

Research Article

Theory of Breakdown of an Arbitrary Gas-dynamic Discontinuity-2D Flows Interaction

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Abstract: We have considered the theory of breakdown of an arbitrary gas-dynamic discontinuity for the space-time dimension equal to two. The link of this task with the geometrical theory of reconfiguration of shock-waves and wave fronts is shown. We consider the Riemann problem of the breakdown of an arbitrary discontinuity of parameters at angular collision of two flat flows. The problem is solved as accurate stated. We consider the solution region with different types of the shock-wave structure. The Mach number region is discovered and the angles of flows interaction for which there is no solution. We demonstrate the generality of solutions for one-dimensional non-stationary and two-dimensional stationary cases.

Keywords: Computational gas dynamics, contact discontinuity, discontinuity breakdown scheme, Riemann wave, shock-wave

INTRODUCTION

The problem considered here is breakdown of an arbitrary gas-dynamic discontinuity in the space-time with the dimension equal to two. Let's remind the basic concepts and terminology. The zero-order gas-dynamic discontinuity is the area of dramatic, discontinuous changes of the gas-dynamic variables. A non-stationary discontinuity, through which the flow goes, is traditionally called a shock-wave, a stationary one is called as a compression shock. There are discontinuities, through which the gas does not flow and the pressure on its sides is the same, the density and other parameters can differ. Such a discontinuity is called tangential (slip line), if its surface is parallel to the velocity vectors on its sides. In the other cases it is called a contact discontinuity. A simple compression wave or depression wave is the area of smooth variation of parameters, restricted on two sides by the first-order discontinuities where the first-order derivatives of gas-dynamic discontinuities jump.

If in the space-time there is a discontinuity of parameters occurred due to for some reasons, at variation of one of the parameters, for example, the time or dimensional coordinate, it transforms (a breakdown happens) into the shock-wave structure composed of a few waves and discontinuities (Fig. 1). At the same time, from the kinematic point of view, the time and space coordinates are absolutely equal and there is no difference between the breakdown of one-dimension non-stationary and two-dimensional stationary discontinuity. In studied in general the problem of an arbitrary discontinuity breakdown (Kotchine, 1927) for polytropic gases. In 1946-1953

Landau and Lifshits (1953) carried out more total study of the given problem and in the modern terms it was made by Kobzeva and Moiseev (2003).

An arbitrary discontinuity of the gas-dynamic parameters takes place, for example, at breakthrough of the shock tube diaphragm, at two shock-waves interaction with each other or with the contact discontinuity.

MATERIALS AND METHODS

Geometrical model of the discontinuity breakdown: We would like to remind how the geometrical concept of the gas-dynamic discontinuity and its modifications (breakdown into new discontinuities and waves) is introduced when a parameter is changing.

Let's consider, for example, a one-dimensional medium of particles moving along a straight line at a constant velocity. The particle free motion law looks as $x = \phi(t) = x_0 + ut$, where u is the particle velocity. Function ϕ satisfies the Newton equation:

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial u}{\partial x} \phi + \frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

On the other hand, the Euler equation describing the field of noninteracting particles has got the same form:

$$u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Thereby, the motion descriptions with the help of the Euler equation for the field of gas-dynamic

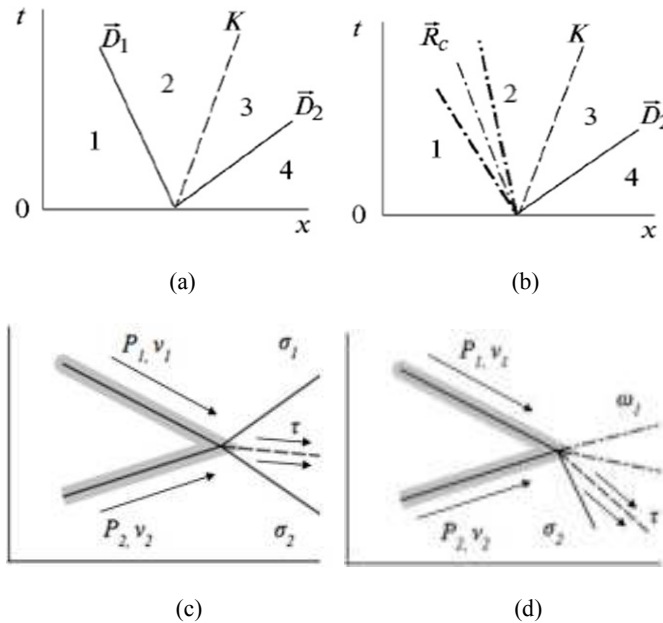


Fig. 1: Equivalence of the problems of an arbitrary discontinuity breakdown; (a): One-dimensional problem of two flows collision with formation in time t of two shock-waves $D_{1,2}$ and a contact discontinuity K , 1 and 4 are gas undisturbed zones, 2 and 3 are the cocurrent flow caused by a wave areas; (b): One-dimensional problem of pressure equalization with formation of a shockwave D_2 , simple Riemann wave R_c and contact discontinuity K ; c) two-dimensional problem of interaction of two inclined supersonic jets with pressure $P_{1,2}$ and velocity $v_{1,2}$, with formation of two outgoing jumps $\sigma_{1,2}$ and tangential discontinuity τ , d) two-dimensional problem of interaction of two inclined supersonic jets with formation of outgoing jump σ_2 , Prandtl-Meyer wave ω_1

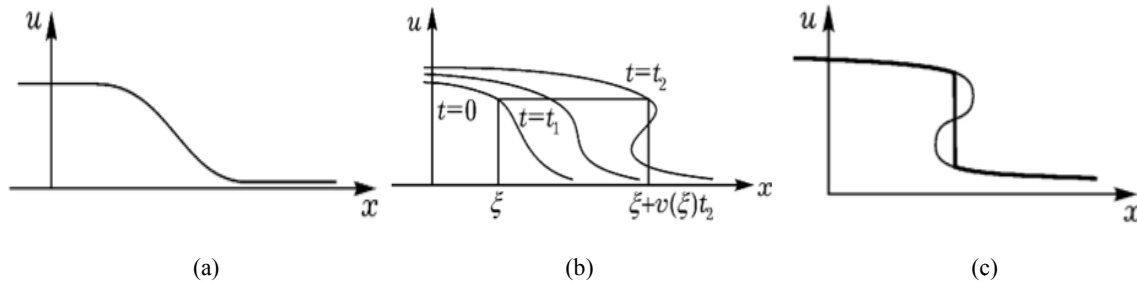


Fig. 2: Geometrical model of shock-wave formation; (a): Velocity initial distribution; (b): Solution peculiarity formation; (c): Shock-wave model

variables and with the help of the Newton equation for particles are equal. We know that quasilinear differential equations in partial derivatives are solved by means of characteristics construction. The Euler equation characteristics are equivalent to the Newton law for a moving particle (Arnold, 1978). In such a way, the problem of wave propagation can be solved by construction of the characteristics along which material particles move. In the example under consideration, they are horizontal lines. Let a velocity initial distribution $u(x)$ be given (Fig. 2a).

When you plot horizontal lines from this curve, particles move at their constant velocity along every line. Then, at some points of time $t = t_1, t_2 \dots t_n$ the velocity distribution form $u(x)$ changes (Fig. 2b). At a point of time ($t = 2$ in Fig. 2b) the reflection $u(x)$ ceases

to be the function graph, i.e., there are values x which are met by some values u . In this field the physical condition of particle interaction absence means their motion through each other, which is not physically. You need to deduce a model of their interaction. For example, the model of the Universe creation offered by (Zeldovich, 1970) covers the Universe expansion and the gravitational interaction. Addition of such conditions leads to peculiarities in the solution, i.e., the areas where the particle (galaxy) concentration is maximal. Such areas (set of critical values) are called caustics (Fig. 3).

Once appeared, the caustic can transform, decompose with formation of new features, but it cannot disappear. This model well describes the formation of uneven (cellular) structure of the Universe

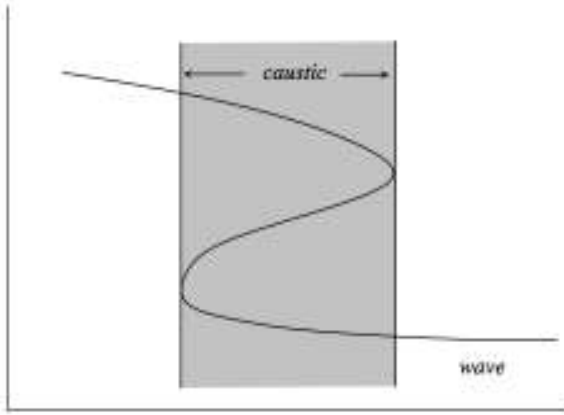


Fig. 3: Wave caustic

from the initial sudden fluctuations of the substance and energy density. In the sample at hand, it is necessary to introduce a model of inelastic collision of particles. Then, in the point of this collision a shock-wave appears which means discontinuity of parameters of particle movement (Fig. 2c). The primitive model of the particles inelastic collision is the Burgers equation (Karman and Burgers, 1939) which describes the gas-dynamic field in the smooth areas of the space and interaction of gas particles inside the shock-wave.

At low viscosity it approximates the Euler equation in the areas of parameter smooth change. On the shock-wave right and left, the flow is described with the Euler equations, inside the shock-wave (gas-dynamic discontinuity) -with an equation similar to the heat conduction equation. We do not give here the complete geometrical theory of the reflection peculiarities specified with hyperbolic equations in partial derivatives, we just mark the remarkable fact that for both caustics and shock-waves all their possible transformation are completely mapped. This fact can't be emphasized too strongly. In the classical theory of gas-dynamic discontinuities all solutions follow the conservation laws. There can be a few solutions and for selection of one of them which is realized physically, it is necessary to attract extra considerations. The geometrical theory allows to do that and to take into account the direction of change of the parameter which influences onto transformation (hysteresis).

Model of a discontinuity breakdown at angle interaction of two flat supersonic flows: Here we consider interaction of flat supersonic jets of non-viscous gas with different gas-dynamic parameters meeting at angle β_0 (Fig. 1c and d). In consequence of interaction, the outgoing from the point of interaction waves 1 and 2 appear, which can be shocks (σ) or isentropic waves (ω), as well as tangential discontinuity τ .

The problem of breakdown of an arbitrary stationary discontinuity is set as follows: according to the given values of gas-dynamic variables before discontinuities 1 and 2 to specify gas-dynamic

parameters following these discontinuities. This problem solution is built based on fulfillment of the conditions of dynamic compatibility on tangential discontinuity τ consisting of the equality of static pressure and collinearity of velocity vectors over and under τ :

$$\hat{p}_1 = \hat{p}_2, \beta_{1-2} - \beta_1 = \beta_2 \quad (1)$$

Here \hat{p}_1 and \hat{p}_2 are the pressures following waves 1 and 2, β_1 and β_2 are flow turning angles on these waves. For definiteness we consider that static pressure \hat{p}_1 before 1 is more or equal to static pressure \hat{p}_2 before 2. Introducing the intensity of interaction J_{1-2} and intensities J_1 and J_2 of discontinuities in 1 and 2:

$$J_{1-2} = \frac{p_1}{p_2}, J_1 = \frac{\hat{p}_1}{p_1}, J_2 = \frac{\hat{p}_2}{p_2} \quad (2)$$

The conditions can be rewritten as follows:

$$J_1 J_{1-2} = J_2, \beta_0 - \beta_f = \beta_g \quad (3)$$

Whereupon $J_{1-2} \geq 1$. On the compression shock the dependence $\beta(J)$ has the appearance:

$$\beta(J) = \arctg \left[\sqrt{\frac{(1+\varepsilon)M^2 - (J+\varepsilon)}{J+\varepsilon} \frac{(1-\varepsilon)(J-1)}{(1+\varepsilon)M^2 - (1-\varepsilon)(J-1)}} \right] \quad (4)$$

On the isentropic depression wave the flow turning angle is calculated according to the formula:

$$\beta = \omega(M) - \omega(M_1) \quad (5)$$

Here $\omega(M)$ is the prandle-meyer function:

$$\omega(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctg \left(\sqrt{\frac{\gamma+1}{\gamma-1} (M^2 - 1)} \right) - \arctg \sqrt{M^2 - 1} \quad (6)$$

where, M and M_1 of the Mach number before and after the wave, $\varepsilon = (\gamma-1)/(\gamma+1)$, γ is the adiabatic index. Relation of the Mach numbers M and M_1 is specified with the help of common for the isentropic and shock waves relation:

$$\mu / \mu_1 = JE \quad (7)$$

where,

$$\mu = 1 + \varepsilon(M^2 - 1), E = \rho / \rho_1$$

Values E and J is connected with the Rankin-Hugoniot adiabats on shock-waves and with the Laplace-Poisson adiabats isentropic waves:

$$E_D = (1 + \varepsilon J) / (J + \varepsilon), E_R = J^{-1/\gamma} \quad (8)$$

Consequently, value M_1 in (5) is expressed with account of (7), (8) by means of the wave intensity J according to formula:

$$\mu_1 = \mu J^{-1/\eta}, (\eta = (1 + \varepsilon) / 2\varepsilon)$$

Thereby, (3) represents a system of two equations in the two unknowns J_1 and J_2 . As intensities J_1 and J_2 are related with dependence $J_2 = J_{1-2} J_1$, the system (3) can be reduced to one equation in intensity $J \equiv J_2$:

$$\beta_2(J, M_2, \gamma_1) = -\beta_1(J / J_{1-2}, M_1, \gamma_1) + \beta_{1-2} \quad (9)$$

Knowing the wave intensity, it is possible, due to the known values of gas-dynamic variables before the wave, to specify any gas-dynamic parameters after it.

RESULTS AND DISCUSSION

Analysis of the region of different solutions for the discontinuity breakdown problem for flat cases: It is possible to analyze the system solution (9) on the plane of cordiform curves $\Lambda = \ln J, \beta$, which are also called as *shock polars*. Each curve has got its top in the point meeting the jump maximal intensity:

$$J_m = (1 + \varepsilon) M^2 - \varepsilon$$

It is accepted to select two points important for the analysis of the region of the discontinuity breakdown solutions. The first is the intensity J_1 , respondent to the maximal angle of the flow turning on the single compression shock:

$$J_1 = \frac{M^2 - 2}{2} + \sqrt{\left(\frac{M^2 - 2}{2} + (1 + 2\varepsilon)(M^2 - 1) + 2\right)} \quad (10)$$

The flow turning angle on the compression shock with intensity J_1 is determined form the relation:

$$\beta_1(J_1) = \arctg \left(\sqrt{\frac{(1 - E_l)}{1 - \varepsilon}} \frac{(1 - E_l)}{2\sqrt{E_l}} \right) \quad (11)$$

where, $E_l = E_D(J_1)$ is expression of the Rankin-Hugoniot adiabat (8). Point l divides the cordiform curve into two parts, one meets the range $[1, J_1]$ and is called a weak branch of the shock polar and its second part is called as its strong branch.

The second special point on the shock polar is connected with the concept of cordiform curve

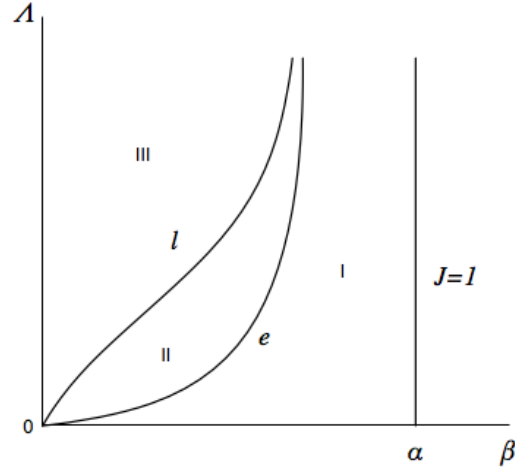


Fig. 4: Region of the solution existence l -curve meeting the limit angle on the jump; e -curve meeting the envelope of cordiform curves; α -angle of the characteristic which the jump degenerates into at $J = 1$. I-outgoing discontinuity 2-compression shock; II-outgoing discontinuity 2 dependently on the value of the Mach number M_2 -either a compression shock or a depression wave; III-outgoing discontinuity 2-a depression wave

envelope limiting the area of single compression shock on the plane $\Lambda - \beta$. The envelope is determined from the condition $\frac{\partial \beta}{\partial M} = 0$:

$$\beta_e(J) = \arctg \left(\frac{1 - E}{2\sqrt{E}} \right) \quad (12)$$

where, $E_e = E_D(J_e)$ is the expression of the Rankin-Hugoniot adiabat (8). One can see from the Eq. (12) that the envelope exists for $M \geq \sqrt{2}$ only.

Curves $J_1 - \beta_1(M)$ and $J_e - \beta_e(M)$ divide the first quadrant of the plane Λ, β into three subareas (1, 2, 3 in Fig. 4), where outgoing discontinuity 2 is a jump (I), depression wave (III) and, dependently on the Mach number M_2 , either a jump or a wave (II).

In such a way, the analysis of the regions of different shock-wave structures appearing at discontinuity breakdown in the point of interaction at the given angle of two flat supersonic jets consists of construction on the plane of shock polars $\Lambda - \beta$ of the curves $\Lambda_1 - \beta_1, \Lambda_e - \beta_e$ and of determination of the special Mach number dividing the region II into two subarea. This problem is similar analysis in the form of theorems and their proving is given by V. Uskov in his work (Uskov *et al.*, 2000).

Let's consider different solutions on the plate of polars at the given M . If you construct the polar 1 ($M = M_1$) from the origin of coordinates and the polar 2 ($M = M_2$), corresponding the wave 2, from the point with coordinates $\{\Lambda_{1-2}, \beta_{1-2}\}$, both conditions are met in the polar cross point (3). Therefore, the cross points of the

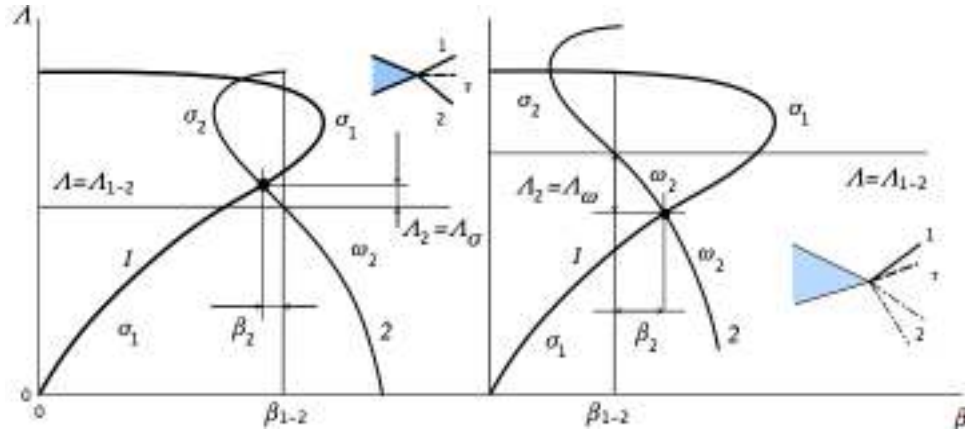


Fig. 5: Two possible cases of breakdown of arbitrary discontinuity on the plate of shock polars

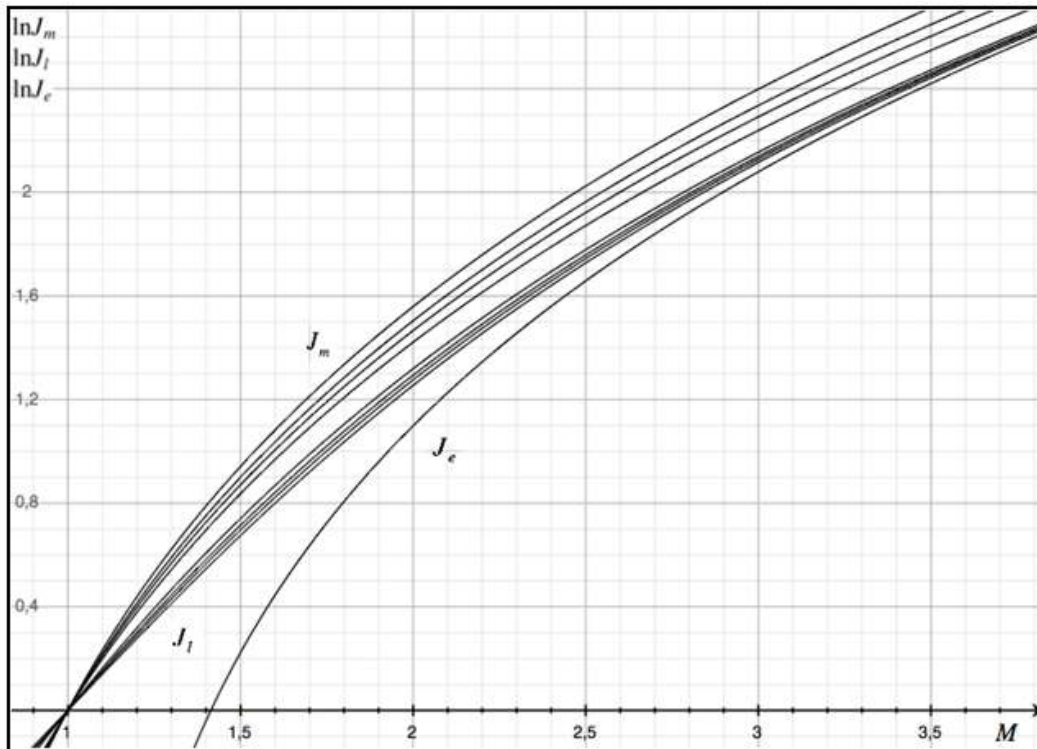


Fig. 6: Dependences of special intensities of jumps on the Mach number. J_m -maximal intensity for the given Mach number; J_l -intensity meeting the maximal; for the given Mach number; the flow turn angle β , J_e -envelope of shock polars; $\gamma = 1.1, 1.25, 1.4, 1.67$. Groups of curves are situated according to increase of γ bottom-upwards; i.e., the upper curve meets $\gamma = 1.67$

given curves reflect on the plane of cordiform curves the problem mathematical solution. One can see in Fig. 5 that in the point of discontinuity breakdown two different shock-wave structures can arise. Obviously, that these two cases are divided with the structure where the outgoing discontinuity 2 degenerates into the discontinuity characteristic.

Further, Fig. 6 and 7 show the dependences of special intensities J_m, J_l, J_e on the Mach number and adiabatic index γ and also special curves on the plate of polars $J_l-\beta_l, J_e-\beta_e$. The dependence of the limit angles of

the flow turn at the jump $-\beta_l$, angles of the flow turn in the point of contact if the shock polar with the envelope β_e on the Mach number is given in Fig. 8.

Figure 6 the definition area meeting the solutions of physical sense is limited with conditions $J > 1, M > 1$. It is also necessary to mark that for the Mach numbers lower than $M = \sqrt{2}$, the envelope of the shock polar family is absent.

Figure 7 shows parametric curves $J_l-\beta_l, J_e-\beta_e$. The parameter is the Mach number which changes within M

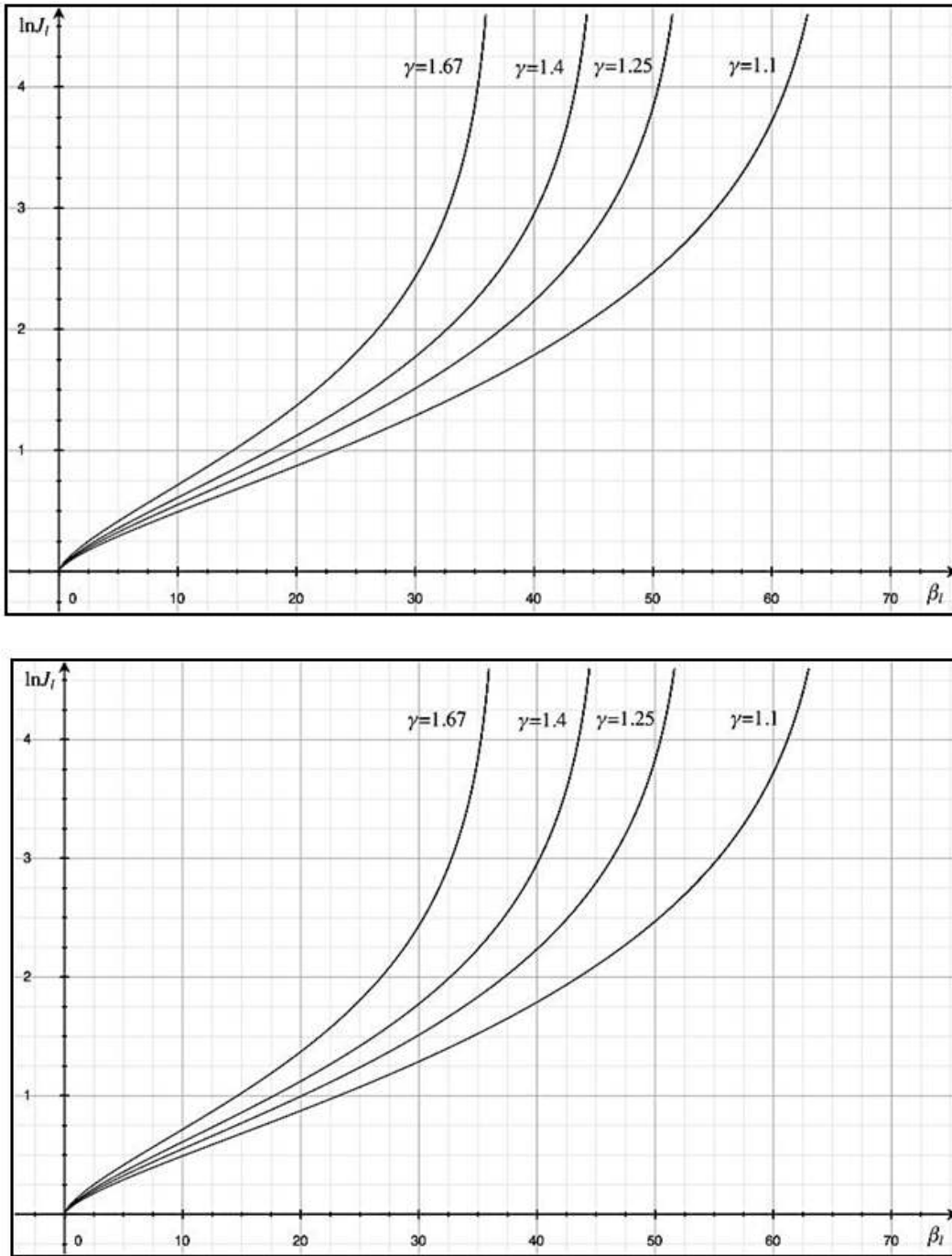


Fig. 7: Special curves on the plane of polars $J_r\text{-}\beta_l, J_e\text{-}\beta_e, M = 1 \dots 10$

$= 1 \dots 10$. The envelope $J_e\text{-}\beta_e$ like in the previous figure is specified only for the Mach numbers higher than $M = \sqrt{2}$, therefore the section below the abscissa has no physical sense.

Figure 8 shows dependences of special limit angle of the flow turn at jump $-\beta_l$, angles of the flow turn in the point of contact of the shock polar with envelope β_e on the Mach number. In the same way in Fig. 7, the envelope $J_e\text{-}\beta_e$ is determined only for the Mach

numbers higher than $M = \sqrt{2}$. One can see that $\beta_e > \beta_l$ and this difference first increases, reaches its maximal value at the medium Mach numbers and then decreases. Both special angles of turn asymptotically tend, at $M \rightarrow \infty$, to the limit for this adiabatic index value (Uskov, 1980):

$$\beta_{\text{lim}}(\gamma) = \text{arctg} \frac{1 - \varepsilon}{2\sqrt{\varepsilon}}$$

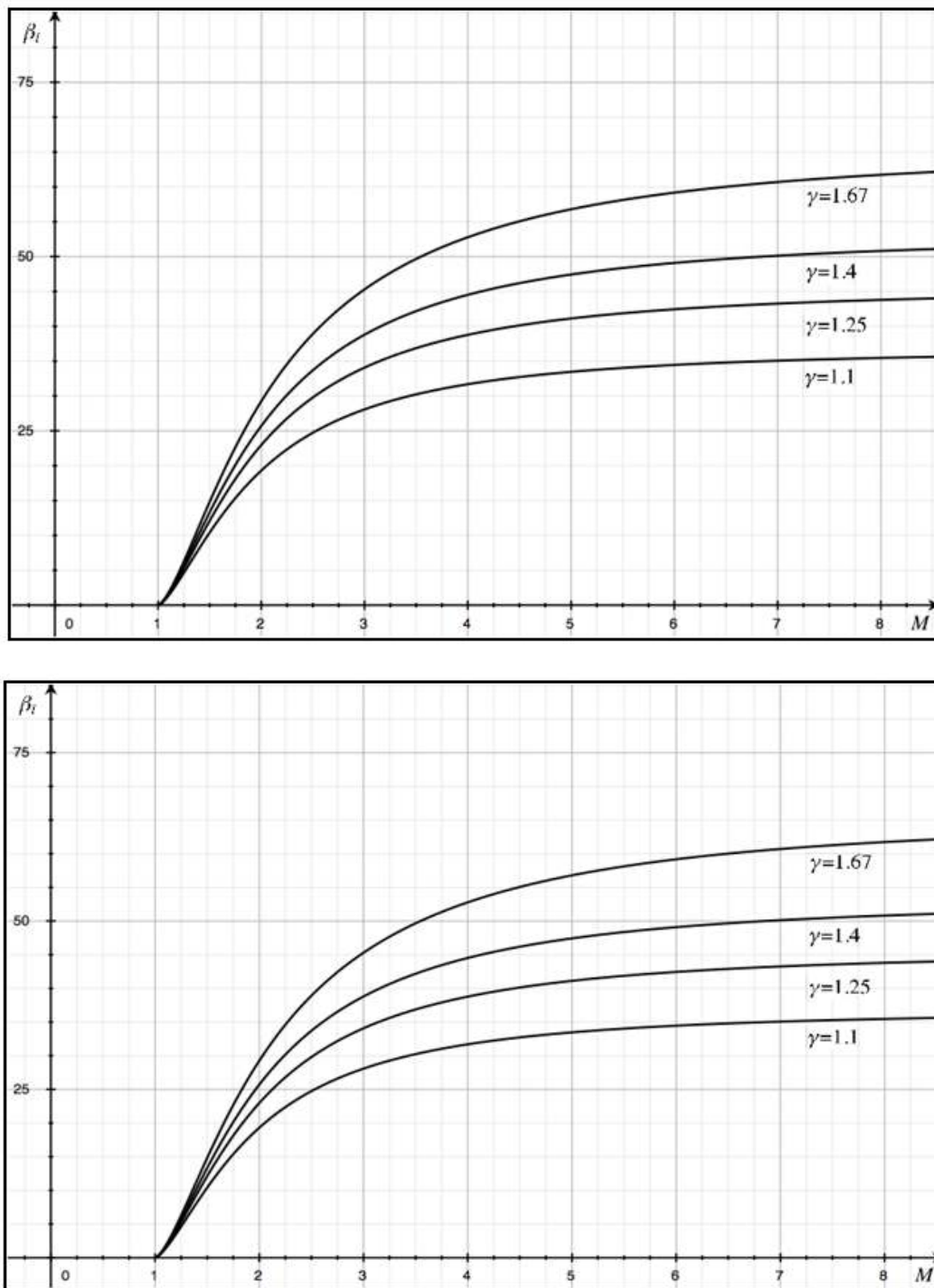


Fig. 8: Dependence of special limit angles of the flow turn on the jump β_e , turn angles in the point of contact of the shock polar and envelope β_e on the Mach number. $M = 1 \dots 10$

The given dependences completely describe two-dimensional interaction of two angle supersonic jets.

CONCLUSION

The development of new computation algorithms for the regions of existence of different solutions is

actual. In 2-D case the conditions of dynamic compatibility is not enough for solution selection. The developed by Russian mathematicians geometrical theory of transformation of shock-waves and wave fronts allows both to select physically realized solutions from the set of solutions meeting the dynamic compatibility and to take into account the hysteresis

depending on the direction of variation of the problem parameter.

For the whole range of technical applications (flow of the airfoil acute edge, obstacle reflection of a shock-wave, shock-wave processes in jet streams) it is necessary to solve the problem of discontinuity breakdown in fine setting, with no simplifications. In the present work the regions of existence of such solutions for a 2-D case is analyzed. In the form of easy-to-use diagrams here are given the basic dependences allowing completely to define the type of outgoing discontinuities and the solution pattern for the problem of an arbitrary discontinuity breakdown.

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