2 > 0$. When component A of i-th system fails due to m_i-th shock, we have $X_{ij} < u$, j = 1,2,…,m_i-1 and $X_{im_i} > u_1$ (fatal shock). Similarly, $Y_{ij} < u$, j = 1,2,…,n_i-1 and $Y_{in_i} > u_2$ (fatal shock). Let $s_i = \max(m_i, n_i)$ and $t_{i1}, t_{i2}, \ldots, t_{is_i}$ be the time epochs at which the system experiences the shocks. The inter-arrival times $(t_{i1}-t_{i0})$, $(t_{i2}-t_{i1})$, …, $\left(t_{is_i}-t_{is_i-1}\right)$ are exponential random variables with parameter λ, $\lambda>0$. The joint pdf $f(\bar{s},\bar{t},\bar{x},\bar{y}|\lambda,\theta_1,\theta_2)$ of the random variables $s_i, t_{i1}, t_{i2}, \ldots, t_{is_i}, X_{i1}, X_{i2}, \ldots X_{im_i-1}$ and $Y_{i1}, Y_{i2}, \ldots, Y_{in_i-1}$ for all the r systems is given by

$$L=\lambda^{\dot{s}}e^{-\lambda t}\left[\frac{1}{\theta_1}\right]^{\dot{m}-r}\left[\frac{1}{\theta_2}\right]^{\dot{n}-r}.e^{\frac{1}{\theta_1}(\ddot{x}+\dot{m}_1)-\frac{1}{\theta_2}(\ddot{y}+\dot{m}_2)} \tag{5}$$

with $\dot{s}=\sum_{i=1}^{r}s_i$, $\dot{m}=\sum_{i=1}^{r}m_i$, $\dot{n}=\sum_{i=1}^{r}n_i$, $t=\sum_{i=1}^{r}t_{is_i}$,

$$\ddot{x}=\sum_{i=1}^{r}\sum_{j=1}^{m_i-1}x_{ij} \text{ and } \ddot{y}=\sum_{i=1}^{r}\sum_{j=1}^{n_i-1}y_{ij}$$

Using (5), the maximum likelihood estimators (MLE)s of parameters are obtained as

$$\hat{\lambda}=\frac{\dot{s}}{\dot{t}},\ \hat{\theta}_1=\frac{\ddot{x}+ru_1}{\dot{m}-r} \text{ and } \hat{\theta}_2=\frac{\ddot{y}+ru_2}{\dot{n}-\dot{r}} \tag{6}$$

RESULTS AND DISCUSSIONS

Estimators of Reliability Function: Using the invariance property of MLEs, the MLE of reliability function can be obtained.

In order to obtain the Bayes estimator of the reliability function $R(\tau)$, consider the prior distributions of λ, θ_1 and θ_2 (Box and Tiao, 1973) as

$$g(\lambda)=\mu e^{-\mu\lambda},\ \mu>0 \tag{7}$$

$$g_1(\theta_1)=\frac{k_1 e^{-\frac{\alpha_1}{\theta_1}}}{\theta_1^{c_1}},\ \theta_1>0 \tag{8}$$

and

$$g(\theta_2)=\frac{k_2 e^{-\frac{\alpha_2}{\theta_2}}}{\theta_2^{c_2}},\ \theta_2>0 \tag{9}$$

where k_i's are obtained such that $\int_0^{\infty} g_i(\theta_i)d\theta=1$, i = 1,2.

Using (5), (7), (8) and (9), the joint pdf of $\bar{s},\bar{t},\bar{x},\bar{y},\lambda,\theta_1$ and θ_2 is given by

$$f_1(\bar{s},\bar{t},\bar{x},\bar{y},\lambda,\theta_1,\theta_2)=f(\bar{s},\bar{t},\bar{x},\bar{y}|\lambda,\theta_1,\theta_2)\,g(\lambda)g_1(\theta_1)g_2(\theta_2) \tag{10}$$

Integrating $f_1(\bar{s},\bar{t},\bar{x},\bar{y},\lambda,\theta_1,\theta_2)$ with respect to λ, θ_1 and θ_2 over their respective ranges, we get the distribution of $(\bar{s},\bar{t},\bar{x},\bar{y})$ as $f_2(\bar{s},\bar{t},\bar{x},\bar{y})$. Dividing $f_1(\bar{s},\bar{t},\bar{x},\bar{y},\lambda,\theta_1,\theta_2)$ by $f_2(\bar{s},\bar{t},\bar{x},\bar{y})$, we get the posterior distribution of λ, θ_1 and θ_2 as

$$f_3(\lambda,\theta_1,\theta_2|\bar{s},\bar{t},\bar{x},\bar{y})$$

Table 1: ML $\left(\hat{R}(\tau)\right)$ and Bayes $\left(R_B(\tau)\right)$ Estimates of Reliability Function for r=5 $\theta_1 = 3.0\ \theta_2 = 4.0\ u_1 = 3.0\ u_2 = 4.0$

λ	τ	R(τ)	$\dot{m}$	$\dot{n}$	$\dot{s}$	$\dot{t}$	$\ddot{x}$	$\ddot{y}$	$\hat{R}(\tau)$	$R_B(\tau)$
1.0	0.5	0.923336	10	10	13	12.4280	20.8479	36.5819	0.934617	0.931290
1.0	1.0	0.835859	14	13	18	13.4484	58.0395	41.7865	0.853412	0.843165
1.0	1.5	0.745601	13	15	19	22.4426	36.5732	46.8039	0.761387	0.751539
1.0	2.0	0.657538	16	18	22	16.2156	46.7337	58.3465	0.674114	0.667124
1.5	0.5	0.880388	13	10	16	38.7023	34.7397	38.9291	0.901435	0.893712
1.5	1.0	0.745601	17	14	18	31.3741	42.0189	56.7635	0.758610	0.752301
1.5	1.5	0.615569	16	17	20	28.5975	37.4217	63.0554	0.629468	0.621752
1.5	2.0	0.498264	18	18	21	31.1767	47.1506	56.4729	0.521027	0.510712
2.0	0.5	0.835859	14	12	17	40.1067	29.0491	30.0447	0.854241	0.846290
2.0	1.0	0.657538	17	18	20	54.7594	46.5177	53.1609	0.671283	0.664745
2.0	1.5	0.498264	16	19	21	40.0278	65.8046	60.6969	0.523316	0.512532
2.0	2.0	0.369335	19	18	23	42.3420	74.5037	54.7478	0.382510	0.379254

Table 2: ML $\left(\hat{R}(\tau)\right)$ and Bayes $R_B(\tau)$ Estimates of Reliability Function for r=10 $\theta_1 = 3.0\ \theta_2 = 4.0\ u_1 = 3.0\ u_2 = 4.0$

λ	τ	R(τ)	$\dot{m}$	$\dot{n}$	$\dot{s}$	$\dot{t}$	$\ddot{x}$	$\ddot{y}$	$\hat{R}(\tau)$	$R_B(\tau)$
1.0	0.5	0.923336	17	15	22	25.1719	40.3888	59.05271	0.932453	0.930148
1.0	1.0	0.835859	30	33	40	36.6543	73.4136	106.6064	0.846291	0.839618
1.0	1.5	0.745601	22	34	37	30.1548	65.1581	106.7447	0.760012	0.750227
1.0	2.0	0.657538	29	35	46	38.2738	72.5011	112.4792	0.674134	0.663217
1.5	0.5	0.880388	24	27	33	37.7926	77.4924	84.8767	0.900601	0.889216
1.5	1.0	0.745601	32	33	38	47.4437	92.9269	102.4893	0.758238	0.751423
1.5	1.5	0.615569	39	31	47	67.4226	98.9399	98.2047	0.625618	0.620008
1.5	2.0	0.498264	46	41	53	75.0378	109.8564	124.3527	0.513602	0.509453
2.0	0.5	0.835859	30	33	41	72.4744	72.1991	98.0504	0.847652	0.842130
2.0	1.0	0.657538	40	39	49	84.6124	87.7343	100.9998	0.661325	0.660254
2.0	1.5	0.498264	43	37	52	92.5674	90.7577	109.4305	0.518763	0.511643
2.0	2.0	0.369335	47	51	58	98.7631	101.8654	146.8477	0.37326	0.371252

$$= \left(\Gamma(\dot{s}+1)\Gamma\left(\dot{m}+c_1-r-1\right)\Gamma\left(\dot{n}+c_2-r-1\right)\right)^{-1}$$

$$.\lambda^{\dot{s}}\left(\frac{1}{\theta_1}\right)^{\dot{m}+c_1-r}\left(\frac{1}{\theta_2}\right)^{\dot{n}+c_2-r}\left(\dot{t}+\mu\right)^{\dot{s}+1} \quad (11)$$

$$.\left(\ddot{x}+\alpha_1+ru_1\right)^{\dot{m}+c_1-r-1}\left(\ddot{y}+\alpha_2+ru_2\right)^{\dot{n}+c_2-r-1} e^{-\left(\lambda(\dot{t}+\mu)+\frac{1}{\theta_1}\left[\ddot{x}+\alpha_1+ru_1\right]+\frac{1}{\theta_2}\left[\ddot{y}+\alpha_2+ru_2\right]\right)}$$

The Posterior expectation of R(τ) is the Bayes estimator of reliability function and is given by

$$R_B(\tau) = \int_{\lambda}\int_{\theta_1}\int_{\theta_2} R(\tau) f_3\left(\lambda, \theta_1, \theta_2 \mid \bar{s}, \bar{t}, \bar{x}, \bar{y}\right) d\theta_2\, d\theta_1\, d\lambda \quad (12)$$

Computation of Estimators: For the i-th system, the random variables m_i, n_i, s_i, t_{i1}, t_{i2}, ..., t_{is_i}, x_{i1}, x_{i2}, ..., x_{im_i-1}, y_{i1}, y_{i2}, ..., y_{in_i-1} are generated as follows:

Step1: Initialize m_i with zero.

A uniform random number V_1 is generated from U(0,1). m_i is incremented by 1. For given value of $\theta_1 = \theta_1^0$ of , an exponential random variable x_{i1} is obtained as $x_{ij} = -\theta_1^0 \ln(1-v_1)$. Here x_{i1} denotes the amount of damage due to 1-st shock to the 1-st component of the i-th system. If ($x_{i1} < u_1 = u_1^0$), then the procedure is repeated until we generate an exponential random variable $x_{ij} > u_1^0$. The values of m_i and x_{ij}, j = 1, 2, . . . m_i -1 are noted.

Step2: Step 1 is repeated with , $u_2 = u_2^{\circ}$ and the values of n_i and y_{ij}, j = 1, 2, . . . n_i -1 are noted.

Step 3: Set $s_i = \max(m_i, n_i)$ and s_i number of exponential random variables each with parameter are generated. That is s_i number of inter-arrival times ($t_{i1} - t_{i0}$), ($t_{i2} - t_{i1}$), . . . $\left(t_{is_i} - t_{is_i-1}\right)$ are generated. Using these inter-arrival times, the time epochs t_{i1}, t_{i2}, ..., t_{is_i} at which the shocks arrive are generated. Thus we have obtained data on m_i, n_i, s_i, t_{i1}, t_{i2}, ..., t_{is_i}, x_{i1}, x_{i2}, ..., x_{im_i-1}, y_{i1}, y_{i2}, ..., y_{in_i-1}. The whole procedure is repeated r number of times and the statistics

$$\dot{m} = \sum_{i=1}^{r} m_i,\ \dot{n} = \sum_{i=1}^{r} n_i,\ \dot{s} = \sum_{i=1}^{r} s_i,$$

$$\ddot{x} = \sum_{i=1}^{r}\sum_{j=1}^{m_i-1} x_{ij},\ \ddot{y} = \sum_{i=1}^{r}\sum_{j=1}^{n_i-1} y_{ij},\ \dot{t} = \sum_{i=1}^{r} t_{is_i},$$

are computed. With the help of these statistics, the MLE of parameters of the model are obtained. Using these MLEs in the expression for reliability function, the MLE of reliability function is obtained for τ = 0.5 (0.5) 2.0,

λ=0.5(0.5)1.0 with u_1=3.0, u_2=4.0, ,. For these parametric values, the reliability function is also obtained.

The Bayes estimator of reliability function is obtained using the simulated values of $\dot{m}, \dot{n}, \dot{s}, \dot{t}, \ddot{x}$ and $\ddot{y}$ and in (12) with μ =1.0, $\alpha_1 = \alpha_2$ = 1.0 and $c_1 = c_2 = 2$.
Table 1 and Table 2 give the results of the above simulation experiment for r =5 and r =10 respectively.

CONCLUSIONS

From the tables it is clear that for greater value of r (large sample size) both MLE and Bayes estimators perform better. Though in majority of the cases both the estimators overestimate the Reliability Function R(t), the Bayes estimator is a better estimator in terms of bias for this data set.

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