Research Journal of Applied Sciences, Engineering and Technology 10(1): 22-28, 2015
DOI:10.19026/rjaset.10.2549
ISSN: 2040-7459; e-ISSN: 2040-7467

Submitted: November 10, 2014 Accepted: February 8, 2015 Published: May 10, 2015

**Research Article**

# Bayesian Estimation of a Mixture Model

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**Abstract:** We present the properties of a bathtub curve reliability model having both a sufficient adaptability and a minimal number of parameters introduced by Idée and Pierrat (2010). This one is a mixture of a Gamma distribution G(2, (1/θ)) and a new distribution L(θ). We are interesting by Bayesian estimation of the parameters and survival function of this model with a squared-error loss function and non-informative prior using the approximations of Lindley (1980) and Tierney and Kadane (1986). Using a statistical sample of 60 failure data relative to a technical device, we illustrate the results derived. Based on a simulation study, comparisons are made between these two methods and the maximum likelihood method of this two parameters model.

**Keywords:** Approximations, bayes estimator, mixture distribution, reliability, weibull distribution

## INTRODUCTION

Mixture models play a vital role in many applications. For example, direct applications of finite mixture models are in economics, botany, medicine, psychology, agriculture, zoology, life testing and reliability engineering. However, a modelling by a finished mixture of Weibull distributions can be more relevant and physically significant. Several authors are interesting by Bayesian estimation in mixture models like Tsionas (2002) and Gelman *et al*. (2003). Modelling by a Weibull mixture distributions was often occulted by the survival analysis experts to the profile of a simple model of Weibull, because of the high number of parameters to be estimated and absence of an effective estimation method of the parameters.

Several methods are used for estimating the parameters of the mixture. The graphic approach was used by Jiang and Murty (1995) and Jiang and Kececioglu (1992) initially to adjust data with a mixture of two Weibull distributions, then to estimate its parameters. The two methods of the moments and the maximum likelihood were largely used while being based on the use of EM algorithm (Expectation-Maximization), this algorithm is not only one digital technique, it gives also useful statistical data, for more detail to see Dempster *et al*. (1977). A modification of this algorithm was introduced by Wei and Tanner (1990), where in the step E, the expectation is calculated by Monte Carlo simulations, this algorithm noted MCEM was used by Levine and Casella (2001) and more recently by Elmahdy and Aboutahoun (2013).

The model in which, we are interested is a mixture model of two distributions introduced by Idée and Pierrat (2010). These authors showed that this model of two parameters is equivalent to a mixture of two Weibull distributions. Equivalence was shown using Kolmogorov Smirnov distance. We propose to use a Bayesian approach, with a non-informative prior and a squared-error loss function.

The main objective of this study is to estimate the two parameters and the survival function of the new model. The methods under consideration are: Maximum likelihood Estimation, Bayesian Methods with two types of approximation: Lindley's approximation and Tierney-Kadane's approximation. The methods are compared using a real data analysis and simulation. Conclusions based on the study are given.

## PROPERTIES OF THE MODEL

The model Probability Density Function (PDF) is given as:

$$\begin{cases} f(x,\theta,\varepsilon) = (1-\varepsilon)\frac{x}{\theta}\frac{exp\left[-\left(\frac{x}{\theta}\right)\right]}{\theta} + \varepsilon\frac{\left(-1+\frac{x}{\theta}\right)\ln\left(\frac{x}{\theta}\right)exp\left[-\left(\frac{x}{\theta}\right)\right]}{\theta} \\ x>0, \theta>0, 0\le\varepsilon\le 1 \end{cases} \tag{1}$$

The survival function is:

$$\begin{cases} R(t,\theta) = \left(1+(1-\varepsilon)\left(\frac{t}{\theta}\right)+\varepsilon\left(\frac{t}{\theta}\right)\ln\left(\frac{t}{\theta}\right)\right)exp\left[-\left(\frac{t}{\theta}\right)\right] \\ t>0, \theta>0, 0\le\varepsilon\le 1 \end{cases} \tag{2}$$

The failure rate is also given as:

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$$\begin{cases} h(t,\varepsilon,\theta) = \frac{1}{\theta}\left(1 - \frac{1+\varepsilon\ln\left(\frac{t}{\theta}\right)}{1+(1-\varepsilon)\left(\frac{t}{\theta}\right)+\varepsilon\left(\frac{t}{\theta}\right)\ln\left(\frac{t}{\theta}\right)}\right) \\ t>0, \theta>0, 0 \le \varepsilon \le 1 \end{cases} \quad (3)$$

**Maximum likelihood:** We lay out of a sample of n independent and complete observations of the same distribution given in (1) noted $(x_1, x_2, \dots, x_n)$. The likelihood of the sample is written:

$$l(\theta,\varepsilon/ech) = \frac{exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)\right)}{\theta^n}\prod_{i=1}^{n} \left((1-\varepsilon)\left(\frac{x_i}{\theta}\right) + \varepsilon\left(\frac{x_i}{\theta}-1\right)\ln\left(\frac{x_i}{\theta}\right)\right) \quad (4)$$

We pose:

$$u_i(\varepsilon,\theta) = \left((1-\varepsilon)\left(\frac{x_i}{\theta}\right) + \varepsilon\left(\frac{x_i}{\theta}-1\right)\ln\left(\frac{x_i}{\theta}\right)\right); \quad S = \sum_{i=1}^{n} x_i$$

Then:

$$l(\theta,\varepsilon/ech) = \frac{exp\left(-\sum_{i=1}^{n}\frac{S}{\theta}\right)}{\theta^n}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right)$$

The log- likelihood is:

$$L(\theta,\varepsilon/ech) = -n\ln(\theta) - \frac{S}{\theta} + \sum_{i=1}^{n}\ln(u_i) \quad (5)$$

Differentiating (5) with respect to $\varepsilon$ and $\theta$ and equating to zero, we have:

$$\frac{\partial L}{\partial \theta} = \frac{S}{\theta} - n + \sum_{i=1}^{n}\left(\frac{-x_i+\varepsilon\theta}{\theta} - \varepsilon\left(\frac{x_i}{\theta}\right)\ln\left(\frac{x_i}{\theta}\right)\right) u_i^{-1} = 0 \quad (6)$$

$$\frac{\partial L}{\partial \varepsilon} = \sum_{i=1}^{n}\left(\frac{-x_i}{\theta} + \left(\frac{x_i}{\theta}-1\right)\ln\left(\frac{x_i}{\theta}\right)\right) u_i^{-1} = 0 \quad (7)$$

From where, we obtain this system:

$$\begin{cases} \frac{S}{\theta} - n + \sum_{i=1}^{n}\left(\frac{-x_i+\varepsilon\theta}{\theta} - \varepsilon\left(\frac{x_i}{\theta}\right)\ln\left(\frac{x_i}{\theta}\right)\right) u_i^{-1} = 0, \\ \sum_{i=1}^{n}\left(\frac{-x_i}{\theta} + \left(\frac{x_i}{\theta}-1\right)\ln\left(\frac{x_i}{\theta}\right)\right) u_i^{-1} = 0 \end{cases}$$

The resolution of the system is done using the iterative methods and provides us estimators of the maximum of likelihood of $\theta$ and $\varepsilon$ noted respectively by $\theta_{MV}$ and $\varepsilon_{MV}$. To have an estimator of the survival function, it is enough to replace $(\varepsilon, \theta)$ in the expression (2) by $(\theta_{MV}, \varepsilon_{MV})$, we obtain an estimator of the survival function noted $R_{MV}(t)$.

**Bayesian estimation:** Given a random sample, $(x_1, x_2, \dots, x_n)$ of size n, from a mixture distribution and $\underline{x}$ the vector of the observed data. The likelihood function can be constructed as follows:

$$l(\theta,\varepsilon/ech) = \frac{exp\left[-\left(\frac{S}{\theta}\right)\right]}{\theta^n}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right)$$

Using the non-informative prior:

$$\pi(\varepsilon,\theta) = \frac{1}{\varepsilon\theta}$$

The posterior distribution is given by:

$$\pi(\varepsilon,\theta|\underline{x}) = \frac{l(\theta,\varepsilon/ech)\pi(\varepsilon,\theta)}{\int_0^\infty\int_0^1 l(\theta,\varepsilon/ech)\pi(\varepsilon,\theta)d\varepsilon d\theta} \propto \frac{exp\left[-\left(\frac{S}{\theta}\right)\right]}{\varepsilon\theta^{n+1}}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right) \quad (8)$$

The Bayesian estimates of parameters under the loss function are:

$$\hat{\varepsilon} = \frac{\int_0^\infty\int_0^1 \frac{exp\left[-\left(\frac{S}{\theta}\right)\right]}{\theta^{n+1}}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right)d\varepsilon d\theta}{\int_0^\infty\int_0^1 l\left(\theta,\frac{\varepsilon}{ech}\right)\pi(\varepsilon,\theta)d\varepsilon d\theta}$$

$$\hat{\theta} = \frac{\int_0^\infty\int_0^1 \frac{exp\left[-\left(\frac{S}{\theta}\right)\right]}{\varepsilon\theta^{n}}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right)d\varepsilon d\theta}{\int_0^\infty\int_0^1 l(\theta,\varepsilon/ech)\pi(\varepsilon,\theta)d\varepsilon d\theta}$$

The Bayesian estimator of $R_t$ under the loss function is the posterior mean:

$$R_t^* = E(R_t|x)$$

$$\frac{\int_0^\infty\int_0^1\left(1+(1-\varepsilon)\frac{t}{\theta}+\varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)\right)exp\left(-\frac{t}{\theta}\right)\frac{exp\left[-\left(\frac{S}{\theta}\right)\right]}{\varepsilon\theta^{n+1}}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right)d\varepsilon d\theta}{\int_0^\infty\int_0^1\frac{exp\left[-\left(\frac{S}{\theta}\right)\right]}{\varepsilon\theta^{n+1}}\prod_{i=1}^{n}\left(u_i(\varepsilon,\theta)\right)d\varepsilon d\theta}$$

To solve this ratio of integrals explicitly, we will use approximation methods.

**Lindley's procedure:** Lindley (1980) developed approximate procedures for the evaluation of the ratio of integrals of the form:

$$\frac{\int w(\theta)exp\{L(\theta)\}d\theta}{\int v(\theta)exp\{L(\theta)\}d\theta} \quad (9)$$

In our case, $m = 2$ and $\lambda = (\varepsilon, \theta)$

$$\rho(\varepsilon,\theta) = \log\{\pi(\varepsilon,\theta)\}$$

$$\wedge(\varepsilon,\theta) = \log\{\pi(\varepsilon,\theta|x)\} = L(\varepsilon,\theta) + \rho(\varepsilon,\theta) = = -\ln(\varepsilon\theta) - n\ln(\theta) - \frac{S}{\theta} + \sum_{i=1}^{n}\ln\left(u_i(\varepsilon,\theta)\right) \quad (10)$$

Denoting $\Phi(t) = R_t$ and using Lindley's method, expanding about the posterior mode, the Bayesian estimator of $R_t$ in Eq. (2) becomes:

$$R_t^* = \Phi(\hat{\lambda}) + \frac{1}{2}\sum\Phi_{ij}(\hat{\lambda})\tau_{ij} + \frac{1}{2}\sum\wedge_{ijk}\Phi_l(\hat{\lambda})\tau_{ij}\tau_{kl} \quad (11)$$

where,

$$\Phi_{ij} = \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j}, \Lambda_{ijk} = \frac{\partial^3 \Lambda}{\partial \lambda_i \partial \lambda_j \partial \lambda_k}, etc \tag{12}$$

The $\tau_{ij}$ values are defined as the $(i, j)$th element of the negative of the inverse of the Hessian matrix of signe second derivatives of $\Lambda$: $\{\tau_{ij}\} = -\{\Lambda_{ij}\}^{-1}$. Expanding Eq. (7), we obtain:

$$R_B^*(t) = \Phi(\hat{\lambda}) + \frac{1}{2}\Phi_{11}(\hat{\lambda})\tau_{11} + \Phi_{12}(\hat{\lambda})\tau_{12} + \frac{1}{2}\Phi_{22}(\hat{\lambda})\tau_{22} + \frac{1}{2}\Lambda_{111}\{\Phi_1(\hat{\lambda})\tau_{11}^2 + \Phi_2(\hat{\lambda})\tau_{11}\tau_{12}\} + \frac{1}{2}\Lambda_{112}\{3\Phi_1(\hat{\lambda})\tau_{11}\tau_{12} + \Phi_2(\hat{\lambda})(\tau_{11}\tau_{22} + 2\tau_{12}^2)\} + \frac{1}{2}\Lambda_{122}\{3\Phi_2(\hat{\lambda})\tau_{22}\tau_{12} + \Phi_1(\hat{\lambda})(\tau_{11}\tau_{22} + 2\tau_{12}^2)\} + \frac{1}{2}\Lambda_{222}\{\Phi_2(\hat{\lambda})\tau_{22}^2 + \Phi_1(\hat{\lambda})\tau_{22}\tau_{12}\} \tag{13}$$

In our case, with $m = 2$ and $\lambda = (\varepsilon, \theta)$, we obtain from Eq. (2), with $\Phi(t) = R_t$:

$$\Phi_1 = \frac{\partial \Phi}{\partial \varepsilon} = \frac{t exp\left[-\left(\frac{t}{\theta}\right)\right]}{\theta}\left(-1 + \ln\left(\frac{t}{\theta}\right)\right), \Phi_{11} = \frac{\partial^2 \Phi}{\partial \varepsilon^2} = 0,$$
$$\Phi_{12} = \frac{\partial^2 \Phi}{\partial \varepsilon \partial \theta} = \frac{t exp\left[-\left(\frac{t}{\theta}\right)\right]}{\theta^2}\left(-\frac{t}{\theta} + \left(\frac{t}{\theta} - 1\right)\ln\left(\frac{t}{\theta}\right)\right), \Phi_2 = \frac{\partial \Phi}{\partial \theta} = \left(\frac{t u_t}{\theta^2}\right) exp\left[-\left(\frac{t}{\theta}\right)\right],$$
$$\Phi_{22} = \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{t^2 exp\left[-\left(\frac{t}{\theta}\right)\right]}{\theta^4} \left(u_t + \frac{\theta\varepsilon - \theta - 4t + 3\varepsilon t}{t} + \left(\frac{2\theta\varepsilon}{t} - 4\varepsilon\right)\ln\left(\frac{t}{\theta}\right)\right) + \frac{t}{\theta^3}\Phi(t)$$

With $\Phi(t) = R_t$. The joint posterior mode of Eq. (10) is obtained by solving the system of equations:

$$\begin{cases} -\frac{1}{\varepsilon} + \sum_{i=1}^{n} s_i u_i^{-1} = 0, \\ -(1+n) + \frac{S}{\theta} + \sum_{i=1}^{n} v_i\, u_i^{-1} = 0, \end{cases}$$

The partial derivatives of the log-posterior density function are:

$$\Lambda_{11} = \frac{\partial^2 \Lambda}{\partial \varepsilon^2} = \frac{1}{\varepsilon^2} + \sum_{i=1}^{n}(-s_i^2 u_i^{-2}),$$

$$\Lambda_{22} = \frac{\partial^2 \Lambda}{\partial \theta^2} = \frac{1+n}{\theta^2} - \frac{2S}{\theta^3} + \frac{1}{\theta^2}\sum_{i=1}^{n} \left(u_i^{-1}\left(\frac{2\varepsilon x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right) + \frac{2x_i + \varepsilon x_i - \varepsilon\theta}{\theta}\right) - v_i^2 u_i^{-2}\right)$$

$$\Lambda_{12} = \frac{\partial^2 \Lambda}{\partial \varepsilon \partial \theta} = \sum_{i=1}^{n}\frac{1}{\theta}\left(u_i^{-1}\left(1 - \frac{x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right)\right) - v_i s_i u_i^{-2}\right),$$

$$\Lambda_{111} = \frac{\partial^3 \Lambda}{\partial \varepsilon^3} = -\frac{2}{\varepsilon^3} + 2\sum_{i=1}^{n} s_i^3\, u_i^{-3},$$

$$\Lambda_{222} = \frac{-2(n+1)}{\theta^3} + \frac{6S}{\theta^4} + \frac{1}{\theta^3}\sum_{i=1}^{n}\left(u_i^{-1}\left(\frac{-6x_i + 2\varepsilon\theta - 5\varepsilon x_i}{\theta} - 6\varepsilon\frac{x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right)\right) - 3v_i u_i^{-2}\left(\frac{2\varepsilon x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right) + \frac{2x_i + \varepsilon x_i - \varepsilon\theta}{\theta}\right) + 2v_i^3 u_i^{-3}\right)$$

$$\Lambda_{112} = \frac{\partial^3 \Lambda}{\partial \varepsilon^2 \partial \theta} = \frac{2}{\theta}\sum_{i=1}^{n}\left(s_i u_i^{-2}\left(1 - \frac{x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right)\right) + v_i s_i^{-2} u_i^{-3}\right),$$

$$\Lambda_{122} = \frac{\partial^3 \Lambda}{\partial \varepsilon \partial \theta^2} = \frac{1}{\theta^2}\sum_{i=1}^{n}\left(u_i^{-1}\left(\frac{x_i - \theta}{\theta} + \frac{2x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right)\right) - s_i u_i^{-2}\left(\frac{2\varepsilon x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right) + \frac{2x_i + \varepsilon x_i - \varepsilon\theta}{\theta}\right) + 2v_i u_i^{-2}\left(1 - \frac{x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right)\right) + 2s_i v_i^2 u_i^{-3}\right)$$

where,

$$v_i = \left(\frac{-x_i + \varepsilon\theta}{\theta}\right) - \varepsilon\left(\frac{x_i}{\theta}\right)\ln\left(\frac{x_i}{\theta}\right); s_i = -\frac{x_i}{\theta} + \left(\frac{x_i}{\theta} - 1\right)\ln\left(\frac{x_i}{\theta}\right)$$

**Tierney and Kadane's method:** Lindley's approximation requires the evaluation of the third derivatives of the likelihood function or the posterior density which can be very tedious and requires great computational precision. Tierney and Kadane (1986) gave an alternative method of evaluation of the ratio of integrals, by writing the two expressions:

$$l = \frac{\log v(\lambda) + L(\lambda|x)}{n}, l^* = \frac{\log \Phi(\lambda) + \log v(\lambda) + L(\lambda|x)}{n}, \tag{14}$$

So that:

$$E(\Phi(\lambda)|x) = \frac{\int exp(nl^*)d\lambda}{\int exp(nl)d\lambda}. \tag{15}$$

The Bayesian estimator of $\Phi(\lambda)$ takes the form:

$$\hat{E}(\Phi(\lambda)|x) = \left(\frac{|\Xi^*|}{|\Xi|}\right)^{1/2} exp\left[n\{l^*(\hat{\lambda}^*) - l(\hat{\lambda})\}\right] \tag{16}$$

where, $\hat{\lambda}^*$ and $\hat{\lambda}$ maximize $l^*$ and $l$ respectively
$\Xi^*$ And $\Xi$ are negatives of the inverse Hessians of $l^*$ and $l$ at $\hat{\lambda}^*$and $\hat{\lambda}$, respectively. The matrix $\Xi$ takes the form:

$$\Xi = \begin{pmatrix} -\frac{\partial^2 l}{\partial \varepsilon^2} & -\frac{\partial^2 l}{\partial \theta \partial \varepsilon} \\ -\frac{\partial^2 l}{\partial \varepsilon \partial \theta} & -\frac{\partial^2 l}{\partial \theta^2} \end{pmatrix}$$

To apply the method, we need to maximize:

$$l = \frac{-\ln(\varepsilon)}{n} - \frac{(1+n)\ln(\theta)}{n} - \frac{S}{n\theta} + \frac{1}{n}\sum_{i=1}^{n} \ln(u_i):$$
$$:l^* = \frac{1}{n}\ln\left(1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)\right) - \frac{(t+S)}{n\theta} - \frac{\ln(\varepsilon)}{n} - \frac{(1+n)\ln(\theta)}{n} + \frac{1}{n}\sum_{i=1}^{n} \ln(u_i)$$

Here:

$$\begin{cases} -\frac{1}{\varepsilon n} + \frac{1}{n}\sum_{i=1}^{n} s_i u_i^{-1} = 0, \\ -\frac{(1+n)}{n\theta} + \frac{S}{n\theta^2} + \frac{1}{n\theta}\sum_{i=1}^{n} v_i u_i^{-1} = 0 \end{cases}$$

Also:

$$\frac{\partial^2 l}{\partial \varepsilon^2} = \frac{1}{n\varepsilon^2} + \frac{1}{n}\sum_{i=1}^{n}(-s_i^2 u_i^{-2}),$$

$$\frac{\partial^2 l}{\partial \varepsilon \partial \theta} = \frac{1}{n\theta}\sum_{i=1}^{n}\left(u_i^{-1}\left(1 - \frac{x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right)\right) - s_i v_i u_i^{-2}\right),$$

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{(1+n)}{n\theta^2} - \frac{2S}{n\theta^3} + \frac{1}{n\theta^2}\sum_{i=1}^{n}\left(u_i^{-1}\left(\frac{2\varepsilon x_i}{\theta}\ln\left(\frac{x_i}{\theta}\right) + \frac{2x_i + \varepsilon x_i - \varepsilon\theta}{\theta}\right) - v_i^2 u_i^{-2}\right)$$

Partial derivatives of $l$ produce the system of equations:

$$\frac{\partial^2 l^*}{\partial \varepsilon^2} = \frac{\partial^2 l}{\partial \varepsilon^2} - \frac{1}{n}\left(\frac{-\frac{t}{\theta} + \frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}{1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}\right)^2,$$

$$\frac{\partial^2 l^*}{\partial \varepsilon \partial \theta} = \frac{\partial^2 l}{\partial \varepsilon \partial \theta} - \frac{t}{n\theta^2}\left(\frac{\ln\left(\frac{t}{\theta}\right)}{1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}\right) + \frac{t}{n\theta^2}\frac{\left(-\frac{t}{\theta} + \frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)\right)\left(1 + \varepsilon\ln\left(\frac{t}{\theta}\right)\right)}{\left(1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)\right)^2},$$

$$\frac{\partial^2 l^*}{\partial \theta^2} = \frac{\partial^2 l}{\partial \theta^2} - \frac{2t}{n\theta^3} + \frac{t}{n\theta^3}\left(\frac{2 + \varepsilon + 2\varepsilon\ln\left(\frac{t}{\theta}\right)}{1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}\right) + \frac{t^2}{n\theta^4}\left(\frac{1 + \varepsilon\ln\left(\frac{t}{\theta}\right)}{1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}\right)^2.$$

Partial derivatives of $l^*$ produce the system of equations:

$$\begin{cases} \frac{1}{n}\left(\frac{-\frac{t}{\theta} + \frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}{1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}\right) - \frac{1}{\varepsilon n} + \frac{1}{n}\sum_{i=1}^{n} s_i u_i^{-1} = 0, \\ -\frac{(1+n)}{n\theta} + \frac{t+S}{n\theta^2} + \frac{1}{n\theta}\sum_{i=1}^{n}\left(v_i u_i^{-1}\right) + \frac{t}{\theta^2}\left(\frac{1 + \varepsilon\ln\left(\frac{t}{\theta}\right)}{1 + (1-\varepsilon)\frac{t}{\theta} + \varepsilon\frac{t}{\theta}\ln\left(\frac{t}{\theta}\right)}\right) = 0. \end{cases}$$

**Data analysis:** We take the data $x_i$ used by Lawless (2003, 2002):

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 14 | 34 | 59 | 61 | 69 | 80 | 123 | 142 | 165 | 210 | 381 |
| 464 | 479 | 556 | 574 | 839 | 917 | 969 | 991 | 1064 | 1088 | 1091 |
| 1174 | 1270 | 1275 | 1355 | 1397 | 1477 | 1578 | 1649 | 1702 | 1893 | 1932 |
| 2001 | 2161 | 2292 | 2326 | 2337 | 2628 | 2785 | 2811 | 2886 | 2993 | 2993 |
| 3248 | 3715 | 3790 | 3857 | 3912 | 4100 | 4106 | 4116 | 4315 | 4510 | 4584 |
| 5267 | 5299 | 5583 | 6065 | 9701 | | | | | | |

These data correspond to an empirical failure rate "out of bath-tub" that the author proposed to identify a balanced additive mixture of two Weibull distributions having this survival function:

$$R(t, \alpha_1, \beta_1, \alpha_2, \beta_2, p) = p exp\left[-\left(\frac{t}{\alpha_1}\right)^{\beta_1}\right] + (1-p) exp\left[-\left(\frac{t}{\alpha_2}\right)^{\beta_2}\right]$$

By maximization of likelihood:

$$\beta_{1MV} = 1.66(0.49), \alpha_{1MV} = 95.4(25.8), \beta_{2MV} = 1.40(0.18), \alpha_{2MV} = 2774.5(314.2), \quad p_{MV} = 0.137(0.051)$$

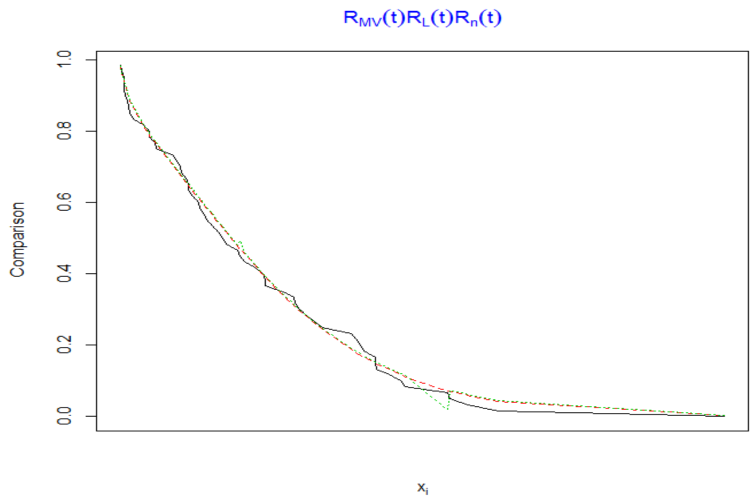


Fig. 1: Empirical reliability (black solid line), $R_{MV}$ (red dotted lines), $R_L$(green dotted lines)

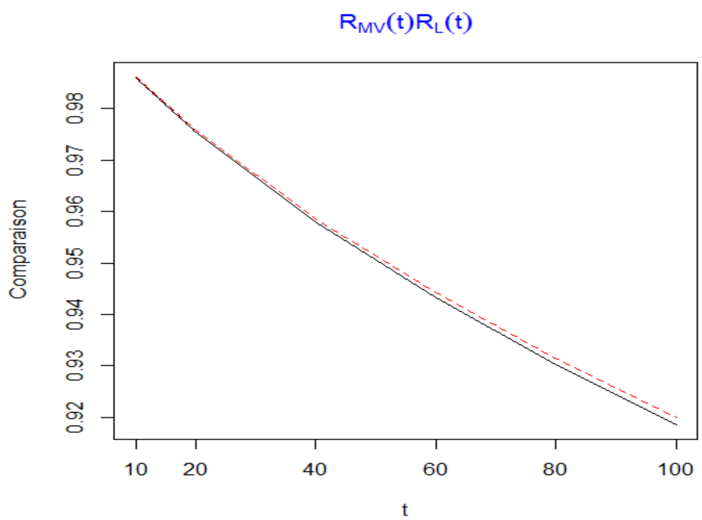


Fig. 2: $R_{MV}(t)$ (black solid line), $R_L$ (red dotted lines)

(The values between brackets are the standard deviations associated to the estimators). The distance of Kolmogorov-Smirnov between empirical reliability and estimated reliability by this model of mixture with 5 parameters is $D_K = 0.0417$. As alternative, we propose a model of mixture with 2 parameters of the type (2) by noticing that the first term of the density (1) translated the fact that a small number of components of the population presents defects known as of "youth" or with "infant mortality". The maximization of likelihood provides the following estimators:

$$\varepsilon_{MV} = 0.291(0.087), \theta_{MV} = 1190.4(78.7)$$

Also Kolmogorov-Smirnov distances between empirical reliability and the reliability estimated by maximum likelihood and Lindley's method of the new model are, respectively: $D_{KMV} = 0.0460$ and $D_{KM,L} = 0.0490$. The three distances of Kolmogorov-Smirnov are very close what translates the great proximity of the survival functions estimated starting from the traditional model of mixture and of the new model and the fact that they are practically indistinguishable. This is illustrated in the Fig. 1 and 2. The Kolmogorov Smirnov distances are presented in Table 1 and 2.

**Remark:** Estimators of the survival function obtained by maximum likelihood and Lindley's approximation for any value of t; are practically indistinguishable and very close to the true values of the survival function; this can be explained by the use of the non-informative prior.

**Simulation study:** In trying to illustrate and compare the methods as described above, a random sample of size, $n = 25, 50, 100$ and $1000$ were generated from (1) with $\varepsilon = 0.5$ and $\theta = 1.5$. These were replicated 1000 times (Table 3 and 4).

Table 1: Kolmogorov-Smirnov distances

| $D_K$z | $D_{KMV}$ | $D_{KM,L}$ |
|---|---|---|
| 0.042 | 0.046 | 0.049 |

Table 2: The estimators "MLE and Lindley of $R_t$"

| Method | $t = 10$ | $t = 20$ | $t = 40$ | $t = 60$ | $t = 80$ | $t = 100$ |
|---|---|---|---|---|---|---|
| MLE | 0.9860 | 0.9754 | 0.9579 | 0.9432 | 0.9302 | 0.9185 |
| Lindley | 0.9862 | 0.9759 | 0.9587 | 0.9442 | 0.9315 | 0.9200 |

Table 3: Estimates of parameters

| | MLE | | LINDLEY | |
|---|---|---|---|---|
| N | $\hat{\varepsilon}$ | $\hat{\theta}$ | $\hat{\varepsilon}$ | $\hat{\theta}$ |
| 25 | 0.5951 | 1.2436 | 0.5291 | 1.1766 |
| | ($9.044616\times10^{-6}$) | ($7.967312\times10^{-5}$) | ($8.461261\times10^{-7}$) | ($4.635779\times10^{-5}$) |
| 50 | 0.5000 | 1.2950 | 0.4675 | 1.2949 |
| | ($2.456328\times10^{-12}$) | ($4.202017\times10^{-5}$) | ($1.058368\times10^{-6}$) | ($4.20737\times10^{-5}$) |
| 100 | 0.4965 | 1.3287 | 0.4832 | 1.3243 |
| | ($1.238528\times10^{-8}$) | ($2.9349\times10^{-5}$) | ($2.836454\times10^{-7}$) | ($3.088453\times10^{-5}$) |
| 1000 | 0.4655 | 1.4917 | 0.4632 | 1.5040 |
| | ($1.193768\times10^{-6}$) | ($6.869766\times10^{-8}$) | ($1.35127\times10^{-6}$) | ($1.63151\times10^{-8}$) |

Table 4: Survival function estimation

| | $\theta$ | | θ = 1.5 | |
|---|---|---|---|---|
| $t$ | | 0.5 | 1.5 | 2.5 |
| $R_t$ | | 0.7048 | 0.5518 | 0.4267 |
| | MLE | 0.6320 | 0.4858 | 0.3549 |
| $n = 25$ | LINDLEY | 0.6367 | 0.4923 | 0.3600 |
| | T-K | 0.6601 | 0.4875 | 0.3266 |
| | MLE | 0.6861 | 0.5226 | 0.3772 |
| $n = 50$ | LINDLEY | 0.6946 | 0.5319 | 0.3870 |
| | T-K | 0.2706 | 0.1986 | 0.1317 |
| | MLE | 0.6911 | 0.5292 | 0.3867 |
| $n = 100$ | LINDLEY | 0.6945 | 0.5327 | 0.3901 |
| | T-K | 0.1703 | 0.1261 | 0.0860 |
| | MLE | 0.7214 | 0.5634 | 0.4302 |
| $n = 1000$ | LINDLEY | 0.7232 | 0.5658 | 0.4334 |
| | T-K | 0.1123 | 0.0757 | 0.0367 |

The comparisons were based on values from Mean Square Error (MSE).
Where,

$$MSE(\hat{\theta}) = (1000)^{-1} \sum_{i=1}^{1000} (\hat{\theta}_i - \theta)^2.$$

The results of the survival function are given in the Table 4.

## CONCLUSION

The model used in this study, is a mixture of a $G\left(2, \frac{1}{\theta}\right)$ and $L(\theta)$ distribution which depends on $\theta$. The mixture parameter is $\varepsilon$. We have shown the equivalence between this model and a classical mixture model of two Weibull distributions with a Kolmogorov Smirnov distance which is the same result found by Idée and Pierrat (2010); Idée (2006) in addition, the estimators of parameters and the survival function of this model obtained by the maximum of likelihood and Bayesian methods with a non-informative prior and a quadratic loss function are equivalent. Tierney-Kadane gives bad results for the large samples ($n \geq 50$).

## REFERENCES


Dempster, A.P., N.M. Laird and D. Rubin, 1977. Maximum likelihood from incomplete data via the EM algorithm. J. Roy. Stat. Soc. B Met., 39(1): 1-38.

Elmahdy, E.E. and A.W. Aboutahoun, 2013. A new approach for parameter estimation of finite Weibull mixture distributions for reliability modelling. Appl. Math. Model., 37: 1800-1810.

Gelman, A., J.B. Carlin, H.S. Stern and D.B. Rubin, 2003. Bayesian Data Analysis. 2nd Edn., Chapman and Hall, New York.

Idée, E., 2006. Loi de probabilité obtenue par perturbation aléatoire simple de l'un des paramètres de la loi de référence. Prépublication du LAMA-Université de Savoie, N° 07-06a.

Idée, E. and L. Pierrat, 2010. Genèse et estimation d'un modèle de fiabilité en Baignoire à Taux de Défaillance Borné. 42ème Journées de Statistique, Marseille, France.

Jiang, R. and D.N.P. Murthy, 1995. Modeling failure-data by mixture of 2 Weibull distributions: A graphical approach. IEEE T. Reliab., 44(3): 477-488.

Jiang, S.Y. and D. Kececioglu, 1992. Maximum likelihood estimates from censored data for mixed-Weibull distribution. IEEE T. Reliab., 41: 248-255.

Lawless, J.F., 2002. Statistical Models and Methods for Lifetime Data. 2nd Edn., A John Wiley and Sons, New York.

Lawless, J.F., 2003. Statistical Models and Methods for Lifetime Data. Wiley Series in Probability and Statistics, John Wiley and Sons, Hoboken, NJ.

Levine, R.A. and G. Casella, 2001. Implementation of the Monte-Carlo EM algorithm. J. Comput. Graph. Stat., 10: 422-439.

Lindley, D.V., 1980. Approximate Bayesian methods. Trabajos Estadist. Invest. Oper., 31: 232-245.

Tierney, L. and J.B. Kadane, 1986. Accurate approximations for posterior moments and marginal densities. J. Am. Stat. Assoc., 81: 82-86.

Tsionas, E.G., 2002. Bayesian analysis of finite mixtures of Weibull distributions. Commun. Stat. A-Theor., 31(1): 37-48.

Wei, G.C.G. and M.A. Tanner, 1990. A monte carlo implementation of the EM Al-gorithm and the poor man's data augmentation algorithms. J. Am. Stat. Assoc., 85: 699-704.