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Parallel Iterations for Cholesky and Rohn's Methods for Solving over Determined Linear Interval Systems

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Abstract: The aim is to present Krawczyk's iteration in the context of parallel Rohn's method and another parallel Interval Cholesky method to improve results for the solution of over determined linear interval system. We compare note with results obtained from using the Block Krawczyk iterative method introduced by Rump, and presented in details by Popova, which forms the basis of discussions. It is shown that parallel Rohn's method which is independent of interval data input gave narrowest interval width of the solution set to the over determined interval linear system followed by Interval Cholesky method while Block Krawczyk's method given in Popova, has the largest interval width.

Key words: AMS Subject Classification (2000), interval Cholesky method, Krawczyk's method, over determined linear interval system, Rohn's method, 65G20 and 65G30

INTRODUCTION

We signify with the following notation R^n, IR^n, R^{nxn}, IR^{nxn} as representing the sets of real vectors, interval vectors, real matrices and interval matrices, respectively. We assume that the interval matrix $A \in [A]$ is an mxn matrix where m > n.

We are interested in the solution of over determined linear system with interval data:

$$Ax = b \quad (1)$$

where the interval entries of A and b are described by:

$A = (A_c - \Delta, A_c + \Delta)$
$b = (b_c - \delta, b_c - \delta)$

Such that:

$$A_c = \tfrac{1}{2}\left(\underline{A} + \overline{A}\right)$$

$$x_c = \tfrac{1}{2}\left(\underline{x} + \overline{x}\right)$$

$$b_c = \tfrac{1}{2}\left(\underline{b} + \overline{b}\right)$$

The terms Δ and δ denote respectively the radius an interval matrix and interval vector. Such situation arises when we describe a dynamical system from state

$S(a) = (S_1(a), \ldots, S_n(a))$ at a given time to the state
$S(a+1) = (S_1(a-1), \ldots, S_n(a+1))$
at the epoch of time a+1 which leads to dependence
$S_n(a+1) = f_i(S_1(a), \ldots, S_n(a))$

This means that the parameters describing the coefficients of the linear system are afflicted with uncertainties. We define the set:

$$\Sigma_{m\times n} = \left\{x_c \in R^n \middle| A \in [A], \exists b \in [b], x_c = \arg\min_{x\in R} \|b - Ax\|\right\}$$
$$= \left\{x \in R^n \middle| A \in [A], \exists b \in [b], A^H Ax = A^H b\right\}$$

(Rump, 1993) proposed and solved the over determined system by considering the augmented square linear system:

$$\begin{pmatrix} [A] & -I \\ 0 & [A]^H \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} [b] \\ 0 \end{pmatrix}, m > n \quad (2)$$

Following the approach presented in (Rump, 1993), the following theorem was proposed by (Popova, 2006).

Theorem 1: (Popova, 2006).

Let $[A] \in IR^{m\times n}, [b] \in IR^n, m > n$

Defining:

$$[B] \in IR^{((m+n)x(m+n))}, [V] \in IR^{m+n}$$

representing a square interval matrix, respectively, the interval vector in (2) and letting,

$$R \in R^{(m+n)x(m+n)}, \overline{U} \in R^{m+n}$$

(i) Let $[Z] \in IR^{m+n}$ be defined in the form:

$$[Z]_{l=1}^{m+n} = \sum_{\mu}^{m} r_{iu}\left((b_u) + \overline{\mu}_{n+m}\right)$$

$$- \sum_{w=l}^{} \sum_{v=l}^{} \left(r_{iu}\overline{U}_v + r_{i,m+v}\overline{U}_{n+\mu}\right) R_e\left(\left[a_{\mu v}\right]\right)$$

$$- \sum_{\mu=1}^{m} \sum_{v=1}^{n} \left(r_{iu}\overline{\mu}_v - r_{i,m+v}\overline{\mu}_{n-w}\right) I_m\left(\left[a_{uv}\right]\right)$$

(ii) Let $[C] \in IR^{(m+n)x(m+n)}$ be defined by I-R[B] and let $[u] \in IR^{m+n}$.

Assuming $[w] \in IR^{m+n}$, where

$$[w_i]_{i=1}^{m+n} = \left\{[Z] + [C].[uu]\right\}_{il}$$

$$[uu] = \left([w_1], \ldots, [w_{i-1}], [u_i], \ldots, [u_{m+n}]\right)^T$$

If $[w] \subsetneq [u]$ then every matrix A ∈ [A] has full rank n, and for every A ∈ [A], b ∈ [b] with x_c minimizing $||b - Ax||$, the unique solution x_c satisfies $\tilde{x} \in x_c + [x]$ where $\tilde{x}$ and [x] are the first n components of u_c, resp [w].

(iii) With $[\Delta] = [C][w] \in IR^{m+n}$ and $[xx] \in IR^n$ defined by

$$[xx_i] = \inf([z_i]) + Sup([\Delta_i])$$

$$\sup\left([Z_i] + \inf([\Delta_i])\right), 1 \le i \le n,$$

The following inner and outer estimations were obtained by (Popova, 2006) in the form:

$$x_c + [xx] \subseteq (\Sigma_{m \times n}) \subseteq x_c + [x]$$

In this work, we are interested in using parallel Cholesky Factorization for the algebraic inclusion of over determined linear interval systems of equations. A procedure for speeding up the Interval Cholesky Factorization using a method described in (Gargantini, 1980) in conjunction with (Carstensen and Petkovic, 1994) and of course (Rump, 1999) to narrow the overestimation of resulting disks is implied.

We introduce a theorem due to (Rohn, 1996) which is free of interval input in section three which can be interpreted as a variant of (Rump, 1993) approach. The convergence of Rohn's method is dependent on the existence of an approximation matrix of the Moore penrose inverse of A.

Finally, numerical example taken from (Popova, 2006) is used to demonstrate the quality of inclusion bounds provided by these methods. We concluded the paper based on the findings with results from these methods.

MATERIALS AND METHODS

We use parallel interval Cholesky method and parallel (Rohn, 1996)'s method as our basis of experiments. Below we give mathematical details of the methods.

Parallel interval Cholesky method: We discuss parallel interval Cholesky method by replacing real entries and arithmetic operations with interval ones. The idea is based on the extended interval arithmetic, which dates backs to (Gargantini, 1976) and further studied by (Rump, 1993). Using the notation

$$[A] = \left[\underline{A}, \overline{A}\right] = a_{ij} = \left(\left[\underline{a}_{ij}, \overline{a}_{ij}\right]\right)$$

where, we can convert the over determined system (1) into equivalent interval linear system of size nxn coefficient matrix by a certain matrix R called a preconditioner by multipling through both sides of system (1).

The choice of R is guided by the following definition:

$$R = \left(A_C^T A_C\right)^{-1} A_C^T \quad (3)$$

where, A^T denotes the transpose of A.
Assuming the system (1) is in preconditioned form,

$$Mx = r \quad (4)$$

where, $M_{i,j} \in M$ and $r_i \in [b]$, we introduce the interval Cholesky method (see also (Alefeld and Mayer, 2007) in the form

$$[l]_{jj} = \left([m]_{jj} - \sum_{k=1}^{j-1} [l]_{kj}^2 \right)^{1/2} \quad (5)$$

$$[l]_{ij} = \left([m]_{ij} - \sum_{k=1}^{j-1} [l]_{ik}[l]_{jk} \right) \Big/ [l]_{jj} \quad (6)$$

$$i = j+1,\ldots,n, \quad j = 1,2,\ldots,n$$

$$[l]_i = \left([r]_i - \sum_{j=i}^{i-1} [l]_{ij} \right) \Big/ [l]_{ii} \quad (7)$$

$$[x]_i^C = \left([y]_i - \sum_{j=i+1}^{n} [l]_{ji}[x]_j^C \right) \Big/ [l]_{ii} \quad (8)$$

$$i = n, n-1,\ldots,1$$

It is assumed that $0<\underline{L}_{il}$ (i=1,2,…,n) and that interval Cholesky method is feasible.

Definition 1: (Alefeld and Mayer, 2007): Assuming that M has an interval Cholesky Factorization and given that, if either n = 1,

$$[L] = \left(\sqrt{[m]_{ii}} \right)$$

or n>1 and ,

$$[L] = \begin{pmatrix} \sqrt{[m]_{11}} & 0 \\ [C] \Big/ \sqrt{[m]_{11}} & [L]^1 \end{pmatrix}$$

we then define the interval Cholesky Schur complement by the equation:

$$\Sigma_{[m]}^C = [m]^1 - [c][c]^T \Big/ [m]_{11}$$

if n>1 and $0 \notin [m]_{11}$

We call the pair ([L] $[L]^T$) as the interval Cholesky factorization of $\Sigma_{[m]}^C$ Obtaining interval Cholesky factorization of M via interval Gaussian elimination algorithm IGA (M), it is proved by (Alefeld and Mayer, 2007) that:

$$\Sigma_{[m]}^C \subseteq \Sigma_{[m]}^G$$

whenever the interval matrix M:

$$\in [A] = \begin{pmatrix} [m]_{11} & [C]^T \\ [V] & [m]^1 \end{pmatrix}$$

with Schur complement:

$$\Sigma_{[m]}^G = [M]^1 - [V][C]^T \Big/ [m]_{11}$$

and [V] $[C]^T$ = [V] $[C]^T$ exists, where

$$\Sigma_{[m]}^C, \Sigma_{[m]}^G$$

signify Interval Cholesky Factorization and Interval Gaussian algorithm for interval matrix M.

The parallelism of interval Cholesky method emanates from parallel computation of disk inversion via the inclusion isotoniciy of square root by using the procedure described in Petkovic (1989) and Gargantini (1980).

To this end, we define disk $[M_{ij}]$ as the parametric notation:

$$\left([m_{ij}] - m_{ijc} < \eta_j \right) \quad (9)$$

where m_{ijc} is the centre and η_j is the radius of the matrix M_{ij}.

Assuming the disk does not contain the origin i.e. $|m_{ijc}| > \eta_j$, the square root of $[m_{ij}]$ is defined by the equation (Gargantini, 1976; 1980) given that R_1 and R_2 are the roots of the disk in the form:

$$R_1 = \left\{ \sqrt{|m_{ijc}|}\, e^{i\theta/2}, \frac{\eta_j}{\sqrt{|m_{ijc}|} + \sqrt{\left(|m_{ijc}| - \eta_j\right)}} \right\} \quad (10)$$

$$R_2 = \left\{ \sqrt{|m_{ijc}|}\, e^{i(\theta/2+\pi)}, \frac{\eta_j}{\sqrt{|m_{ijc}|} + \sqrt{\left(|m_{ijc}| - \eta_j\right)}} \right\} \quad (11)$$

Where R_1 and R_2 are mutually disjoint and none of them contains the origin.

We can shorten notation for R_1 and R_2 for brevity in the form:

$$R_1 = \left(\psi_1,\ w_1\right), R_2 = \left(-\psi_1, w_1\right)$$

The choice of the roots to be taken in the square root formulas expressed in Eq. (10) and (11) is guilded by the procedures discussed in (Petkovic, 1989).

For the above square root Eq. in (10) and (11) to be effective, the centered disk inversion due to (Carstensen and Petkovic, 1994) has been introduced in the form:

$$m_{ij} = \left\{\psi_1, w_1\right\} = \left\{\frac{1}{m_{ijc}}, \frac{2w_1}{\left|m_{ijc}\right|^2 - \alpha_1^2}\right\} \qquad m_{ijc} > w_1 \tag{12}$$

Let us note that θ appearing in Eq. (10) and (11) denotes the value of the argument m_{ij}.

The existence verification for $\Sigma_{[m]}^{C}$ relies solely on the existence verification for IGA(M) whenever the matrix M is symmetric. This is also not surprising since the matrix M is an M-matrix.

Defining the matrix D by:

$$D = diag\left(L_{11}, L_{22}, \ldots, L_m\right)$$

and since D has positive diagonal entries, we can decompose M in the form $M \equiv LL^T = (LD^{-1})(DL^T)$ where from again:

$$\left[M\right] \subseteq \left(LD^{-1}\right)\left(DL^T\right)$$

Comparison with Rohn's method: The Rohn's method is implied by the following theorem:

Theorem 2: (Rohn, 1996)
Let R be an arbitrary nxm matrix, suppose x_o and d>0 be arbitrary n-vectors such that:

$$Gd+g<d \tag{13}$$

holds where,

$$G = \left|I - RA_c\right| + \left|R\right|\Delta \tag{14}$$

$$g = \left|R\left(A_c x_o - b_c\right)\right| + \left|R\right|\left(\Delta\left|x_o\right| + \delta\right) \tag{15}$$

Then an inclusion for the linear interval system is given by:

$$x \subseteq \left[x_o - d, x_o + d,\right] \tag{16}$$

The x_o is taken to be Rb_c where d_o may be assumed to be 0.

A variant of Rohn's method is the interval version proposed in (Rump, 1993). It is based on the work of (Krawczyk, 1969). Iteratively, Rohn's method (13) is written in the form:

$$d_{k+1} = Gd_k + g + f \tag{17}$$

where f is an arbitrary small vector.The vector f plays a very crucial role in the convergence of Rohn's theorem.

RESULTS AND DISCUSSION

The numerical test is taken from (Popova, 2006) given as problem 1. Consider a parameterized system (1.1) of order 6x4 matrix

$$A = \begin{pmatrix} a & a+1 & a+2 & a \\ a & a+2 & a+3 & a+1 \\ a+1 & a+2 & a+3 & a+2 \\ a+2 & a+3 & a+4 & a+3 \\ a+3 & a+4 & a+5 & a+5 \\ a+4 & a+5 & a+6 & a+7 \end{pmatrix}$$

$$b = \begin{pmatrix} 3 \\ 1 \\ -2 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \quad ex = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

We are requested to solve $[A]x = b$, $[A] = ([A(a)], | a \in [0.999, 1.001]])$

The following results were obtained by (Popova, 2006) using block Krawaczyk's algorithm

$$[X] = ([0.8629, 1.1371], [-1.8273, -0.1727], [0.4924, 1.5076], [-1.2742, -0.7257])^T$$

When the same problem 1 above was solved with Parallel Rohn's method of Eq. (13), the following results were obtained:

$$[X] = ([0.999, 1.001], [-1.00997, -0.999003], [0998997, 1.001003], [-1.001, -0.999003])^T$$

The Interval Cholesky Method represented in midpoint –radius form gave the following results:

Interval Cholesky Method

Midpoint X	Radius X
1	0
-1.013684744	0.028789846
1	0.009105466
-1	0.009113722

CONCLUSION

The interval widths produced by Rohn's method of (13) are by far the narrowest than those obtained via interval Cholesky method. The block Krawczyk's method by (Popova, 2006) used in the above example gave inclusion bounds whose widths have the largest values. It follows that Rohn's algorithm is the best as far as these investigations are concerned.

REFERENCES

Alefeld, G. and G. Mayer, 2007. New Criteria for the Feasibility of the Cholesky Method with Interval Data. Preprint Nr 07/07, Institut fur Wissenschaftliches Rechnen und Mathematische model building, Universitat Karlsruhe (TH), Karlsruhe. Retrieved from: www.mathematik. karlsruhe.de/iwrmm/seite/preprints.

Carstensen, C. and M.S. Petkovic, 1994. An improvement of Gargantini simultaneous inclusion method for polynomial roots by Schroder's correction. Appl. Numer. Math., 13: 453-468.

Gargantini, I., 1976. Comparing parallel Newton's method with parallel Laguerrel's method. Comp. Math. Appl., 2: 201-206.

Gargantini, I., 1980. Parallel square root iterations for multiple roots. Comp. Math. Appl., 6: 279-288.

Krawczyk, R., 1969. Newton-Algorithmen zur bestimmung von nullstellen mit fehllerschranken. Computing, 4: 187-201.

Petkovic, M.S., 1989. Iterative methods for simultaneous inclusion of Polynomial zeros. Springer Verlag, Berlin- Heidelberg- New York.

Popova, E.D., 2006. Improved Solution Enclosures for Over- and Underdetermined Interval Linear Systems. In: LSSC 2005. I. Lirkov, S. Margenov and J. Wasniewski, (Eds.), lncs., 3743: 305-312.

Rump, S.M., 1993. Validated solution of large Linear systems. In: Validation Numerics: Theory and Applications. R. Albrocht, *et al.* (Eds.), 9(Computing Supplementum): 191-212.

Rump, S.M., 1999. Fast and parallel interval arithmetic. BIT, 39(3): 534-554.

Rohn, J., 1996. Enclosing solutions of over determined systems of linear interval equations. Reliable Comput., 2: 167-171.