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Research Article

A New Finite Element Based on the Strain Approach for Linear and Dynamic Analysis

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Abstract: In this study, linear and dynamic analysis of 2-D structures using a new strain based quadrilateral finite element is addressed. The developed element has the three Degrees of Freedom (DOF) at each of the four corner nodes (two general external degrees of freedom and the in-plane rotation). The displacement functions of the developed element satisfy the exact representation of the rigid body modes. Both linear and dynamic analyses are considered. For the dynamic analysis lumped mass and explicit time integration are employed. For the purposes of validation some selected numerical examples are solved using this developed element and the obtained results show its good performance.

Keywords: Drilling rotation, dynamic analysis, linear analysis, lumped mass, strain approach

INTRODUCTION

For many years researchers have investigated strain based finite element approach for structural analysis (Ashwell and Sabir, 1972). The most important tasks in this research are the development of reliable elements and the improvement of the accuracy and computational efficiency of these elements. Compared to the displacement based method which has been recognized that for some type of problems provides poor results, the strain based method has become among the most widely used for the analysis of solid and structures. Since the appearance of this approach, many researchers have tried to develop finite elements that are accurate and robust. The first developed elements were only concerned with curved ones (Ashwell *et al*., 1971). This approach was later extended to plane elasticity elements (Sabir, 1985b; Sabir and Salhi, 1986; Belarbi and Maalam, 2005), for three-dimensional elasticity (Belarbi and Charif, 1999), for cylindrical shells (Sabir and Lock, 1972) and for plate bending (Belounar and Guenfoud, 2005). Compared with displacement-based method, the advantages of the strain based approach have been illustrated on several elements (Sabir and Charchafchi, 1982; Rebiai and Belounar, 2014, 2013).

The ability to solve linear, dynamic and dynamic elastoplastic problems is more important in many aspects of finite element work. In fact, exact solutions for these problems exist only for a few simple cases, so the use of the finite element method is required. However the use of the classical displacement-based elements becomes increasingly inefficient and leads to a considerable gain on computing times for this type of analysis.

In this study, the strain based approach which was recently extended to the material nonlinear analysis of 2-D structures (Rebiai and Belounar, 2014, 2013) is used to examine linear and dynamic behavior (free and forced vibration analyses) of membrane structures through a new strain based element with drilling rotation named "SBE" Strain Based Element. A number of benchmarks with classical conditions and load conditions are considered which were already used in the validation of new finite elements.

METHODOLOGY

Formulation of the developed element: The element SBE with three degrees of freedom (U_i, V_i and in plane rotation θ_i) at each of the four corner nodes is shown in Fig. 1.

In a 2-D analysis the relationship between strains and displacements are given by:

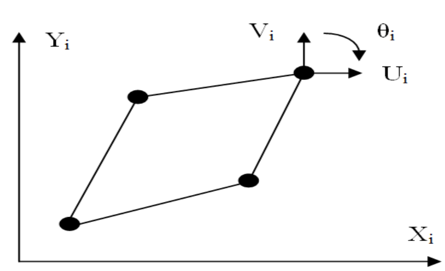

Fig. 1: SBE finite element

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$$\varepsilon_x = \frac{\partial U}{\partial x}$$
$$\varepsilon_y = \frac{\partial V}{\partial y}$$
$$\gamma_{xy} == \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \qquad (1)$$

The strains given by Eq. (1) must satisfy the compatibility Eq. (2) which can be formed by the eliminating U, V from Eq. (1), hence:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (2)$$

We first integrate Eq. (1) with all three strains equal to zero to obtain:

$$U = a_1 - a_3 y$$
$$V = a_2 + a_3 x$$
$$\theta = a_3 \qquad (3)$$

Equation (3) gives the three components of rigid body displacements. We use three independent constants (a_1, a_2, a_3) for the representation of the rigid body components, it remains nine constants (a_4, a_5, ….., a_{12}) for expressing the displacement due to straining of the element. These are given as:

$$\varepsilon_x = a_4 + a_6 y + a_7(x+1) + a_{10} y^2 + 2a_{11} x y^3$$
$$\varepsilon_y = a_7 + a_8 x + a_9 y - a_{10} x^2 - 2a_{11} y x^3$$
$$\gamma_{xy} = 2a_5(y+1) + 2a_6 x + 2a_7 x + 2a_8 y + a_9 y + 2a_{12} x \qquad (4)$$

The strains given by Eq. (4) satisfy the compatibility equation. Equations (4) are equated to the equations in terms of U, V from Eq. (1) and the resulting equations are integrated, to give:

$$U = a_4 x + a_5(y + y^2) + a_6 xy + a_7(0.5x^2 + x) + 0.5a_8 y^2$$
$$+ 0.5a_9 y^2 + a_{10} x y^2 + a_{11} x^2 y^3$$
$$V = a_5 x + 0.5a_6 x^2 + a_7(x^2 + y) + a_8 xy + 0.5a_9 y^2 - a_{10} x^2 y$$
$$- a_{11} x^3 y^2 + a_{12} x^2$$
$$\theta = -a_5 y + a_7 x - 0.5a_9 y - 2a_{10} xy - 3a_{11} x^2 y^2 + a_{12} x \qquad (5)$$

The final displacement functions are obtained by adding Eq. (3) and (5) to obtain the following:

$$U = a_1 - a_3 y + a_4 x + a_5(y + y^2)$$
$$+ a_6 xy + a_7(0.5x^2 + x) + 0.5a_8 y^2$$
$$+ 0.5a_9 y^2 + a_{10} x y^2 + a_{11} x^2 y^3$$
$$V = a_2 + a_3 x + a_5 x + 0.5a_6 x^2 + a_7(x^2 + y)$$
$$+ a_8 xy + 0.5a_9 y^2 - a_{10} x^2 y - a_{11} x^3 y^2 + a_{12} x^2$$
$$\theta = a_3 - a_5 y + a_7 x - 0.5a_9 y - 2a_{10} xy$$
$$- 3a_{11} x^2 y^2 + a_{12} x \qquad (6)$$

The present element SBE has the four corner nodes and 12 degrees of freedom and since the matrix [C] of the developed element is not singular, its inverse exists and the stiffness matrix [K^e] for the present element is given by:

$$\left[K^e\right] = [C]^{-T} \left(\int_{-1}^{1}\int_{-1}^{1} [Q]^T [D][Q] \det|J| \, d\xi \, d\eta\right)[C]^{-1} \qquad (7)$$

where, [Q], [J] and [D] are the strain, the Jacobean and the elasticity matrices respectively and [C] is the matrix which relates the 12 nodal displacements to the 12 constants a_1 to a_{12}. These are given respectively in appendix.

DYNAMIC NUMERICAL VALIDATION

A computer program is prepared for studying the behavior of the developed element. Three example problems are presented to demonstrate the robustness and accuracy of this element in dynamic analysis. The elements used in comparison are listed in the appendix.

Eigenvalues of a rectangular solid with lumped mass: Nominally this test as shown in Fig. 2 treated in Smith and Griffith (2004, 1988) representing an elastic solid cantilever beam with flexural rigidity of 1/12 and Poisson's ratio set to 0.3.

The results of the eigenvalues are shown in Table 1 for plane strain conditions. The fundamental frequency ω_1 and the axial frequency ω_2 are calculated with different elements (Q4, Q8 and SBE). The results obtained by the element SBE are in good agreement with those obtained by the Q8 element and with those of the analytical solution. The fundamental frequency obtained by the Q4 element is considerably greater than that of the analytical solution which shows that the Q4 is a poor representation of the solid (beam), at least in the bending modes.

Forced vibration of rectangular solid in plane strain conditions: In this example Fig. 3 the forced vibration analysis uses the complex response method described in reference (Smith and Griffith, 1988). The cantilever beam is subjected to a harmonic vertical force (cos ωt) at the end of the beam. The damping ratio γ is 0.005 or 5% applied to all modes of the system, the Young's modulus is E = 1 kN/m^2, Poisson's ratio v = 0.3, the forcing frequency ω = 0.3, the time step is DT = 1/20 of forcing period (2 π/ω) and the mass density per unit volume is ρ = 1 t/m^3. The problem is in plane strain conditions.

The results presented in Fig. 4 show the displacements at the end of the beam versus time-step, the two elements Q8 and SBE are used in this analysis. We can see clearly that the behavior of the SBE is strictly similar to the Q8 element in forced vibration analysis but this later uses more degrees of freedom.

Table 1: Eigenvalues of the cantilever beam

Frequencies		Q4	Q8	SBE
Mesh 3×1	ω_1	0.080	0.064	0.064
	ω_2	0.353	0.413	0.410
Mesh 5×1	ω_1	0.068	0.060	0.062
	ω_2	0.391	0.391	0.393
Exact solution	ω_1	$1.875^4 \times EI/\rho AL^4$		0.063
	ω_2	$\pi/2L\sqrt{E/\rho}$		0.393

Table 2: Forced vibration of a rectangular elasto-plastic solid (displacement versus time)

Time	Displacements Q8	SBE
0	0	0
$0.5000.10^{-4}$	$-0.2995.10^{-3}$	$-0.3090.10^{-3}$
$0.1000.10^{-3}$	$-0.1214.10^{-2}$	$-0.1223.10^{-2}$
$0.1500.10^{-3}$	$-0.2684.10^{-2}$	$-0.2581.10^{-2}$
$0.2000.10^{-3}$	$-0.4867.10^{-2}$	$-0.3974.10^{-2}$
$0.2500.10^{-3}$	$-0.8084.10^{-2}$	$-0.8098.10^{-2}$
$0.3000.10^{-3}$	$-0.1231.10^{-1}$	$-0.1239.10^{-1}$

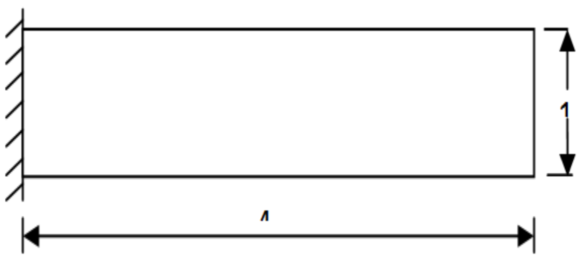

Fig. 2: Geometry of cantilever beam

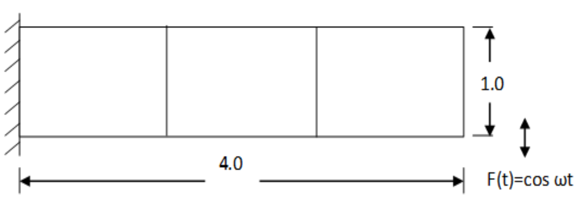

Fig. 3: Geometry and mesh of the cantilever beam subjected to forced vibration

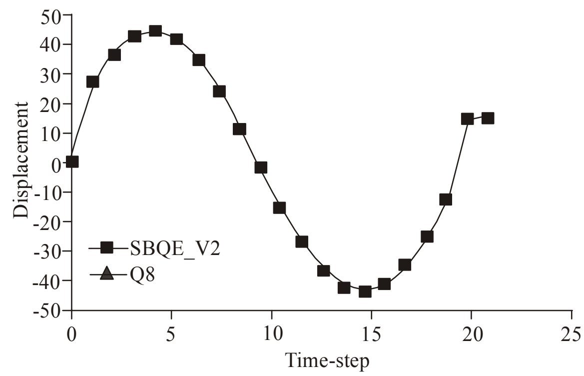

Fig. 4: Forced vibration of a rectangular solid 'displacements versus time-step'

Forced vibration of a rectangular elasto-plastic solid with lumped mass: In this example Fig. 5 explicit integration method is used in this analysis. The maximum stress VonMises is σ_{max} = 50.000 kN/m^2, the Young's modulus E = 3.10^7 kN/m^2, Poisson's ratio v = 0.3, load multiplier PL = 180, the number of step is 700 and the mass density per unit volume $\rho = 0.7333.10^{-3}$ t/m^3.

The results presented in Table 2 show the displacements at the end of the beam. These results show clearly that the behavior of the new element is similar to the Q8 but the SBE is more economic.

LINEAR NUMERICAL VALIDATION

Linear MacNeal beam: Three types of examples for plane elasticity problems are presented to validate the present element. We consider a slender beam of MacNeal and Harder (1985). The geometrical and materials characteristics of the structure are shown in Fig. 6. The deflection results are listed in Table 3. From the linear deflection results we can see that the SBE is insensitive to mesh distortion. For the regular

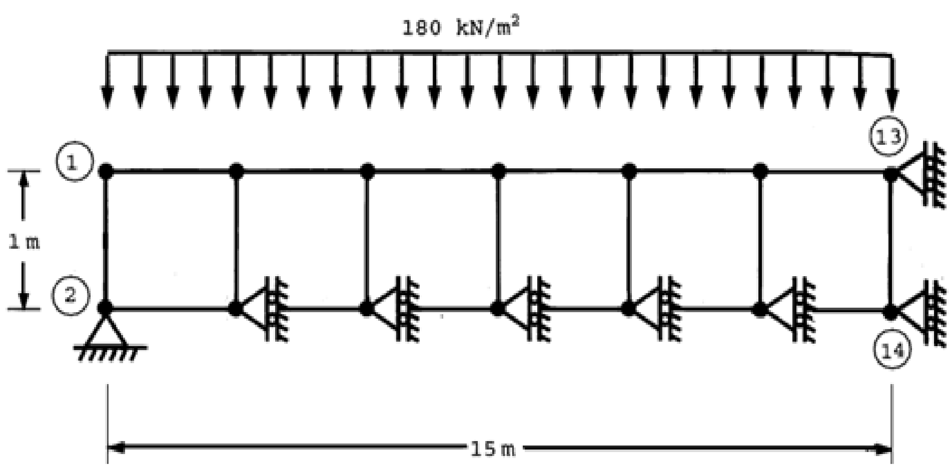

Fig. 5: Geometry and mesh of the elastoplastic cantilever beam

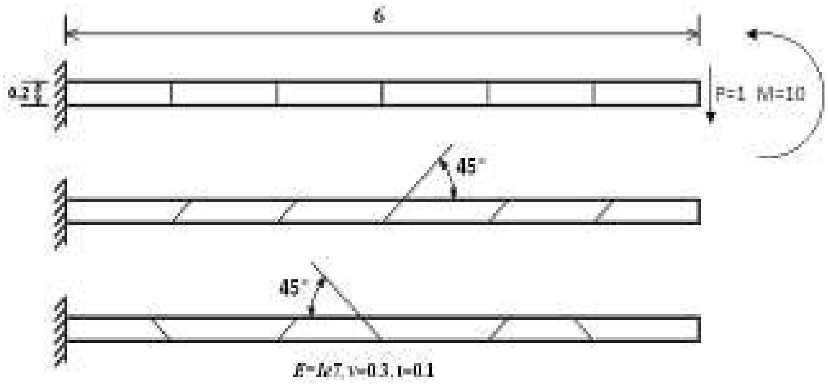

Fig. 6: MacNeal and harder patch tests: geometry, mesh and boundary conditions

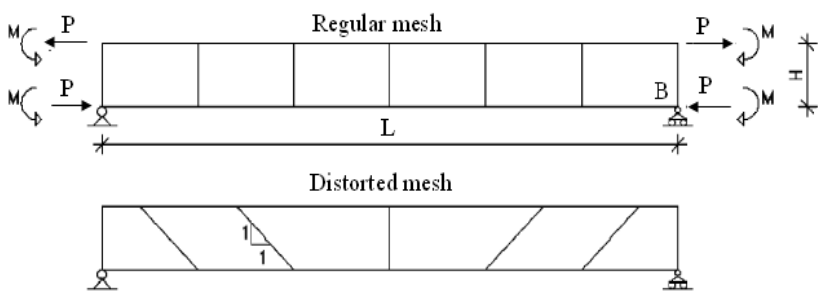

Fig. 7: Geometry and mesh of a simple beam

Table 3: MacNeal-harder cantilever beam: numerical results of deflection for different load cases and mesh geometry

	Shear P = 1			Bending M = 10		
Element	Regular	Parallel	Trapezoidal	Regular	Parallel	Trapezoidal
P5Sβ	0.1081	0.07848	0.004970	0.26800	0.17000	0.01404
SBTIEIR	0.0050	0.00390	0.000054	0.03186	0.02727	0.00108
SBT2V	0.1020	0.09440	0.090000	0.25500	0.25400	0.25700
HQ4-9β	0.1072	0.10570	0.105800	0.26900	0.26600	0.26600
SBE	0.1081	0.10550	0.105700	0.27000	0.26700	0.26700
Analytical		0.10810		0.27000		

Table 4: Vertical displacement and rotation at the point B of the simple beam

		ITW		Pimp		SBE	
Load case	Mesh	Vert. dis.	End rot.	Vert. dis.	End rot.	Vert. dis.	End rot.
Forces	Reg.	1.50	0.60	1.50	0.60	1.50	0.59
Forces	Dist.	1.14	0.57	1.39	0.54	1.49	0.59
Couple	Reg.	1.50	0.62	1.51	1.44	1.50	0.60
Couple	Dist.	1.39	0.49	1.39	1.28	1.50	0.60
Analytical		1.50		0.60			

Table 5: Vertical displacement at point A

	Normalized displacement at point A				
Mesh	Q4	SBRIE	ALLMAN	SBRIEIR	SBE
4×1	0.2412	0.3293	0.3027	0.3300	0.3347
Ref (timoshenko)	1,000 (0.3553)				

mesh all the results are in good agreement with the exact solution. Compared with the Strain Based Elements (SBTIEIR, SBT2V) we can see that the developed element is more accurate in both cases of loads.

A simple beam:

The higher-order patch test: This problem is treated by Ibrahimobigovic *et al*. (1990) and it is relative to a beam fixed by a minimum number of constraints. The beam is subjected to a pure bending state as shown in

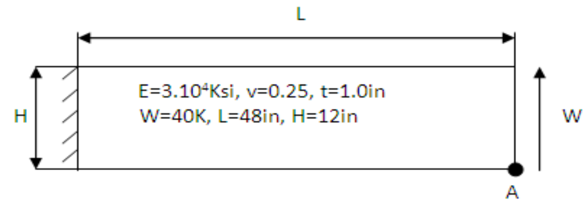

Fig. 8: Cantilever beam under a tip load

Fig. 7. The first load condition is constituted by a unit couple of forces applied at the free end of the beam whereas the second load case is still a moment but it is applied as a concentrated couple at the end. The geometrical and mechanical characteristics are as follow:

$$E = 100,\ v = 0\ P = 1,\ M = 0.5,\ L = 10,\ H = 1,\ t = 1$$

Both regular and distorted meshes are considered in this example. The vertical displacement and the rotation at the point B are computed.

The good behavior of developed element relatively to the insensitivity to distortion is shown in Table 4. The results in terms of the drilling rotations show a significant improvement with those of Ibrahimobigovic *et al*. (1990) and Pimpinelli (2004).

Short cantilever beam of Allman: A short cantilever beam is subjected to uniform vertical load as shown in Fig. 8. It is modeled by 4 elements. The results of the displacement presented in Table 5 for the SBE show the good agreement with those of the exact solutions given by Timoshenko and Goodier (1951).

CONCLUSION

Here the finite element named SBE for linear and dynamic analysis of 2-D structures is a relatively straightforward one with 12 degrees of freedom. Using this element the numerical obtained results agree reasonably well with all those from others research and from the exact solutions. This element is simple and contains higher order of polynomial terms. The convergence rate is shown to be quite rapid and usually a coarse mesh will give satisfactory results. This element can be conveniently applied to the solution of elastic and dynamic engineering problems.

APPENDIX

Components of the matrix [C] of the dimension (12×12) for the SBE are:

$$[C]=\begin{bmatrix} 1 & 0 & -y & x & y+y^2 & xy & x+0.5x^2 & 0.5y^2 & 0.5y^2 & xy^2 & x^2y^3 & 0 \\ 0 & 1 & x & 0 & x & x^2/2 & y+x^2 & xy & 0.5y^2 & -x^2y & -x^3y^2 & x^2 \\ 0 & 0 & 1 & 0 & -y & 0 & x & 0 & -0.5y & -2xy & -3x^2y^2 & x \end{bmatrix}$$

where x_i and y_i are the coordinates of node i (i = 1, 4), the matrix [C] is given by:

$$[C] = [[C_1][C_2][C_3][C_4]]^T$$

For the case of plane stress problems the elasticity matrix [D] is:

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

For the case of plane strain problems the elasticity matrix [D] is:

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{(1-2v)}{2} \end{bmatrix}$$

The strain matrix is given by:

$$[Q]=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & y & x+1 & 0 & 0 & y^2 & 2xy^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & -x^2 & -2x^3y & 0 \\ 0 & 0 & 0 & 0 & 2(y+1) & 2x & 2x & 2y & y & 0 & 0 & 2x \end{bmatrix}$$

A brief notes on the elements to be compared are given:

- **Q8:** The eight nodes quadrilateral element with sixteen Degrees of Freedom (DOFs)
- **SBRIEIR and SBTIEIR:** The four and three node strain based rectangular and triangular in-plane elements with in-plane rotation with twelve DOFs (Sabir, 1985a)
- **SBT2V:** The Improved three node strain based triangular in-plane element with drilling rotation with nine DOFs (Belarbi and Bourezane, 2005)
- **HQ4-9β:** Isostatic quadrilateral membrane finite element with drilling rotation (Madeo *et al*., 2012)
- **P5Sβ:** Pian's hybrid element with four node (Pian and Sumihara, 1984)
- Quadrilateral element with drilling ITW DOFS (Ibrahimobigivic *et al*., 1990)
- Quadrilateral element with drilling rotation Pimp (Pimpinelli, 2004)
- **Q4:** Quadrilateral element with four node
- **SBRIE:** Strain based rectangular inplane element (Sabir and Sfendji, 1995)
- **Allman:** (Allman, 1988)

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