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Research Article

A New Homotopy Analysis Method for Approximating the Analytic Solution of KdV Equation

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Abstract: In this study a new technique of the Homotopy Analysis Method (nHAM) is applied to obtain an approximate analytic solution of the well-known Korteweg-de Vries (KdV) equation. This method removes the extra terms and decreases the time taken in the original HAM by converting the KdV equation to a system of first order differential equations. The resulted nHAM solution at third order approximation is then compared with that of the exact soliton solution of the KdV equation and found to be in excellent agreement.

Keywords: Approximate analytic solution, ħ-curve, KdV equation, new homotopy analysis method, system of first order differential equation

INTRODUCTION

It is usually difficult to find an analytic solution for most of nonlinear partial differential equations. Nonetheless, some analytical techniques have been applied to approximate the analytic solution of some nonlinear problems, such as the use of perturbation techniques (Cole, 1968; Von-Dyke, 1975; Nayfeh, 1981, 1985; Hinch, 1991; Murdock, 1991; Bush, 1992; Kahn and Zarmi, 1998; Nayfeh, 2000), Adomian's decomposition method (Adomian, 1976, 1994) and etc., Liao in 1992 introduced the Homotopy Analysis Method (HAM) (Liao, 1992, 2003), which is seemingly a practical method for approximating analytic solutions of nonlinear problems. Recently, Hassan and El-Tawil (2011, 2012) applied a new technique of HAM to obtain an approximation of some high-order nonlinear partial differential equations.

The well-known Korteweg-deVries equation (KdV) is given by:

$$u_t - 6uu_x + u_{xxx} = 0, \quad x,t \in R \tag{1}$$

with initial condition:

$$u(x,0) = f(x) \tag{2}$$

an approximate analytic solution of KdV equation by HAM is given by Nazari *et al*. (2012a, b). In this study, instead we use a new technique of Homotopy Analysis Method (nHAM) to generate an approximate analytic solution of Eq. (1).

MATERIALS AND METHODS

Important idea of the Homotopy Analysis Method (HAM): To explain the main ideas of HAM, we consider a nonlinear equation in general form:

$$N[u(r,t)] = 0 \tag{3}$$

where N is a nonlinear operator $u(r,t)$ is an unknown function, $u_0(r,t)$ denotes an initial guess of the exact solution $u(r,t)$, $\hbar \neq 0$ an auxiliary parameter, $H(r,t)$ auxiliary function and L an auxiliary linear operator, p∈ (0, 1) as an embedding parameter, we construct the so-called zeroth-order deformation equation by means of HAM:

$$(1-p)L\left[\varphi(r,t;p) - u_0(r,t)\right] = p\hbar H(r,t) N\left[\varphi(r,t;p)\right]. \tag{4}$$

We have a great freedom to choose auxiliary parameters in HAM. Obviously, if we let p = 0, 1 then respectively we have $\varphi(r,t;0) = u_0(r,t),\ \varphi(r,t;1) = u(r,t)$, i.e., when p grows from 0 to 1, the solution $\varphi(r,t;p)$ changes from initial guess u_0 (r, t) to exact solution u (r, t).

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According to Liao (1992), $\varphi(r,t;p)$ can be rewritten in a power series of the form below:

$$\varphi(r,t;p) = \varphi(r,t;0) + \sum_{m=1}^{\infty} u_m(r,t)p^m \tag{5}$$

where,

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \varphi(r,t;p)}{\partial p^m}\bigg|_{p=0} \tag{6}$$

The convergence of the series (5) depends upon the auxiliary parameter $\hbar$, auxiliary function H (r, t), initial guess u_0 (r, t) and auxiliary linear operator L. If they were chosen properly, the series (5) is convergence at $p = 1$, one has:

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) \tag{7}$$

According to definition (6), the governing equation can be inferred from the zeroth-order deformation Eq. (4). We define the vector:

$$\overrightarrow{u_n(r,t)} = \{u_0(r,t), u_1(r,t), ..., u_n(r,t)\},$$

and if we take m-times differentiating with respect to p from zeroth-order deformation Eq. (4) and dividing them by $m!$ then setting p = 1 we obtain the so-called m^{th}-order deformation equation as below:

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\overline{u}_{m-1}(r,t)) \tag{8}$$

where,

$$\chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases} \tag{9}$$

and

$$R_m(\overline{u}_{m-1}(r,t)) = \frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial p^{m-1}} N\left[\sum_{m=0}^{\infty} u_m(r,t)p^m\right]\right\}\bigg|_{p=0} \tag{10}$$

Theorem 1: As long as the series (7) is convergent, it is convergent to the exact solution of (3).

Note that the HAM contains the auxiliary parameter $\hbar$, that we can control and adjustment the convergence of the series solution (7).

Exact solution of KdV: The Korteweg-de Vries equation (KdV equation) describes the theory of water wave in shallow channels, such as a canal. It is a nonlinear equation which governed by (1) and (2). We assume that the exact solution and its derivatives tend to zero (Ablowitz and Segur, 1981; Ablowitz and Clarkson, 1991) when $|x| \to \infty$.

An exact solution of KdV equation as below (Wazwaz, 2001):

$$u(x,t) = -2k^2 \frac{e^{k(x-k^2 t)}}{(1+e^{k(x-k^2 t)})^2} \tag{11}$$

we will compare our final results with (11):

Analysis of HAM: We reformatted (3) in the form as below:

$$Lu(x,t) + Au(x,t) + Bu(x,t) = 0 \tag{12}$$

with initial conditions:

$$u(x,0) = f_0(x) \quad \frac{\partial u(x,t)}{\partial t}\bigg|_{t=0} = f_1(x) \tag{13}$$

where, L = $\frac{\partial u(x,t)}{\partial t}$, $Au(x,t), Bu(x,t)$ are linear and nonlinear parts of equation respectively.
The zero-order deformation equation is:

$$(1-p)L[\varphi(x,t;p) - u_0(x,t)] =$$

$$p\hbar H(x,t)(Lu(x,t) + Au(x,t) + Bu(x,t)) \tag{14}$$

The m^{th}-order deformation equation is obtained by taking m times derivative of (14), i.e.:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] =$$

$$\hbar H(x,t)(Lu_{m-1}(x,t) + Au_{m-1}(x,t) + Bu_{m-1}(x,t)) \tag{15}$$

and then:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[H(x,t)(Lu_{m-1}(x,t) +$$

$$Au_{m-1}(x,t) + Bu_{m-1}(x,t))] \tag{16}$$

We consider $H(r,t) = 1$ and L^{-1} is an integral operator and u_0 (r, t) is an initial guess of the approximation of exact solution (11) and (16) then becomes:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\int_0^t \left(\frac{\partial u_{m-1}(x,t)}{\partial t} + Au_{m-1}(x,t) + Bu_{m-1}(x,t)\right)dt \tag{17}$$

For m = 1, $\chi_1 = 0$ and u_0 (x, 0) u_0 (x, t) f_0 (x) from (17), we obtain:

$$u_1(x,t) = \hbar\int_0^t (Au_0(x,t) + Bu_0(x,t))dt \tag{18}$$

For m > 1, $\chi_m = 1$ and u_{m-1} (x, 0) = 0:

$$u_m(x,t) = (1+h)u_{m-1}(x,t) + \int_0^t (Au_{m-1}(x,t) + Bu_{m-1}(x,t))dt \tag{19}$$

New technique of HAM: We rewrite Eq. (3) in a system of first order differential equation as below:

$$u_t(x,t) - v(x,t) = 0 \tag{20}$$

$$v(x,t) + Au(x,t) + Bu(x,t) = 0 \tag{21}$$

we consider the initial approximation and an auxiliary linear operator, respectively in the form below:

$$u_0(x,t) = f_0(x), v_0(x) = f_1(x),$$

$$Lu(x,t) = \frac{\partial u(x,t)}{\partial t} \quad .$$

From (18) and (20) we have, respectively as:

$$u_1(x,t) = \hbar\int_0^t (-v_0(x,t))\partial t \tag{22}$$

$$v_1(x,t) = \hbar(Au_0(x,t) + Bu_0(x,t)) \quad . \tag{23}$$

In (19) for m>1, $\chi_m = 1$ and $u_m(x, 0) = 0$, $v_m(x, 0) = 0$, we obtain the following results:

$$u_m(x,t) = (1+\hbar)u_{m-1}(x,t) + \hbar\int_0^t (-v_{m-1}(x,t))dt \tag{24}$$

$$v_m(x,t) = (1+h)v_{m-1}(x,t) + \hbar(Au_{m-1}(x,t) + Bu_{m-1}(x,t)). \tag{25}$$

RESULTS AND DISCUSSION

Using new technique of HAM for KdV: We rewrite KdV equation in a system of (24) and (25) as below:

$$u_t(x,t) = v(x,t),$$

$$v(x,t) = 6u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^3 u(x,t)}{\partial x^3}. \tag{26}$$

we choose:

$$u_0(x,t) = \frac{-2e^x}{(1+e^x)^2}, \quad v_0(x,t) = \frac{-4e^{2x}}{(1+e^x)^3} + \frac{2e^x}{(1+e^x)^2}$$

and

$$Lu(x,t) = \frac{\partial u(x,t)}{\partial t} \; \Rightarrow \; L^{-1} = \int (.)dt + c_0$$

From (23) we obtain:

$$u_1(x,t) = \hbar\int_0^t (-v_0(x,t)dt,$$

$$v_1(x,t) = \hbar(\frac{\partial^3 u_0(x,t)}{\partial x^3} - 6u_0(x,t)\frac{\partial u_0(x,t)}{\partial t})$$

and for m>1, $\chi_m = 1$ and:

$$Av_{m-1}(x,t) = \frac{\partial^3 u(x,t)}{\partial x^3}$$

$$Bv_{m-1}(x,t) = -6\sum_{i=0}^{m-1} u_i(x,t)\frac{\partial u_{m-1-i}}{\partial x},$$

we obtain:

$$u_m(x,t) = (1+\hbar)u_{m-1}(x,t) + \hbar\int_0^t (-v_{m-1}(x,t)\partial t \tag{27}$$

$$v_m(x,t) = (1+\hbar)v_{m-1}(x,t) + \hbar(\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3} - 6\sum_{i=0}^{m-1} u_i(x,t)\frac{\partial u_{m-1-i}(x,t)}{\partial x}) \tag{28}$$

The results of new technique of HAM for KdV: The results are generated as below:

$$u_1(x,t) = (\frac{4e^{2x}}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2})\hbar t \tag{29}$$

$$v_1(x,t) = \frac{12e^x\hbar(\frac{4e^{2x}}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2})}{(1+e^x)^2} + \hbar(\frac{12e^{2x}}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2} - 6e^x(\frac{6e^{2x}}{(1+e^x)^4} - \frac{2e^x}{(1+e^x)^3}) - 2e^x(-\frac{24e^{3x}}{(1+e^x)^5} + \frac{18e^x}{(1+e^x)^4} - \frac{2e^x}{(1+e^x)^3})) \tag{30}$$

$$u_2(x,t) = (\frac{4e^{2x}}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2})\hbar(1+\hbar)t + \hbar t(-\frac{12e^x\hbar(\frac{4e^{2x}}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2})}{(1+e^x)^2} - \hbar(\frac{12e^{2x}}{(1+e^x)^3} - \frac{2e^x}{(1+e^x)^2} - 6e^x(\frac{6e^{2x}}{(1+e^x)^4} - \frac{2e^x}{(1+e^x)^3}) - 2e^x(-\frac{24e^{3x}}{(1+e^x)^5} + \frac{18e^x}{(1+e^x)^4} - \frac{2e^x}{(1+e^x)^3}))), \tag{31}$$

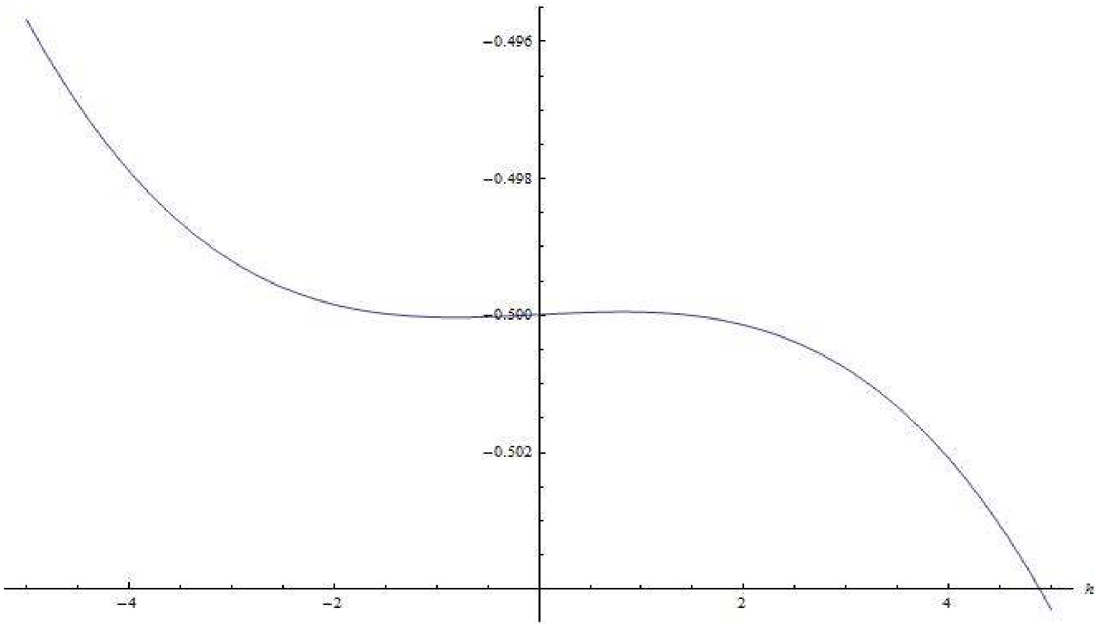
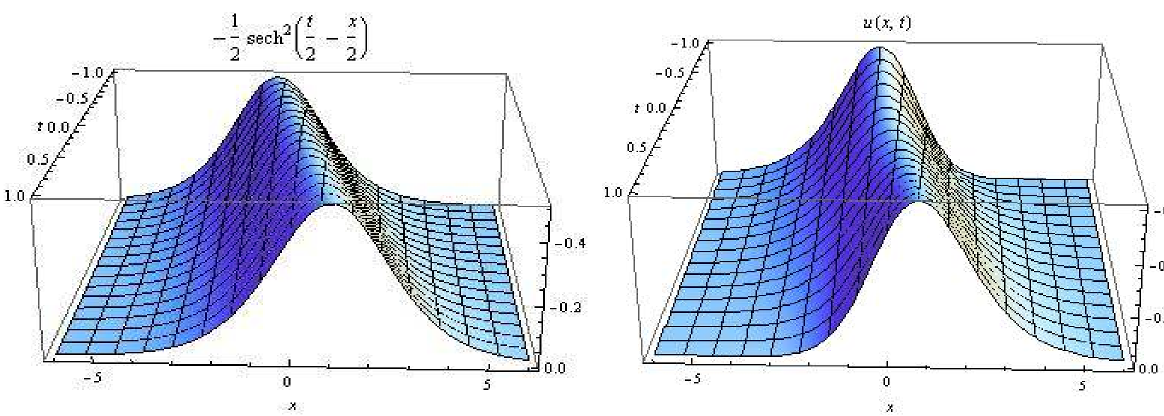
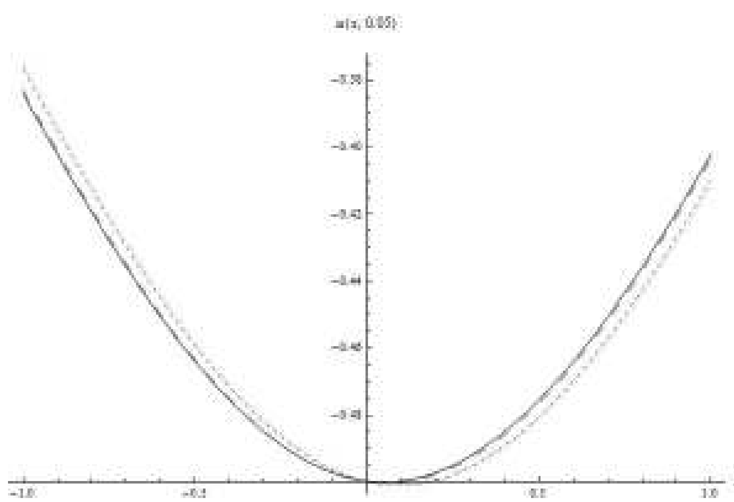


Fig. 1: The $\hbar$-curve of at 3[th]-order approximation



(a) (b)

Fig. 2: Comparison between exact solution, (a) and nHAM solution (b) for $\hbar = -0.4$



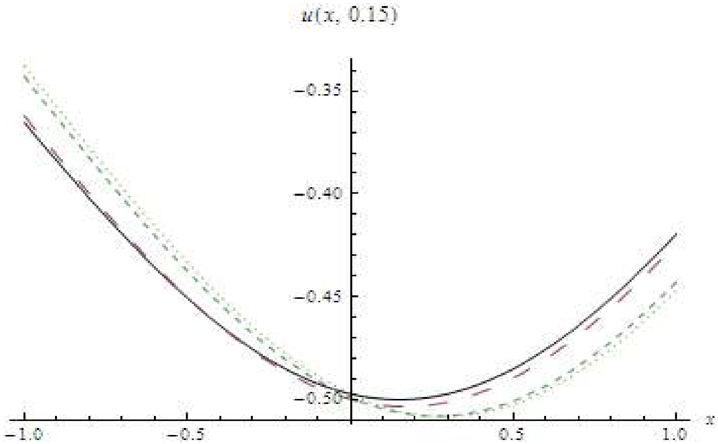


Fig. 3: Comparison between shapes of u (x, 0 .05) and u (x, 0 .15) for black line refers to exact solution; large dash line refers to $\hbar = -0.4$; tiny dash line refers $\hbar = -1$ and dash line refers to $\hbar = -0.75$

Table 1: Comparison between results of exact solution and results of nHAM

t	x	Exact solution	nHAM solution	Absolute error
0.01	-10	−0.00008988830501180548	−0.00008976060682025857	$1.27698191546 \times 10^{-7}$
	-6	−0.004884174555553007	−0.004877272170776152	$6.902384776854 \times 10^{-6}$
	2	−0.21159029899433188	−0.21180416748168646	2.13868×10^{-4}
	10	−0.00009170400269620042	−0.00009182320503164965	$1.19202335449 \times 10^{-7}$
0.05	-10	−0.00008636403856823246	−0.00008564238129433044	$7.216572739020 \times 10^{-7}$
	-6	−0.004693564364865835	−0.004654597563930618	3.89668×10^{-5}
	2	−0.21807963830331245	−0.21907712719380515	9.97489×10^{-4}
	10	−0.0000954461568177339	−0.00009595537235128575	$5.092155335518 \times 10^{-7}$
0.25	-10	−0.0000707100013545677	−0.0000651906716017695	$5.519329752798 \times 10^{-6}$
	-6	−0.0038460447137296156	−0.003548689512075873	2.97355×10^{-4}
	2	−0.25225845037331557	−0.25556127807710666	3.30283×10^{-3}
	10	−0.00011657573557899152	−0.00011675562688654592	$1.798913075544 \times 10^{-7}$
0.50	-10	−0.00005506986580060628	−0.00003995279527609886	1.51171×10^{-5}
	-6	−0.0029978574171384237	−0.002183800499693992	8.14057×10^{-4}
	2	−0.29829290414067045	−0.30144619868758066	3.15329×10^{-3}
	10	−0.00014968125110570107	−0.00014308270584565167	6.59855×10^{-6}
0.75	-10	−0.00004288897707021007	−0.000015077986494906601	2.7811×10^{-5}
	-6	−0.0023362850215911525	−0.0008383515455749411	1.49793×10^{-3}
	2	−0.346209573984998	−0.34764193263844034	1.43236×10^{-3}
	10	−0.00019218635965044794	−0.0001697728523492358	2.24135×10^{-5}

$$u(x,t) = u_0(x,t) + u_1(x,t) + \dots \quad . \tag{32}$$

We can calculate $u_m(x,t)$, $m = 3, 4, \dots$ and then this can be written in the form $u_m(x,t,\hbar) = \sum_{m=0}^{M} u_m(x,t,\hbar)$. We can control the rate of convergence of this approximation by the auxiliary parameter $\hbar$. If we let $x = t = 0.01$, then it is obvious from Fig. 1 that the best reliable region for this analytical solution is in the interval $-1.2 < \hbar < 0$. According to theorem 1 the series solution (32) must be the exact solution, as long as the series is convergent. In this case, when $-1 < t < 1$ and $\hbar = -0.4$, the exact solution and nHAM are similar in Fig. 2. The numerical results are shown in Table 1. In Fig. 3 we obtained the third-order approximation and we drew the shapes of u (x, 0.05) and u (x, 0.15) for $\hbar = -0.4, -1, -0.75$ and compared them with the exact solution. We can see that the best result is relevant to an approximation that has $\hbar = -0.4$.

CONCLUSION

In this study, a new technique of Homotopy Analysis Method (nHAM) is applied to obtain an approximated analytic solution of KdV equation. The resulted nHAM solution at the third order approximation is compared with the exact soliton solution of the KdV equation and it is shown that the result is a good approximation in comparison with the exact solution. Clearly this technique provides a way for eliminating some extra terms of the original homotopy analysis method and then decreases the time consumed for obtaining the final result. We have used the auxiliary parameter $\hbar$ for controlling the convergence of the approximation series which is a fundamental qualitative difference in analysis between nHAM and other methods.

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