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Research Article

Application of Order Nine Block Method for Solving Second Order Ordinary Differential Equations Directly

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Abstract: In this study, a new block method of order nine is proposed to solve second order initial value problems of ordinary differential equations directly. The method is developed via interpolation and collocation approach where the use of power series approximate solution as a basis function is considered. The properties of the developed block method which includes zero-stability, order, consistency and convergence are also established. The numerical results reveal that the new method performs better than the existing method when applied for solving second order ordinary differential equations.

Keywords: Block method, collocation, interpolation, initial value problems, ordinary differential equations

INTRODUCTION

In this study, numerical method for the direct solution of initial value problems of ordinary differential equations of the form

$$y'' = f(x, y, y') \quad y(0) = \alpha, y'(0) = \beta \tag{1}$$

is examined. Efforts in the development of numerical methods for solving higher order initial value problems especially second order ordinary differential equations have been made by eminent scholars such as Awoyemi (2001), Adesanya *et al*. (2008), Kayode (2008) and Yahaya and Badmus (2009).

Recently, much attention had been devoted in developing predictor-corrector methods for solving (1) directly. It is noted that this method is associated with some limitations and these are: computational burden which as a result of many functions to be evaluated, combining of predictors that are of lower order with the correctors. All these highlighted drawbacks affect the accuracy of the method (Kayode, 2008; Adesanya *et al*., 2008).

In order to advance the accuracy of numerical methods, block method was introduced to simultaneously generate numerical results (Adesanya *et al*., 2013; Omar and Suleiman, 1999, 2003, 2005). This method gives better approximation and found to be cost effective because of the evaluation of few functions involved. The derivation of block methods with lower step-length for solving second order ordinary differential equations have been considered by researchers like Adesanya *et al*. (2008) and Mohammed *et al*. (2010) in which the accuracy of the methods are very low.

To bring improvement on the accuracy of block method, this study considers higher step-length $k = 8$ for the development of block method that can solve second order initial value problems of ordinary differential equations directly.

METHODOLOGY

Power series approximate solution of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{2}$$

is considered as an interpolation polynomial. Where r and s are the number of interpolation and collocation points, respectively. Equation (2) is differentiated twice to give:

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} \tag{3}$$

The approximate solution (2) is interpolated at $x = x_{n+i}$, $i = 5(1)6$ and we collocate Eq. (3) at $x = x_{n+i}$, $i = 0(1)8$. The interpolation and collocation equations give:

$$AX = B \tag{4}$$

where, $X = [a_0, a_1, a_2, ..., a_{10}]^T$, $B = [y_{n+5}, y_{n+6}, f_n, f_{n+1}, ..., f_{n+8}]^T$ the value of A is shown in Appendix A.

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Gaussian elimination method is applied to find the values of the unknown variables a_j, $j = 0(1)k+2$ (refer to Appendix B) in (4) which are substituted into Eq. (2) to give a continuous implicit linear multistep method of the form:

$$y(z) = \sum_{j=5}^{k=2} \alpha_j(z) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(z) f_{n+j} \tag{5}$$

where the coefficients $\alpha_j(z)$ and $\beta_j(z)$ are given as:

$$\begin{pmatrix} \alpha_5(z) \\ \alpha_6(z) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \end{pmatrix},$$

$$\begin{pmatrix} \beta_0(z) \\ \beta_1(z) \\ \beta_2(z) \\ \beta_3(z) \\ \beta_4(z) \\ \beta_5(z) \\ \beta_6(z) \\ \beta_7(z) \\ \beta_8(z) \end{pmatrix} = C \begin{pmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \\ z^7 \\ z^8 \\ z^9 \\ z^{10} \end{pmatrix},$$

where $k = 8, z = \dfrac{x - x_{n+k-1}}{h}$ and C is displayed in Appendix A.

Evaluating (5) at the non-interpolating points and its derivative at $x = x_{n+i}$, $i = 0(1)8$ to produce discrete schemes and its derivatives which are combined in a matrix finite difference equation to give a block of the form:

$$A' Y_{N+1} = B' Y_N + h D' Y_N^{'} + h^2 (E' F_{N+1} + E'' F_N) \tag{6}$$

where,

$$Y_{N+1} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ y_{n+8} \end{pmatrix}, Y_N = \begin{pmatrix} y_{n-7} \\ y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, Y'_N = \begin{pmatrix} y'_{n-7} \\ y'_{n-6} \\ y'_{n-5} \\ y'_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}, F_{N+1} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \\ f_{n+8} \end{pmatrix}, F_N = \begin{pmatrix} f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix},$$

$$B' = \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$A' = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$D' = \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&2 \\ 0&0&0&0&0&0&0&3 \\ 0&0&0&0&0&0&0&4 \\ 0&0&0&0&0&0&0&5 \\ 0&0&0&0&0&0&0&6 \\ 0&0&0&0&0&0&0&7 \\ 0&0&0&0&0&0&0&8 \end{pmatrix},$$

$$E'' = \begin{pmatrix} 0&0&0&0&0&0&0&\frac{1624505}{7257600} \\ 0&0&0&0&0&0&0&\frac{58193}{113400} \\ 0&0&0&0&0&0&0&\frac{71661}{89600} \\ 0&0&0&0&0&0&0&\frac{30812}{28350} \\ 0&0&0&0&0&0&0&\frac{398825}{290304} \\ 0&0&0&0&0&0&0&\frac{2325}{1400} \\ 0&0&0&0&0&0&0&\frac{2019731}{1036800} \\ 0&0&0&0&0&0&0&\frac{63296}{28350} \end{pmatrix}$$

and E' is shown in Appendix A.
The derivative of (6) gives:

$$\begin{pmatrix} y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y'_{n+5} \\ y'_{n+6} \\ y'_{n+7} \\ y'_{n+8} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y'_n + G \begin{pmatrix} f_{n+8} \\ f_{n+7} \\ f_{n+6} \\ f_{n+5} \\ f_{n+4} \\ f_{n+3} \\ f_{n+2} \\ f_{n+1} \\ f_n \end{pmatrix}$$

where, G can also be seen in Appendix A.

ANALYSIS OF THE METHOD

Order of the method: In finding the order of our method, the method proposed by Lambert (1973) is adopted. This gives our method to have a uniform order $[9,9,9,9,9,9,9,9]^T$ with error constants:

$$\left[\frac{63}{13346},\frac{140}{11873},\frac{148}{7995},\frac{187}{7388},\frac{155}{4827},\frac{299}{7700},\frac{51}{1111},\frac{187}{3694}\right]^T$$

Zero stability: The block method (6) is zero-stable if the roots $z_p, p = 1,2,.....,N$ of the first characteristic polynomial $\rho(z) = \det(zA' - B')$ satisfy $|z_p| \le 1$ and the root $|z_p| = 1$ having multiplicity not exceeding the order of differential equation under consideration. It is shown below:

$$\rho(z) = \left| z\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \right| = 0$$

This gives $\rho(z) = z^7(z-1)$ which implies z = 0,0,0,0,0,0,0,1. Hence, the new method (6) is convergent because it is zero-stable and having an order greater than one (Henrici, 1962).

NUMERICAL EXPERIMENTS

Problem 1: $y'' = -y + 2\cos x$, $y(0) = 1$, $y'(0) = 0$, $0 \le x \le 1$
Exact solution: $y(x) = \cos x + x \sin x$.
Omar and Suleiman (1999) solved the above differential problem where the maximum errors were also selected. Both results are shown in Table A1 (Appendix A).

Problem 2: $y'' = y$, $y(0) = 1$, $y'(0) = 1$, $0 \le x \le 1$. Exact solution: $y(x) = e^x$.
The above problem 2 was also solved by Omar and Suleiman (1999). We solved the same problem using our new method and maximum errors were selected. The result is shown in Table A2 (Appendix A).

CONCLUSION

A new order nine block method which was applied to second order initial value problems of ordinary differential equations to generate numerical results has been developed in this study. The results generated are shown in Tables A1 and A2 which are better than the results produced by Omar and Suleiman (1999).

Appendix A:

Table A1: Comparison of the new method with Omar and Suleiman (1999)

h	Exact solution	Numerical solution	Error in Omar and Suleiman (1999). K = 8	Error in the new method. K = 8
0.01	7.650397549979526700	7.650397550380509900	6.01615E-04	4.009832E-10
0.001	7.759795937206075900	7.759795937206067000	4.16853E-04	8.881784E-15
0.0001	7.768432159576440700	7.768432159576273800	4.20700E-05	1.669775E-13
0.00001	7.769272793381416500	7.769272793372717700	4.20736E-06	8.698819E-12

Table A2: Comparison of the new method with Omar and Suleiman (1999)

h	Exact solution	Numerical solution	Error in Omar and Suleiman (1999). K = 8	Error in new method. K = 8
0.01	2.944679551065525900	2.944661540405518500	1.06947E-02	1.801066E-05
0.001	2.740115300529943000	2.740113674210062900	5.93077E-04	1.626320E-06
0.0001	2.720457324003748800	2.720455876101673300	5.87659E-05	1.447902E-06
0.00001	2.718499299698846600	2.718497856159629100	5.87601E-06	1.443539E-06

$$A = \begin{pmatrix}
1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+4}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 & x_{n+5}^9 & x_{n+5}^{10} \\
1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 & x_{n+6}^9 & x_{n+6}^{10} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 & 90x_{n+1}^8 \\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 & 90x_{n+2}^8 \\
0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 & 72x_{n+3}^7 & 90x_{n+3}^8 \\
0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 & 90x_{n+4}^8 \\
0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7 & 90x_{n+5}^8 \\
0 & 0 & 2 & 6x_{n+6} & 12x_{n+6}^2 & 20x_{n+6}^3 & 30x_{n+6}^4 & 42x_{n+6}^5 & 56x_{n+6}^6 & 72x_{n+6}^7 & 90x_{n+6}^8 \\
0 & 0 & 2 & 6x_{n+7} & 12x_{n+7}^2 & 20x_{n+7}^3 & 30x_{n+7}^4 & 42x_{n+7}^5 & 56x_{n+7}^6 & 72x_{n+7}^7 & 90x_{n+7}^8 \\
0 & 0 & 2 & 6x_{n+8} & 12x_{n+8}^2 & 20x_{n+8}^3 & 30x_{n+8}^4 & 42x_{n+8}^5 & 56x_{n+8}^6 & 72x_{n+8}^7 & 90x_{n+8}^8
\end{pmatrix}$$

$$C=\begin{pmatrix}
\frac{3142}{7257600} & \frac{11203}{7257600} & 0 & \frac{-21600}{7257600} & \frac{-15660}{7257600} & \frac{1260}{7257600} & \frac{5334}{7257600} & \frac{2400}{7257600} & \frac{495}{7257600} & \frac{50}{7257600} & \frac{2}{7257600}\\
\frac{-7214}{1814400} & \frac{-26199}{1814400} & 0 & \frac{50400}{1814400} & \frac{35940}{1814400} & \frac{-3402}{1814400} & \frac{-12348}{1814400} & \frac{-5400}{1814400} & \frac{-1080}{1814400} & \frac{-105}{1814400} & \frac{-4}{1814400}\\
\frac{29434}{1814400} & \frac{110237}{1814400} & 0 & \frac{-211680}{1814400} & \frac{-147420}{1814400} & \frac{16884}{1814400} & \frac{51198}{1814400} & \frac{2154}{1814400} & \frac{4140}{1814400} & \frac{385}{1814400} & \frac{14}{1814400}\\
\frac{-69098}{1814400} & \frac{-275285}{1814400} & 0 & \frac{529200}{1814400} & \frac{355320}{1814400} & \frac{-51282}{1814400} & \frac{-125076}{1814400} & \frac{-49800}{1814400} & \frac{-9090}{1814400} & \frac{-805}{1814400} & \frac{-28}{1814400}\\
\frac{375700}{7257600} & \frac{1800330}{7257600} & 0 & \frac{-3528000}{7257600} & \frac{-2221800}{7257600} & \frac{430920}{7257600} & \frac{795060}{7257600} & \frac{291600}{7257600} & \frac{49950}{7257600} & \frac{4200}{7257600} & \frac{140}{7257600}\\
\frac{86302}{1814400} & \frac{-376177}{1814400} & 0 & \frac{1058400}{1814400} & \frac{578340}{1814400} & \frac{-169470}{1814400} & \frac{-210756}{1814400} & \frac{-69000}{1814400} & \frac{-10980}{1814400} & \frac{-875}{1814400} & \frac{-28}{1814400}\\
\frac{1522114}{1814400} & \frac{2191489}{1814400} & 0 & \frac{-1058400}{1814400} & \frac{-313740}{1814400} & \frac{184212}{1814400} & \frac{144438}{1814400} & \frac{41100}{1814400} & \frac{6030}{1814400} & \frac{455}{1814400} & \frac{14}{1814400}\\
\frac{163066}{1814400} & \frac{659613}{1814400} & \frac{907200}{1814400} & \frac{481680}{1814400} & \frac{1920}{1814400} & \frac{-114534}{1814400} & \frac{-57708}{1814400} & \frac{-14040}{1814400} & \frac{-1890}{1814400} & \frac{-135}{1814400} & \frac{-4}{1814400}\\
\frac{-19658}{7257600} & \frac{-59845}{7257600} & 0 & \frac{151200}{7257600} & \frac{196020}{7257600} & \frac{118188}{7257600} & \frac{40614}{7257600} & \frac{8400}{7257600} & \frac{1035}{7257600} & \frac{70}{7257600} & \frac{2}{7257600}
\end{pmatrix}$$

$$E'=\begin{pmatrix}
\frac{4124231}{7257600} & \frac{-5225623}{7257600} & \frac{6488191}{7257600} & \frac{-5888311}{7257600} & \frac{3698922}{7257600} & \frac{-1522673}{7257600} & \frac{369744}{7257600} & \frac{-40187}{7257600}\\
\frac{235072}{113400} & \frac{-183708}{113400} & \frac{247328}{113400} & \frac{-227030}{113400} & \frac{143232}{113400} & \frac{-59092}{113400} & \frac{14368}{113400} & \frac{-1563}{113400}\\
\frac{328608}{89600} & \frac{-150624}{89600} & \frac{315000}{89600} & \frac{-281430}{89600} & \frac{177264}{89600} & \frac{-73128}{89600} & \frac{17784}{89600} & \frac{-1935}{89600}\\
\frac{148992}{28350} & \frac{-46400}{28350} & \frac{160256}{28350} & \frac{-118440}{28350} & \frac{76288}{28350} & \frac{-31552}{28350} & \frac{7680}{28350} & \frac{-836}{28350}\\
\frac{1987000}{290304} & \frac{-465000}{290304} & \frac{2294000}{290304} & \frac{-1283750}{290304} & \frac{1020600}{290304} & \frac{-412000}{290304} & \frac{100000}{290304} & \frac{-10875}{290304}\\
\frac{11808}{1400} & \frac{-2196}{1400} & \frac{14208}{1400} & \frac{-6390}{1400} & \frac{7200}{1400} & \frac{-2268}{1400} & \frac{576}{1400} & \frac{-63}{1400}\\
\frac{10388784}{1036800} & \frac{-1575056}{1036800} & \frac{12811736}{1036800} & \frac{-4826010}{1036800} & \frac{7068544}{1036800} & \frac{-1018024}{1036800} & \frac{589176}{1036800} & \frac{-57281}{1036800}\\
\frac{329728}{28350} & \frac{-44544}{28350} & \frac{419840}{28350} & \frac{-145280}{28350} & \frac{251904}{28350} & \frac{-14848}{28350} & \frac{47104}{28350} & 0
\end{pmatrix}$$

$$G=\begin{pmatrix}
\frac{-10004}{1069200} & \frac{92186}{1069200} & \frac{-380447}{1069200} & \frac{927046}{1069200} & \frac{-1482974}{1069200} & \frac{1648632}{1069200} & \frac{-1356711}{1069200} & \frac{1316197}{1069200} & \frac{315273}{1069200}\\
\frac{-833}{113400} & \frac{7624}{113400} & \frac{-31154}{113400} & \frac{74728}{113400} & \frac{-116120}{113400} & \frac{120088}{113400} & \frac{-42494}{113400} & \frac{182584}{113400} & \frac{32377}{113400}\\
\frac{-369}{44800} & \frac{3402}{44800} & \frac{-14062}{44800} & \frac{34434}{44800} & \frac{-56160}{44800} & \frac{79934}{44800} & \frac{3438}{44800} & \frac{70902}{44800} & \frac{12881}{44800}\\
\frac{-214}{28350} & \frac{1952}{28350} & \frac{-7912}{28350} & \frac{18464}{28350} & \frac{-18160}{28350} & \frac{65504}{28350} & \frac{488}{28350} & \frac{45152}{28350} & \frac{8126}{28350}\\
\frac{-1225}{145152} & \frac{11450}{145152} & \frac{-49150}{145152} & \frac{170930}{145152} & \frac{-4000}{145152} & \frac{318350}{145152} & \frac{7550}{145152} & \frac{230150}{145152} & \frac{41705}{145152}\\
\frac{-9}{1400} & \frac{72}{1400} & \frac{158}{1400} & \frac{2664}{1400} & \frac{-360}{1400} & \frac{3224}{1400} & \frac{18}{1400} & \frac{2232}{1400} & \frac{401}{1400}\\
\frac{-8183}{518400} & \frac{223174}{518400} & \frac{522046}{518400} & \frac{736078}{518400} & \frac{54880}{518400} & \frac{1085937}{518400} & \frac{48706}{518400} & \frac{816634}{518400} & \frac{149527}{518400}\\
\frac{7912}{28350} & \frac{47104}{28350} & \frac{-7424}{28350} & \frac{83968}{28350} & \frac{-36320}{28350} & \frac{83968}{28350} & \frac{-7424}{28350} & \frac{47104}{28350} & \frac{7912}{28350}
\end{pmatrix}$$

Appendix B:

The values of $a_j,\ j = 0(1)k+2$

$$a_0 = 6y_{n+5} - 5y_{n+6} + \frac{2935}{48384}h^2 f_n + \frac{33389}{30240}h^2 f_{n+1} + \frac{17819}{10080}h^2 f_{n+2} + \frac{50357}{15120}h^2 f_{n+3} + \frac{89779}{24192}h^2 f_{n+4}$$
$$+ \frac{1035}{224}h^2 f_{n+5} + \frac{3139}{7560}h^2 f_{n+6} - \frac{73}{7560}h^2 f_{n+7} - \frac{19}{80640}h^2 f_{n+8} + \frac{1}{2}x_n^2 f_n + + \frac{x_n}{h}(y_{n+4} - y_{n+5}) +$$
$$\frac{x_n^3}{5040h}(2283f_n - 6720f_{n+1} + 11760f_{n+2} - 15680f_{n+3} + 14700f_{n+4} - 9408f_{n+5} + 3920f_{n+6} - 960f_{n+7}$$
$$+105f_{n+8}) + \frac{x_n^9}{725760h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + 210f_{n+6} - 58f_{n+7}$$
$$+7f_{n+8}) + \frac{x_n^8}{161280h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3} + 2090f_{n+4} - 1564f_{n+5} + 732f_{n+6} -$$
$$196f_{n+7} + 23f_{n+8}) + \frac{x_n^7}{30240h^5}(81f_n - 575f_{n+1} + 1790f_{n+2} - 3195f_{n+3} + 3580f_{n+4} - 2581f_{n+5} +$$
$$1170f_{n+6} - 305f_{n+7} + 35f_{n+8}) + \frac{x_n^5}{28800h^3}(2403f_n - 13960f_{n+1} + 36706f_{n+2} - 57384f_{n+3} + 58280f_{n+4}$$
$$-39128f_{n+5} + 16830f_{n+6} - 4216f_{n+7} + 469f_{n+8}) + \frac{x_n^6}{172800h^4}(3207f_n - 21056f_{n+1} + 61156f_{n+2}$$
$$-102912f_{n+3} + 109930f_{n+4} - 76352f_{n+5} + 33636f_{n+6} - 8576f_{n+7} + 967f_{n+8}) + \frac{x_n^4}{120960h^2}(29531f_n$$
$$-138528f_{n+1} + 312984f_{n+2} - 448672f_{n+3} + 435330f_{n+4} - 284256f_{n+5} + 120008f_{n+6} - 29664f_{n+7} +$$
$$3267f_{n+8}) + \frac{x_n^{10}}{3628800h^6}(f_n - 8f_{n+1} + 28f_{n+2} - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8}) +$$
$$\frac{83287h}{290304}x_n f_n + \frac{1442209h}{907200}x_n f_{n+1} + \frac{10039h}{302400}x_n f_{n+2} + \frac{1019017h}{453600}x_n f_{n+3} - \frac{14743h}{103680}x_n f_{n+4} + \frac{729h}{448}x_n f_{n+5}$$
$$-\frac{45541h}{226800}x_n f_{n+6} + \frac{15187h}{226800}x_n f_{n+7} - \frac{18239h}{2419200}x_n f_{n+8}$$

$$a_1 = \frac{14743}{103680}hf_{n+4} - \frac{1442209}{907200}hf_{n+1} - \frac{10039}{302400}hf_{n+2} - \frac{1019017}{453600}hf_{n+3} - \frac{83287}{290304}hf_n - \frac{729}{448}hf_{n+5}$$
$$+ \frac{45541}{226800}hf_{n+6} - \frac{15187}{226800}hf_{n+7} + \frac{18239}{2419200}hf_{n+8} - x_n f_n - \frac{1}{h}(y_{n+5} - y_{n+6}) - \frac{x_n^2}{1680h}(2283f_n$$
$$-6720f_{n+1} + 11760f_{n+2} - 15680f_{n+3} + 14700f_{n+4} - 9408f_{n+5} + 3920f_{n+6} - 960f_{n+7} + 105f_{n+8})$$
$$-\frac{x_n^8}{80640h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} + 560f_{n+4} - 434f_{n+5} + 210f_{n+6} - 58f_{n+7} + 7f_{n+8})$$
$$-\frac{x_n^7}{20160h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3} + 2090f_{n+4} - 1564f_{n+5} + 732f_{n+6} - 196f_{n+7}$$
$$+23f_{n+8}) - \frac{x_n^6}{4320h^5}(81f_n - 575f_{n+1} + 1790f_{n+2} - 3195f_{n+3} + 3580f_{n+4} - 2581f_{n+5} + 1170f_{n+6}$$
$$-305f_{n+7} + 35f_{n+8}) - \frac{x_n^4}{5760h^3}(2403f_n - 13960f_{n+1} + 36706f_{n+2} - 57384f_{n+3} + 58280f_{n+4} -$$
$$39128f_{n+5} + 16830f_{n+6} - 4216f_{n+7} + 469f_{n+8}) - \frac{x_n^5}{28800h^4}(3207f_n - 21056f_{n+1} + 61156f_{n+2} -$$
$$102912f_{n+3} + 109930f_{n+4} - 76352f_{n+5} + 33636f_{n+6} - 8576f_{n+7} + 967f_{n+8}) - \frac{x_n^3}{30240h^2}(29531f_n$$
$$-138528f_{n+1} + 312984f_{n+2} - 448672f_{n+3} + 435330f_{n+4} - 284256f_{n+5} + 120008f_{n+6} - 29664f_{n+7}$$
$$+3267f_{n+8}) - \frac{x_n^9}{362880h^8}(f_n - 8f_{n+1} + 28f_{n+2}' - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8})$$

$$39128f_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})-\frac{x_n^5}{28800h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-$$
$$102912f_{n+3}+109930f_{n+4}-76352f_{n+5}+33636f_{n+6}-8576f_{n+7}+967f_{n+8})-\frac{x_n^3}{30240h^2}(29531f_n$$
$$-138528f_{n+1}+312984f_{n+2}-448672f_{n+3}+435330f_{n+4}-284256f_{n+5}+120008f_{n+6}-29664f_{n+7}$$
$$+3267f_{n+8})-\frac{x_n^9}{362880h^8}(f_n-8f_{n+1}+28f_{n+2}{}'-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})$$

$$a_2=\frac{1}{2}f_n+\frac{x_n^7}{20160h^7}(9f_n-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+$$
$$7f_{n+8})+\frac{x_n^6}{5760h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}$$
$$+23f_{n+8})+\frac{x_n^5}{1440h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}-$$
$$305f_{n+7}+35f_{n+8})+\frac{x_n^3}{2880h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}$$
$$16830f_{n+6}-4216f_{n+7}+469f_{n+8})-\frac{x_n^3}{8640h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+$$
$$109930f_{n+4}-76352f_{n+5}+133636f_{n+6}-8576f_{n+7}+967f_{n+8})-\frac{x_n^7}{30240h^8}(f_n-8f_{n+1}+28f_{n+2}$$
$$-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})-\frac{x_n}{1680h}(2283f_n-6720f_{n+1}+11760f_{n+2}$$
$$-15680f_{n+3}+14700f_{n+4}-9408f_{n+5}+3920f_{n+6}-960f_{n+7}+105f_{n+8})$$
$$\frac{x_n^4}{11520h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}+33636f_{n+6}$$
$$-8576f_{n+7}+967f_{n+8})+\frac{x_n^2}{20160h^2}(29531f_n-138528f_{n+1}+312984f_{n+2}-448672f_{n+3}+$$
$$435330f_{n+4}-284256f_{n+5}+120008f_{n+6}-29664f_{n+7}+3267f_{n+8})+\frac{x_n^8}{80640h^8}(f_n-8f_{n+1}+28f_{n+2}{}'$$
$$-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})+\frac{x_n}{1680h}(2283f_n-6720f_{n+1}+11760f_{n+2}$$
$$-15680f_{n+3}+14700f_{n+4}-9408f_{n+5}+3920f_{n+6}-960f_{n+7}+105f_{n+8})$$

$$a_3=\frac{-761}{1680h}f_n+\frac{}{3h}f_{n+1}-\frac{7}{3h}f_{n+2}+\frac{28}{9h}f_{n+3}-\frac{35}{12h}f_{n+4}+\frac{28}{15h}f_{n+5}-\frac{7}{9h}f_{n+6}+\frac{4}{21h}f_{n+7}-\frac{1}{48h}f_{n+8}$$
$$-\frac{x_n^6}{8640h^7}(9f_n-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})-$$
$$\frac{x_n^5}{2880h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})$$
$$-\frac{x_n^4}{864h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}-305f_{n+7}+$$
$$35f_{n+8})-\frac{x_n^2}{2880h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}+$$

$$16830f_{n+6}-4216f_{n+7}+469f_{n+8})-\frac{x_n^3}{8640h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+$$
$$109930f_{n+4}-76352f_{n+5}+133636f_{n+6}-8576f_{n+7}+967f_{n+8})-\frac{x_n^7}{30240h^8}(f_n-8f_{n+1}+28f_{n+2}$$
$$-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})-+\frac{x_n}{5760h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}$$
$$-57384f_{n+3}+58280f_{n+4}-39128f_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})$$

$$a_4=\frac{29531}{120960h^2}f_n-\frac{481}{420h^2}f_{n+1}+\frac{207}{80h^2}f_{n+2}-\frac{2003}{540h^2}f_{n+3}+\frac{691}{192h^2}f_{n+4}-\frac{47}{20h^2}f_{n+5}+\frac{2143}{2160h^2}f_{n+6}$$
$$-\frac{103}{240h^2}f_{n+7}+\frac{121}{4480h^2}f_{n+8}-\frac{x_n^5}{5760h^7}(9f_n-70f_{n+1}+238f_{n+2}-462f_{n+3}+560f_{n+4}-434f_{n+5}$$
$$+210f_{n+6}-58f_{n+7}+7f_{n+8})+\frac{x_n^4}{2304h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}$$
$$+732f_{n+6}-196f_{n+7}+23f_{n+8})+\frac{x_n^3}{864h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}$$
$$+1170f_{n+6}-305f_{n+7}+35f_{n+8})+\frac{x_n^2}{11520h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+$$
$$109930f_{n+4}-76352f_{n+5}+133636f_{n+6}-8576f_{n+7}+967f_{n+8})-\frac{x_n^6}{17280h^8}(f_n-8f_{n+1}+28f_{n+2}'-56f_{n+3}$$
$$+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})+\frac{x_n}{5760h^3}(2403f_n-13960f_{n+1}+36706f_{n+2}-57384f_{n+3}+$$
$$58280f_{n+4}-39128f_{n+5}+16830f_{n+6}-4216f_{n+7}+469f_{n+8})$$

$$a_5=\frac{-267}{3200h^3}f_n+\frac{349}{720h^3}f_{n+1}-\frac{18353}{14400h^3}f_{n+2}+\frac{797}{400h^3}f_{n+3}-\frac{1457}{720h^3}f_{n+4}+\frac{4891}{3600h^3}f_{n+5}$$
$$-\frac{187}{320h^3}f_{n+6}+\frac{527}{3600h^3}f_{n+7}--\frac{469}{28800h^3}f_{n+8}-\frac{x_n^4}{5760h^7}(9f_n-70f_{n+1}+238f_{n+2}-$$
$$462f_{n+3}+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})+\frac{x_n^3}{2880h^6}(39f_n-292f_{n+1}+$$
$$956f_{n+2}-1788f_{n+3}+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})-\frac{x_n^2}{1440h^5}(81f_n$$
$$-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+3580f_{n+4}-2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})$$
$$-\frac{x_n^5}{14400h^8}(f_n-8f_{n+1}+28f_{n+2}'-56f_{n+3}+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})-$$
$$\frac{x_n}{28800h^4}(3207f_n-21056f_{n+1}+61156f_{n+2}-102912f_{n+3}+109930f_{n+4}-76352f_{n+5}+$$
$$133636f_{n+6}-8576f_{n+7}+967f_{n+8})$$

$$a_6=\frac{1069}{57600h^4}f_n-\frac{329}{2700h^4}f_{n+1}+\frac{15289}{43200h^4}f_{n+2}-\frac{134}{225h^4}f_{n+3}+\frac{10993}{17280h^4}f_{n+4}-\frac{1193}{2700h^4}f_{n+5}$$
$$+\frac{2803}{14400h^4}f_{n+6}-\frac{67}{1350h^4}f_{n+7}++\frac{967}{172800h^4}f_{n+8}+\frac{x_n^3}{8640h^7}(9f_n-70f_{n+1}+238f_{n+2}-462f_{n+3}$$
$$+560f_{n+4}-434f_{n+5}+210f_{n+6}-58f_{n+7}+7f_{n+8})+\frac{x_n^2}{5760h^6}(39f_n-292f_{n+1}+956f_{n+2}-1788f_{n+3}$$
$$+2090f_{n+4}-1564f_{n+5}+732f_{n+6}-196f_{n+7}+23f_{n+8})-\frac{x_n^4}{17280h^8}(f_n-8f_{n+1}+28f_{n+2}'-56f_{n+3}$$
$$+70f_{n+4}-56f_{n+5}+28f_{n+6}-8f_{n+7}+f_{n+8})+\frac{x_n}{4320h^5}(81f_n-575f_{n+1}+1790f_{n+2}-3195f_{n+3}+$$
$$3580f_{n+4}-2581f_{n+5}+1170f_{n+6}-305f_{n+7}+35f_{n+8})$$

$$a_7 = \frac{-3}{1120h^5} f_n + \frac{115}{6048h^5} f_{n+1} - \frac{179}{3024h^5} f_{n+2} + \frac{71}{672h^5} f_{n+3} - \frac{179}{1512h^5} f_{n+4} + \frac{2581}{30240h^5} f_{n+5} -$$
$$\frac{13}{336h^5} f_{n+6} + \frac{61}{6048h^5} f_{n+7} - \frac{1}{864h^5} f_{n+8} - \frac{x_n^2}{20160h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3} +$$
$$560f_{n+4} - 434f_{n+5} + 210f_{n+6} - 58f_{n+7} + 7f_{n+8}) - \frac{x_n^3}{30240h^8}(f_n - 8f_{n+1} + 28f_{n+2}' - 56f_{n+3} +$$
$$70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8}) - \frac{x_n}{20160h^6}(39f_n - 292f_{n+1} + 956f_{n+2} - 1788f_{n+3}$$
$$+ 2090f_{n+4} - 1564f_{n+5} + 732f_{n+6} - 196f_{n+7} + 23f_{n+8})$$

$$a_8 = \frac{13}{53760h^6} f_n - \frac{73}{40320h^6} f_{n+1} + \frac{239}{40320h^6} f_{n+2} - \frac{149}{13440h^6} f_{n+3} + \frac{209}{16128h^6} f_{n+4} - \frac{391}{40320h^6} f_{n+5}$$
$$+ \frac{61}{134400h^6} f_{n+6} - \frac{7}{5760h^6} f_{n+7} + \frac{23}{161280h^6} f_{n+8} - \frac{x_n^2}{80640h^8}(f_n - 8f_{n+1} + 28f_{n+2}' - 56f_{n+3} +$$
$$70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8}) + \frac{x_n}{80640h^7}(9f_n - 70f_{n+1} + 238f_{n+2} - 462f_{n+3}$$
$$+ 560f_{n+4} - 434f_{n+5} + 210f_{n+6} - 58f_{n+7} + 7f_{n+8})$$

$$a_9 = \frac{-1}{80640h^7} f_n + \frac{1}{10368h^7} f_{n+1} - \frac{17}{51840h^7} f_{n+2} + \frac{11}{17280h^7} f_{n+3} - \frac{1}{1296h^7} f_{n+4} + \frac{31}{51840h^7} f_{n+5}$$
$$- \frac{1}{3456h^7} f_{n+6} + \frac{29}{362880h^7} f_{n+7} - \frac{1}{103680h^7} f_{n+8} - \frac{x_n}{362880h^8}(f_n - 8f_{n+1} + 28f_{n+2}' - 56f_{n+3} +$$
$$70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8})$$

$$a_{10} = \frac{1}{3628800h^8}(f_n - 8f_{n+1} + 28f_{n+2}' - 56f_{n+3} + 70f_{n+4} - 56f_{n+5} + 28f_{n+6} - 8f_{n+7} + f_{n+8})$$

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