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A Class of A-Stable Block Explicit Methods for the Solutions of Ordinary Differential Equations

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Abstract: The search for high order A-Stable numerical methods has been between implicit Rung-Kutta methods and implicit Linear multistep methods. The cost of implementation of these methods is high due to its implicitness. Explicit A-stable multistep methods will be most desirable for the treatment of non-linear Ordinary Differential equations. This study constructed a class of A-stable Block adams bashfort explicit (Babe) methods including their hybrid forms. The new methods tested on non-linear initial value problems show that they perform well and favourably compete with the Block hybrid adams moulton (Bham) methods of a higher order.

Keywords: A-stable, Bham methods, block hybrid, explicit, initial value problems, non-linear

INTRODUCTION

Recently, a class of continuous Adams Formulae including their hybrid forms have been constructed based on the multistep collocation approached in Onumanyi *et al*. (1994, 1999) and further investigated by Chollom and Onumanyi (2004) and Chollom (2005). These methods have been able to provide sufficient number of finite difference equations used simultaneously in block form for the numerical solutions of first order ODE's of the form:

$$y'(x) = f(x,y)$$
$$y(a) = y_0, a \le x \le b \quad (1)$$

The Ode (1) has prescribed initial, boundary or mixed conditions. These methods which have largely belong to the Adams Moulton class have been found to be A-Stable, a property suitable for stiff ODE's. The popular methods for solving stiff equations has been the implicit methods since It has been shown that there is no standard explicit numerical method that is A-stable which has the order of convergence greater than 2 Dahlquist (1963). Also Hairer (1978) shows that the maximum order of an A-stable method cannot exceed 2q, where, q is the number of derivatives or stages used.

As a result, the choice for higher order A-stable methods is restricted to highly implicit methods such as Runge-Kutta and multi derivative methods Gear (1981). The explicit linear multi-step methods are known to have low implementation cost and small error constants. As a result of the Dalhquist barrier, several Authors resulted to the search for other methods with suitable stability for particular problems. Xu and Zhao (2010) developed new explicit methods with large regions of absolute stability of orders three, step four. These methods were shown to have larger stability regions than the classical Adams Bashfort methods. Because of the difficulties of solving non linear equations using implicit methods, Kim (2010) constructed an explicit type stable method for solving stiff initial value problems which avoided the iteration process to get low complexity for the stiff solver.

In this study, we describe a k-step continuous finite difference formulae based on Adams Bashfort methods including their hybrid forms involving one, two or more off mesh points located between $[x_{n+k-1}, x_{n,k}]$. These methods are constructed using the interpolation concept were the continuous interpolants provide simultaneous discrete methods through evaluation.

at grid and off grid points. The outcome of this is the self starting A-stable explicit block Adams Bashfort integrators which provide dense output of solutions for non linear Ordinary Differential Equations.

DERIVATION OF THE (BABE) METHODS

In this section, the new A-stable block Adams Bashfort explicit (Babe) methods are constructed based on the continuous finite difference approximation approach using the interpolation and collocation criteria

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described by Lie and Norsett (1981) called Multi step Collocation (MC) and block multistep methods by Onumanyi *et al*. (1994).
They adopted the notations:

$$\underline{u} = (u_0, u_1, ..., u_{t+m-1})^T,$$
$$\Phi(t) = (\phi_0(x), u_1(x), \ldots, u_{t+m-1}(x))^T$$

with undetermined constants u_r, r = 0, 1 ,…, t + m−1 where the $\phi(x)$ are specified basis functions with t, m interpolation and collocation points, respectively. The interpolant Y(x) is an approximation to y(x) and is given by:

$$Y(x) = \sum_{r=0}^{t+m-1} u_r \Psi_r(x) = \underline{u}^T \Psi(x) \tag{2}$$

Valid in the sub-interval $X_n \le X \le X_{n+k}$. The constant coefficients (2) are determined by imposing the conditions:

$$Y(x_{n+j}) = y_{n+j},\ j = 0,1,...,t-1 \tag{3}$$

$$Y'(\overline{x}_j) = f_{n+j},\ j = 0,1,...,m-1 \tag{4}$$

where,

$$f_{n+j} = f(x_{n+j}, y_{n+j}) \tag{5}$$

Equation (3)-(5) are expressed as a single set of algebraic equations of the form:

$$\underline{\underline{D}}\,\underline{u} = \underline{F} \tag{6}$$

where,

$$\underline{F} = (Y_n, Y_{n+1}, ..., Y_{n+t-1}, f_n, f_{n+1}, ..., f_{n+m-1})^T$$

But:

$$\underline{u} = \underline{\underline{D}}^{-1}\underline{F} \tag{7}$$

D is the non-singular matrix of dimension (t+m) given below:

$$\underline{\underline{D}} = \begin{pmatrix} \psi_0(x_n) & \cdots & \psi_{t+m-1}(x_n) \\ \psi_0(x_{n+1}) & \cdots & \psi_{t+m-1}(x_{n+1}) \\ \vdots & \vdots & \vdots \\ \psi_0(x_{n+t-1}) & \cdots & \psi_{t+m-1}(x_{n+t-1}) \\ \psi_0'(\overline{x}_0) & \cdots & \psi_{t+m-1}'(\overline{x}_0) \\ \psi_0'(\overline{x}_1) & \cdots & \psi_{t+m-1}'(\overline{x}_1) \\ \vdots & \vdots & \vdots \\ \psi_0'(\overline{x}_{n-1}) & \cdots & \psi_{t+m-1}'(\overline{x}_{n-1}) \end{pmatrix} \tag{8}$$

Substituting (8) into (2) yields the MC formula:

$$Y(x) = F^T \underline{\underline{C}}^T \phi(x), X \in [X_n, X_{n+k}] \tag{9}$$

where,

$$\underline{\underline{C}} = \underline{\underline{D}}^{-1} = (c_{ij}), i, j = 1,...,t+m-1$$

$$\underline{\underline{C}} = \begin{pmatrix} c_{11} & \cdots & c_{1t} & c_{1,t+1} & \cdots & c_{1,t+m} \\ c_{21} & \cdots & c_{2t} & c_{2,t+1} & \cdots & c_{2,t+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{t+m,1} & \cdots & c_{t+m,t} & c_{t+m,t+1} & \cdots & c_{t+m,t+m} \end{pmatrix}$$

Expanding $\underline{\underline{C}}^T \psi(x)$ in (9) yields:

$$Y(x) = \sum_{j=0}^{t-1} \alpha_j(x) Y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f_{n+j} \tag{10}$$

$$\alpha_j(x) = \sum_{r=0}^{t+m-1} C_{r+1,j+1} \psi_r(x),\ j = 0,1,...,t-1 \tag{11}$$

$$\beta_j(x) = \sum_{r=0}^{t+m-1} \frac{C_{r+1,j+1}}{h} \psi_r(x),\ j = 0,1,...,m-1 \tag{12}$$

To express Y(x) in the form of (10-12) involves the use of matrix inversion technique and evaluating the continuous interpolant (10) at some mesh points yields simultaneous discrete methods used in block form for integration.
Derivation of the Babe method for k = 2 at μ = 1/2:
The matrix $\underline{\underline{D}}$ for this class is given by:

$$\underline{\underline{D}} = \begin{pmatrix} 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} \\ 0 & 1 & 2x_n & 3x^2_n \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} \end{pmatrix} \tag{13}$$

Inverting the matrix (13) using the maple software yields the elements of $\underline{\underline{C}} = \underline{\underline{D}}^{-1}$. Substituting the resulting solution into (10) gives the continuous interpolant:

$$\bar{y}(x) = y_{n+1} + \left\{\frac{4}{6h^2}(x-x_{n+1})^3 + \frac{3}{6h}(x-x_{n+1})^2\right\} f_n + \left\{-\frac{4}{3h^2}(x-x_{n+1})^3 - \frac{6}{3h}(x-x_{n+1})^2\right\} f_{n+\frac{1}{2}} + \left\{\frac{4}{6h^2}(x-x_{n+1})^3 + \frac{9}{6h}(x-x_{n+1})^2 + (x-x_{n+1})\right\} f_{n+1} \quad (14)$$

Evaluating (14) at $x = x_n$, $x = x_{n+1/2}$, $x = x_{n+2}$ yields the Babe method (15) for k = 2 used in block form for integration:

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_{n+1} + \frac{h}{24}\left(-f_n - 8f_{n+\frac{1}{2}} - 5f_{n+1}\right) \\ y_{n+1} &= y_n + \frac{h}{6}\left(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}\right) \\ y_{n+2} &= y_{n+1} + \frac{h}{6}\left(7f_n - 20f_{n+\frac{1}{2}} + 19f_{n+1}\right) \end{aligned} \quad (15)$$

Derivation of the Babe method for k = 2 at μ = 3/4:
The matrix $\underline{\underline{D}}$ for this class is given by:

$$\underline{\underline{D}} = \begin{pmatrix} 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} \\ 0 & 1 & 2x_n & 3x^2_n \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} \end{pmatrix} \quad (16)$$

Using the procedure in (10) on (16), yields the continuous interpolant below:

$$\bar{y}(x) = y_{n+1} + \frac{1}{18h^2}\left\{18h^2(x-x_{n+1}) - 21h(x-x_{n+1})^2 + 8(x-x_{n+1})^3 - 5h^3\right\} f_n + \frac{1}{9h^2}\left\{-24h(x-x_{n+1})^2 - 16(x-x_{n+1})^3 - 8h^3\right\} f_{n+\frac{3}{4}} + \frac{1}{6h^2}\left\{8(x-x_{n+1})^3 - 9h(x-x_{n+1})^2 + h^3\right\} f_{n+1} \quad (17)$$

Evaluating (17) at $x = x_n$, $x = x_{n+1/2}$, $x = x_{n+2}$ yields the Babe method (18) for k = 2 used in block form for integration:

$$\begin{aligned} y_{n+\frac{3}{4}} &= y_{n+1} + \frac{h}{288}\left(5f_n - 40f_{n+\frac{3}{4}} - 33f_{n+1}\right) \\ y_{n+1} &= y_n + \frac{h}{18}\left(5f_n + 16f_{n+\frac{3}{4}} - 3f_{n+1}\right) \\ y_{n+2} &= y_{n+1} + \frac{h}{18}\left(11f_n - 80f_{n+\frac{3}{4}} + 87f_{n+1}\right) \end{aligned} \quad (18)$$

Derivation of the Babe method for k = 3:
The collocation matrix $\underline{\underline{D}}$ for this class is given by:

$$\underline{\underline{D}} = \begin{pmatrix} 1 & x_{n+2} & x^2_{n+2} & x^3_{n+2} \\ 0 & 1 & 2x_n & 3x^2_n \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} \end{pmatrix} \quad (19)$$

Inverting the matrix (19) using the maple software yields the elements of $\underline{\underline{C}} = \underline{\underline{D}}^{-1}$. Substituting the resulting solution into (10) gives the continuous interpolant:

$$y(x) = y_{n+2} + \left\{\frac{1}{6h^2}(x-x_n)^3 - \frac{3}{4h}(x-x_n)^2 + (x-x_n) - \frac{h}{3}\right\} f_n + \left\{\frac{1}{3h^2}(x-x_n)^3 - \frac{1}{h}(x-x_n)^2 + \frac{4h}{3}\right\} f_{n+1} + \left\{\frac{1}{6h^2}(x-x_n)^3 + \frac{1}{4h}(x-x_n)^2 + \frac{h}{3}\right\} f_{n+2} \quad (20)$$

The continuous interpolant (20) evaluated at $x = x_n$, $x = x_{n+1}$, $x = x_{n+2}$ produces the Babe method (21) for k = 3 used in block form for integration:

$$\begin{aligned} y_{n+1} &= y_{n+2} + \frac{h}{12}\left(f_n + 8f_{n+1} + 5f_{n+2}\right) \\ y_{n+2} &= y_n + \frac{h}{3}\left(f_n + 4f_{n+1} + f_{n+2}\right) \\ y_{n+3} &= y_{n+2} + \frac{h}{12}\left(5f_n - 16f_{n+1} + 23f_{n+2}\right) \end{aligned} \quad (21)$$

ANALYSIS OF THE NEW METHODS

In this section, the analysis of the newly constructed methods is considered. Their convergence is ascertained and regions of absolute stability plotted.

Convergence analysis: The convergence of the new methods is carried using the approach by Fatunla (1991) and considered in Chollom *et al*. (2007) for linear multistep methods, where the block methods are represented in a single block, r point multi-step method of the form:

$$A^{(0)}y_{m+1} = \sum_{i=1}^{k} A^{(i)}y_{m+1} + h\sum_{i=0}^{k} B^{(i)}f_{m-1} \quad (22)$$

h, affixed mesh size within a block, A^i, B^i, i = o(1)k are rxr identity matrix while Y_m , Y_{m-1} and F_{m-1} are vectors of numerical estimates.

Definition 1: Zero stability:
For n = mr, for some integer m ≥ 0, the block method is zero stable if the roots R_j, N = 1(1)k of the first characteristic polynomial ρ(R) given by:

$$\rho(R) = \det\left[\sum_{i=0}^{k} A^{(i)} R^{i}\right] = 0 \tag{23}$$

Satisfies $|R_j| \leq 1$ and for those roots with $|Rj| \leq 1$, the multiplicity must not exceed two.

The block method (14) expressed in the form of (22) gives:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} + h\begin{pmatrix} \frac{-8}{24} & \frac{-5}{24} & 0 \\ \frac{4}{6} & \frac{1}{6} & 0 \\ \frac{-20}{6} & 0 & \frac{19}{6} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+2} \end{pmatrix} + h\begin{pmatrix} 0 & 0 & \frac{1}{24} \\ 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{7}{6} \end{pmatrix}\begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix} \tag{24}$$

where,

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B^{(0)} = \begin{pmatrix} \frac{-3}{24} & \frac{-5}{24} & 0 \\ \frac{4}{6} & \frac{1}{6} & 0 \\ \frac{-20}{6} & 0 & \frac{15}{6} \end{pmatrix}, B^{(1)} = \begin{pmatrix} 0 & 0 & \frac{1}{24} \\ 0 & 0 & \frac{1}{6} \\ 0 & 0 & \frac{7}{6} \end{pmatrix}$$

Using (23) in (22) gives the zero stability polynomial on a parameter R as follows:

$$\rho(R) = \left| R\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| = R(R-1) = 0, \Rightarrow R = 0, R = 1$$

The block metho (15) by definition (22) is zero stable and is of order p ≥ 1 thus by Henrici (1962), the block method (14) is convergent. Using the same approach, the block methods (18) and (21) are also convergent.

Order of the new methods: The order of the newly constructed block methods are determined using the approach in Chollom *et al*. (2007) and it shows that the new methods have the following orders and error constants:

Block	$Bhabe\, k=2, \mu=\frac{1}{2}$	$Bhabe\, k=2, \mu=\frac{3}{4}$	$Bhabe\, k=3$
Order	4	4	4
Error Constants	$\frac{1}{384}, 0, \frac{1}{6}$	$\frac{7}{18432}, \frac{1}{144}, \frac{19}{144}$	$-\frac{1}{24}, 0, \frac{1}{8}$

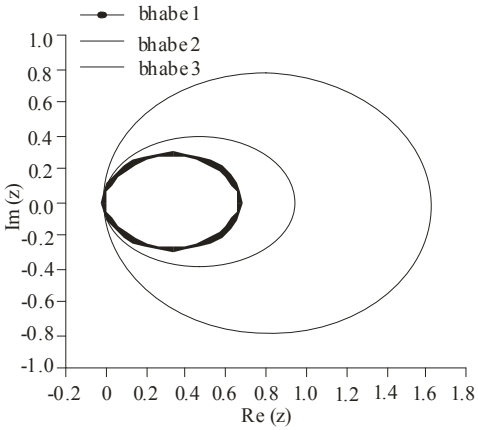

Fig. 1: Absolute stability regions of the Babe methods

Stability regions of the new methods: The absolute stability regions of the new methods are plotted using the approach in Chollom (2005) where the block methods are reformulated as the General Linear Methods of Butchers (1985) expressed as:

$$\begin{bmatrix} Y \\ y_{n+1} \end{bmatrix} = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right]\begin{bmatrix} hf(Y) \\ y_{n-1} \end{bmatrix} \tag{25}$$

Thus the block method (15) expressed in the form of (25) yields:

$$\begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ \underline{y_{n+2}} \\ y_{n+2} \\ y_{n+1} \end{bmatrix} = \left[\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-1}{24} & \frac{8}{24} & \frac{5}{24} & 0 & 1 & 0 \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 & 1 \\ \frac{7}{6} & \frac{-20}{6} & \frac{19}{6} & 0 & 1 & 0 \\ \hdashline \frac{7}{6} & \frac{-20}{6} & \frac{19}{6} & 0 & 1 & 0 \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 & 1 \end{array}\right]\begin{bmatrix} hf_n \\ hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ \underline{hf_{n+2}} \\ y_{n+1} \\ y_n \end{bmatrix} \tag{26}$$

Expressing the block methods (15, 18 and 21) in the form of (25) and plotting in a matlab environment produces the absolute stability regions in Fig. 1.

NUMERICAL EXPERIMENTS

In this section, the new Babe method for k = 2 of order 4 is tested on two non-linear initial value problems of ordinary differential equations taken from Nazeeruddin and Teh (2009) and Jorge and Jesus

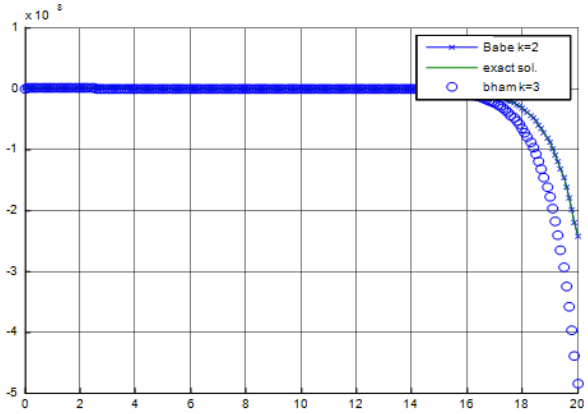

Fig. 2: Solution of example 1

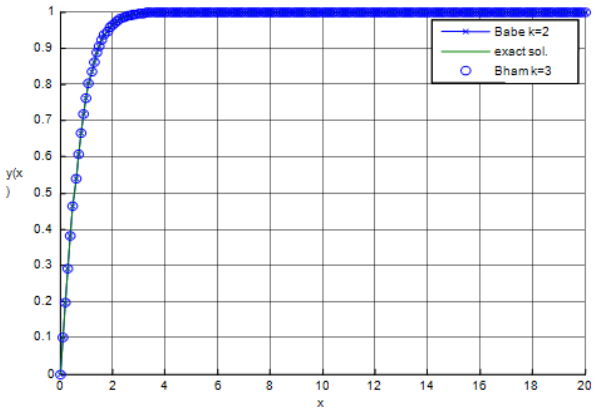

Fig. 3: Solution of example 2

(1999) to ascertain their accuracy. Results are compared to that of the Bham method of order 5:

Example 1:

$$y' = y - x^2 + 1, \quad y(0) = \frac{1}{2}, \quad y(x) = (x+1)^2 - \frac{1}{2}e^x$$

Example 2:

$$y' = 1 - y^2, \quad y(0) = 0, \quad y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

CONCLUDING REMARKS

A-stability is a rare achievable property for linear multistep methods, thus authors resort to the construction of other methods with less severe stability requirement for the solutions of ode. Due to the implementation cost of implicit linear multistep methods, we have in this study constructed a class of A-stable Block adams bashfort explici (Babe) methods including their hybrid forms. These methods have been shown to be A-stable as sdown in Fig. 1. The new method of order four for k = 2 is subjected to a test on two nonlinear ODE's and the results show that the new method competes favourably with the block hybrid Adams Moulton method of order five (Fig. 2 and 3) .Future work in this area will address the construction of high order A-stable explicit block methods.

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