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Research Article

Novel Structures of Normed Spaces with Fuzzy Autocatalytic Set (FACS) of Fuzzy Graph Type-3

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Abstract: In this study, our aim is to explore Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3 particularly on the structures of normed spaces and its relations to cycles in FACS. Several new notions namely FT3-fuzzy detour cycles of FACS, a FT3-cycle space of FACS as a vector space and normed space of FACS of fuzzy graph Type-3 are presented in this study. These notions are implemented in a modeling of an incineration process.

Keywords: Cycle space of FACS, fuzzy autocatalytic set, fuzzy graph, incineration process, normed space of FACS

INTRODUCTION

Fuzzy set theory was proposed by Zadeh (1965) and it has been employed in mathematical modelling of some systems. In Rosenfeld (1975) defined the basic structure of fuzzy graph. The modelling of clinical waste incineration process in Malacca is one of many applications of fuzzy graph (Sabariah, 2005; Tahir *et al*., 2010).

The concept of an Autocatalytic Set (ACS) was introduced by Jain and Krishna (1998, 2003). Nevertheless, ACS was insufficient to explain the incineration process (Sabariah *et al*., 2009) as presented in Fig. 1. Therefore, an integration of fuzzy graph into ACS was finally inspired to produce a new concept known as Fuzzy Autocatalytic Set (FACS) and shown to be a better in explaining the incineration process (Tahir *et al*., 2010). Six main variables specified in the process were modelled as vertices (nodes) and the catalytic relationships were presented as edges (links).

In this study, we focus on the study of FACS of fuzzy graph Type-3 from a viewpoint of normed space and cycles in FACS. A new concept namely normed space of FACS of fuzzy graph Type-3 is defined and implemented in the modelling of the incineration process. A cycle in FACS of fuzzy graph Type-3 and a fuzzy detour cycle are defined. The study has yielded a notion called FT3-cycle space of FACS as a vector space over $\{0,1\}^{|E|}$ whereby $|E|$ is number of its edges. This notion offers a useful method for constructing normed space of FACS. Finally, these new notions are used to introduce a FT3-cycle space associated with a graph of FACS for the incineration process.

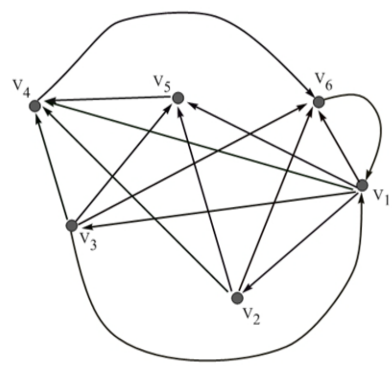

Fig. 1: Crisp graph $DG(V, E)$ for the incineration process where V_1: Waste, V_2: Fuel, V_3: Oxygen, V_4: Carbon Dioxide, V_5: Carbon Monoxide, V_6: Other gases including water (Sabariah *et al*., 2009)

PRELIMINARIES

Some mathematical concepts and terminologies that are needed in this study are reviewed. Fuzzy graph has become a diverse and expanding field. A directed graph $DG(V, E)$ is defined by a relation $E \subset V \times V$ on a set V where V denotes the set of vertices and E denotes the set of its edges (Epp, 1995). A directed graph is also called a crisp graph if all the values of edges are 1 or 0 and it is called a fuzzy graph if its values are between 0 and 1. In other words, a fuzzy graph is $DG(\sigma, \mu)$ with a vertex set V as the underlying set with a pair of functions such that $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ is a fuzzy relation on V with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ and $\sigma(u) \wedge \sigma(v)$ denotes

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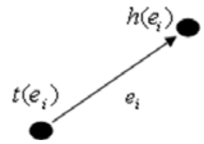


Fig. 2: Fuzzy head and tail of the edge e_i

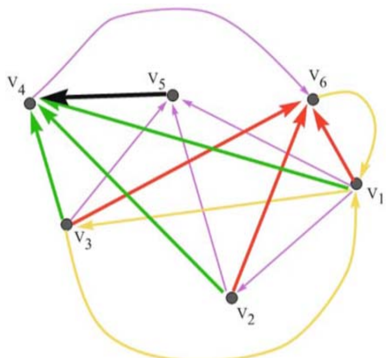


Fig. 3: FACS of fuzzy graph Type-3 $DG(V, E(\mu(e_i)))$ for the incineration process (Tahir *et al*., 2010)

the minimum of $\sigma(u)$ and $\sigma(v)$ (Rosenfeld, 1975). A path $p: v_i = v_0, v_1, v_2, ..., v_{n-1}, v_n = v_j$ from a vertex v_i to a vertex v_j in a fuzzy graph if its sequence of distinct vertices and edges starting from v_i and ending at v_j such that the membership value $\mu((v_{k-1}, v_k)) > 0$ for $k = 1, \cdots, n$. If v_i and v_j coincide in a path then we call p as a cycle. Note that the underlying crisp graph of the fuzzy graph $DG(\sigma, \mu)$ is referred to $DG(V, E)$.

Furthermore, Tahir *et al*. (2010) formalized five types of fuzzy graphs that were given by Blue *et al*. (1997; 2002) into one definition of fuzzy graph and any of the five features could be considered as fuzzy graph. They found that Type-3 is to be the most convenient for the assignment of improving the model of the incineration process as explained in Sabariah *et al*. (2009). The fuzzy graph of this type consists of known vertices and edges, but unknown edge connectivity i.e. the edge has fuzzy head and tail (Fig. 2). These notions namely fuzzy, graph and Autocatalytic Set (ACS) are integrated to formulate the mathematical model of the clinical incineration process. The formal definition of FACS is given as follows: Fuzzy Autocatalytic Set (FACS) is a subgraph where each of whose nodes has at least one incoming link with membership value $\mu(e_i) \in (0,1]$, $\forall\, e_i \in E$ (Tahir *et al*., 2010).

The membership values for the incineration process in Tahir *et al*. (2010) are determined through the chemical reactions taken place between six variables that have major influences in the clinical waste incinerator, namely waste (v_1), fuel (v_2), oxygen (v_3), carbon dioxide (v_4), carbon monoxide (v_5) and other gases including water (v_6). From the explanation given in Sabariah *et al*. (2009) and Tahir *et al*. (2010) pertaining to the construction of FACS of fuzzy graph of Type-3 for the incineration process, the graph is presented as in Fig. 3. Aforementioned, Fig. 3 is a fuzzy graphical for the model of the incineration process with 6 variables and 15 edges which based on the catalytic relationship between the variables. Thus, $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the set of vertices and E is the set of edges where $E = \{e_i = (v_s, v_r): s, r = 1,2, \cdots, 6\}$ for $s \neq r$ and $i = 1,2, \cdots, 15$. Hence, fifteen edges are the connection between these variables in the process and the membership values of each fuzzy connectivity of edge e_i are given as follows:

$\mu(e_1) = \mu((v_1, v_2)) = 0.00001,$
$\mu(e_9) = \mu((v_3, v_1)) = 0.06529,$
$\mu(e_2) = \mu((v_1, v_3)) = 0.15615,$
$\mu(e_{10}) = \mu((v_3, v_4)) = 0.63563,$
$\mu(e_3) = \mu((v_1, v_4)) = 0.51632,$
$\mu(e_{11}) = \mu((v_3, v_5)) = 0.00002,$
$\mu(e_4) = \mu((v_1, v_5)) = 0.00001,$
$\mu(e_{12}) = \mu((v_3, v_6)) = 0.29906,$
$\mu(e_5) = \mu((v_1, v_6)) = 0.32752,$
$\mu(e_{13}) = \mu((v_4, v_6)) = 0.00001,$
$\mu(e_6) = \mu((v_2, v_4)) = 0.68004,$
$\mu(e_{14}) = \mu((v_5, v_4)) = 0.99999,$
$\mu(e_7) = \mu((v_2, v_5)) = 0.00001,$
$\mu(e_{15}) = \mu((v_6, v_1)) = 0.13401.$
$\mu(e_8) = \mu((v_2, v_6)) = 0.31995,$

Umilkeram and Tahir (2014) introduced the concept of fuzzy detour FT3-distance between vertices in FACS by establishing the FT3-length of any fuzzy detour path between two distinct vertices in FACS. Now, we look at a fuzzy detour as above-mentioned in a more systematic way. A cycle in FACS of fuzzy graph Type-3 which will call FT3-cycle of FACS is categorized by this detour. This notion produces a new type of normed space for FACS.

RESULTS AND DISCUSSION

FT3-fuzzy detour cycles of FACS: A cycle in a fuzzy digraph is a directed closed path of its vertices such that each edge and each vertex (except starting point) is visited only once. The FT3-cycle of FACS and the FT3-fuzzy detour cycle of FACS through the specific length of a cycle in FACS are introduced in the following definitions, respectively.

Definition 1: Let $DG_{FT3}(V, E)$ be a FACS of fuzzy graph Type-3. The FT3-cycle $C^n_{e_1, e_2, ..., e_n}$ is a closed path of distinct vertices $v_0, v_1, \ldots, v_n$ (except $v_0 = v_n$) such that the membership value $\mu((v_{k-1}, v_k)) = \mu(e_k) > 0$, $1 \leq k \leq n$, for each fuzzy edge connectivity $e_k \in E$ of FACS and *n* is the number of edges in this cycle. The length of FT3-cycle $|C^n_{e_1, e_2, ..., e_n}|$ in FACS is calculated by $|C^n_{e_1, e_2, ..., e_n}| = \sum_{k=1}^{n} \frac{1}{\mu((v_{k-1}, v_k))} = \sum_{k=1}^{n} \frac{1}{\mu(e_k)} = \frac{1}{\mu(e_1)} + \frac{1}{\mu(e_2)} + \cdots + \frac{1}{\mu(e_n)}$ such that each edge in this cycle is traversed in the right direction (Fig. 4).

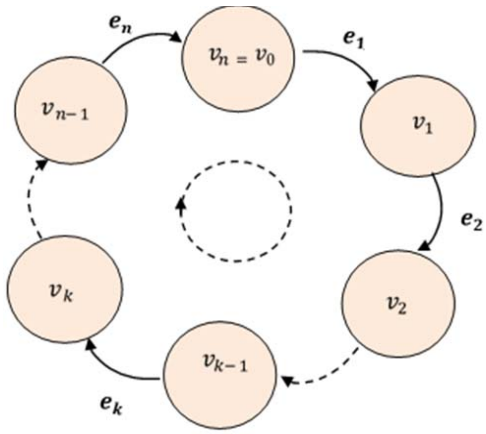


Fig. 4: FT3-cycle $C^n_{e_1,e_2,\ldots,e_n}$ in FACS with the membership value $\mu(e_k) > 0$ for each fuzzy edge connectivity $e_k \in E$ of FACS

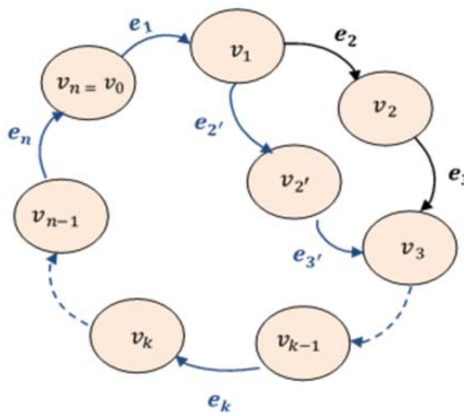


Fig. 5: FT3-fuzzy detour cycle $C^n_{e_k}$ in FACS

Definition 2: Let *n* be the number of all edges in the FT3-cycle containing an edge e_k in FACS. A length of FT3-cycle containing the edge $e_k \in E$ with *n*, denoted by $|C^n_{e_k}|$, in FACS of fuzzy graph Type-3 is defined as the maximum length of any FT3-cycle that goes through the edge e_k with the number of all edges in these FT3-cycles is the same and equal to *n*. A FT3-cycle containing the edge e_k of length $|C^n_{e_k}|$is called FT3-fuzzy detour cycle $C^n_{e_k}$ such that $|C^n_{e_k}| = \sum_{(e_i \in C^n_{e_k})} \frac{1}{\mu(e_i)}$.

Figure 5 illustrates the FT3-fuzzy detour cycle $C^n_{e_k}$ that contained the edge e_k with length $|C^n_{e_k}|$ as follows.

Notice that $C^n_{e_k}$ is one of the two FT3-cycles $C^n_{e_1,e_2,e_3,\ldots,e_k,\cdots,e_n}$, $C^n_{e_1,e_{2'},e_{3'},\ldots,e_k,\cdots,e_n}$ that contains edge e_k with the number of all edges in these FT3-cycles is equal to *n*. The lengths of these FT3-cycles are given in the following:

$$|C^n_{e_1,e_2,e_3,\ldots,e_k,\cdots,e_n}| = \frac{1}{\mu(e_1)} + \frac{1}{\mu(e_2)} + \frac{1}{\mu(e_3)} + \cdots + \frac{1}{\mu(e_k)} + \cdots \frac{1}{\mu(e_n)}$$

And,

$$|C^n_{e_1,e_{2'},e_{3'},\ldots,e_k,\cdots,e_n}| = \frac{1}{\mu(e_1)} + \frac{1}{\mu(e_{2'})} + \frac{1}{\mu(e_{3'})} + \cdots + \frac{1}{\mu(e_k)} + \cdots + \frac{1}{\mu(e_n)}$$

Therefore, by Definition 2, the FT3-fuzzy detour cycle $C^n_{e_k}$ containing the edge e_k is one of these FT3-cycles where its length $|C^n_{e_k}|$ is defined as:

$$|C^n_{e_k}| = \max\left\{|C^n_{e_1,e_2,e_3,\ldots,e_k,\cdots,e_n}|, |C^n_{e_1,e_{2'},e_{3'},\ldots,e_k,\cdots,e_n}|\right\} = \sum_{(e_i \in C^n_{e_k})} \frac{1}{\mu(e_i)}$$

It is worth noted in the above definitions that the starting point of a path of a FT3-cycle $C^n_{e_1,e_2,\ldots,e_n}$ and a FT3-fuzzy detour cycle $C^n_{e_k}$ in FACS is not determined.

Vector space and FACS of fuzzy graph type-3: Let $DG_{FT3}(V,E)$ be a graph of FACS of fuzzy graph Type-3 with $|V|$ is the number of vertices and $|E|$ is the number of edges. One can associate the vector space of all FT3-fuzzy detour cycles and edge-disjoint unions of these cycles in FACS called the FT3-cycle space of $DG_{FT3}(V,E)$ as presented in this section.

For construction this vector space, the process can be explained in the following way. It begins with each FT3-fuzzy detour cycle $C^n_{e_k}$, whereby one can relate a vector representation $\vec{C}^n_{e_k} \in \{0,1\}^{|E|}$, $1 \le k \le |E|$, with entries is equal to 1 if the edge e_k is in this cycle $C^n_{e_k}$. Clearly, its direction agrees with the direction of the cycle $C^n_{e_k}$ due to each edge in this cycle is traversed in the right direction, (see Definition 1). Also, the entries of $\vec{C}^n_{e_k}$ is equal to 0 if the edge e_k is not part of the cycle $C^n_{e_k}$. In other words, each FT3-fuzzy detour cycle $C^n_{e_k}$ has a vector representation in the form of $\vec{C}^n_{e_k} = (c_{e_1}, c_{e_2}, \cdots, c_{e_k}, \cdots, c_{e_{|E|}})$ with entries $c_{e_1}, c_{e_2}, \cdots, c_{e_k}, \cdots, c_{e_{|E|}}$ are equal to 1 or 0 and $c_{e_k} = 1$.

Figure 6 serves as an example to explain the vector representation $\vec{C}^n_{e_k}$ of the FT3-fuzzy detour cycle $C^n_{e_k}$ in FACS.

Consider a Fig. 6 in which a graph of FACS with the number of vertices $|V| = 6$ and the number of edges $|E| = 8$. Then, note that the FT3-fuzzy detour cycle $C^5_{e_4}$ containing the edge e_4 can be written by $C^5_{e_4} = (e_4, e_5, e_1, e_2, e_3)$. Hence, the vector representation of $C^5_{e_4}$, denoted by $\vec{C}^5_{e_4}$, is as follows:

$$\vec{C}^5_{e_4} = (c_{e_1}, c_{e_2}, c_{e_3}, c_{e_4}, c_{e_5}, c_{e_6}, c_{e_7}, c_{e_8}) = (1,1,1,1,1,0,0,0)$$

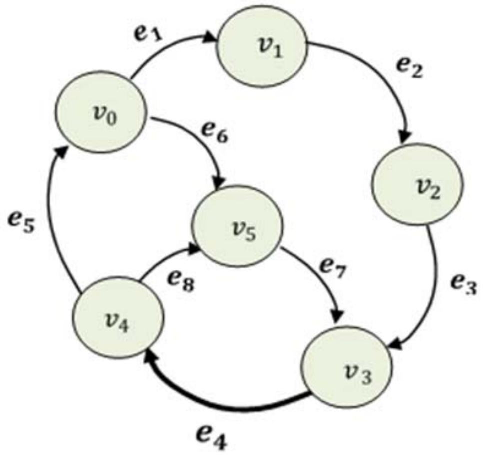


Fig. 6: Vector representation $\vec{C}^n_{e_4}$ of FT3-fuzzy detour cycle $C^n_{e_4}$ in FACS

While, the FT3-fuzzy detour cycle $C^4_{e_4}$ containing the edge e_4 is $C^4_{e_4} = (e_4, e_5, e_6, e_7)$ and the vector representation of $C^4_{e_4}$ is $\vec{C}^4_{e_4} = (0,0,0,1,1,1,1,0)$.

Consequently, by all these vector representation $\vec{C}^n_{e_k} \in \{0,1\}^{|E|}$, the set of all FT3-fuzzy detour cycles and edge-disjoint unions of these cycles in FACS can be extended to form a vector space over $\{0,1\}^{|E|}$. The FT3-cycle space of $DG_{FT3}(V,E)$ is given as follows.

Definition 3: For a graph $DG_{FT3}(V,E)$, let $S_C(DG_{FT3})$ denote the set of all FT3-fuzzy detour cycles and edge-disjoint unions of these cycles in FACS and an empty graph $\emptyset$ which means all vertices separate from each other. Then, $S_C(DG_{FT3})$ is called the FT3-cycle space of $DG_{FT3}(V,E)$.

It is appropriate to see again Fig. 6. It is evident that the unions of edge-disjoint of $C^5_{e_4}$ and $C^4_{e_4}$ entail the set of edges A $=\{e_1, e_2, e_3, e_6, e_7\}$. As a matter of fact, in term of the vector representation of these cycles, it will have an addition modulo 2 such as the sum $\vec{C}^5_{e_4}$ and $\vec{C}^4_{e_4}$ is equal to $(1,1,1,0,0,1,1,0)$ which means the set of edges A in its vector representation.

Then, the next step is to consider $S_C(DG_{FT3})$ as a vector space over $\{0,1\}^{|E|}$. The vector addition of two FT3-fuzzy detour cycles $C^n_{e_k}, C^m_{e_l}$ in $S_C(DG_{FT3})$ is given by symmetric difference:

$$C^n_{e_k} \boxplus C^m_{e_l} = (C^n_{e_k} \cup C^m_{e_l}) \setminus (C^n_{e_k} \cap C^m_{e_l}) \quad (1)$$

The union or intersection of two FT3-fuzzy detour cycles may fail to be FT3-fuzzy detour cycle. However, the symmetric difference operator preserves the character of being FT3-fuzzy detour cycles and edge-disjoint unions of these cycles in FACS. Then, $S_C(DG_{FT3})$ is closed under the operation of symmetric difference. Thereby, a collection of sets closed under this operation can be characterized algebraically as a vector space over $\mathbb{Z}_2$ as shown in Theorem 1.

Theorem 1 asserts that $S_C(DG_{FT3})$ under the sum operation defined in Eq. (1) forms a vector space over {0, 1} (with addition and multiplication modulo 2), where a scalar multiplication $\boxdot$ is given by:

$$1 \boxdot C^n_{e_k} = C^n_{e_k} \text{ and } 0 \boxdot C^n_{e_k} = \emptyset \text{ for each } C^n_{e_k} \in S_C(DG_{FT3}), \emptyset \in S_C(DG_{FT3}) \quad (2)$$

Theorem 1: The FT3-cycle space of $DG_{FT3}(V,E)$, $S_C(DG_{FT3})$, is a vector space over {0, 1} (with addition and multiplication modulo 2).

Proof: For given two FT3-fuzzy detour cycles $C^n_{e_k}, C^m_{e_l}$ in $S_C(DG_{FT3})$, it is clear that the vector addition namely, $C^n_{e_k} \boxplus C^m_{e_l} = (C^n_{e_k} \cup C^m_{e_l}) \setminus (C^n_{e_k} \cap C^m_{e_l})$ is an element in $S_C(DG_{FT3})$. This element may be not FT3-fuzzy detour cycle but is certainly unions of edge-disjoint of the cycles $C^n_{e_k}, C^m_{e_l}$. Then, $S_C(DG_{FT3})$ is closed under the vector addition.

Furthermore, all following constraints of the vector addition are verified for all $C^n_{e_k}, C^m_{e_l}, C^t_{e_j}$ in $S_C(DG_{FT3})$.

- $C^n_{e_k} \boxplus C^m_{e_l} = C^n_{e_l} \boxplus C^m_{e_k}$; i.e., the vector addition is commutative. This is clear from definition of the vector addition as given in Eq. (1).
- $(C^n_{e_k} \boxplus C^m_{e_l}) \boxplus C^t_{e_j} = C^n_{e_k} \boxplus (C^m_{e_l} \boxplus C^t_{e_j})$; i.e., the vector addition is associative.
  Assume that $C^n_{e_k} \boxplus C^m_{e_l}$ is edge-disjoint unions of these cycles $C^n_{e_k}, C^m_{e_l}$ in FACS and denoted by $E_{C^n_{e_k} \boxplus C^m_{e_l}}$. Now, take again a vector addition of $(C^n_{e_k} \boxplus C^m_{e_l})$ and $C^t_{e_j}$ which, clearly, equal to unions of edge-disjoint of $(C^n_{e_k} \boxplus C^m_{e_l})$ and $C^t_{e_j}$ and denoted by $E_{(C^n_{e_k} \boxplus C^m_{e_l}) \boxplus C^t_{e_j}}$ which means unions of edge-disjoint of cycles $C^n_{e_k}, C^m_{e_l}, C^t_{e_j}$.

In the right side of (2), assume that $C^m_{e_l} \boxplus C^t_{e_j}$ is edge-disjoint unions of these cycles $C^m_{e_l}, C^t_{e_j}$ in FACS and denoted by $E_{C^m_{e_l} \boxplus C^t_{e_j}}$. Then, take again a vector addition of $C^n_{e_k}$ and $(C^m_{e_l} \boxplus C^t_{e_j})$ which, evidently, equal to unions of edge-disjoint of $C^n_{e_k}$ and $(C^m_{e_l} \boxplus C^t_{e_j})$ and denoted by $E_{C^n_{e_k} \boxplus (C^m_{e_l} \boxplus C^t_{e_j})}$ which also means unions of edge-disjoint of cycles $C^n_{e_k}, C^m_{e_l}, C^t_{e_j}$; i.e $E_{(C^n_{e_k} \boxplus C^m_{e_l}) \boxplus C^t_{e_j}} = E_{C^n_{e_k} \boxplus (C^m_{e_l} \boxplus C^t_{e_j})}$:

- Define $\emptyset$ is the empty graph which as the additive identity element in $S_C(DG_{FT3})$, since $C^n_{e_k} \boxplus \emptyset = (C^n_{e_k} \cup \emptyset) \setminus (C^n_{e_k} \cap \emptyset) = C^n_{e_k} \setminus \emptyset = C^n_{e_k} = \emptyset \boxplus C^n_{e_k}$.
- Each element in $S_C(DG_{FT3})$, there is the additive inverse in $S_C(DG_{FT3})$, since $C^n_{e_k} \boxplus C^n_{e_k} = \emptyset$ if and only if $C^n_{e_k} = C^n_{e_k}$. Therefore, every element to be its own negative.

Now, consider the multiplication of vector by the scalar 1 is the identity operation, $1 \boxdot C^n_{e_k} = C^n_{e_k}$ and multiplication of vector by the scalar 0 takes every element in $S_C(DG_{FT3})$ to the empty graph, $0 \boxdot C^n_{e_k} = \emptyset$, (see the scalar multiplication Eq. (2)). Then, clearly, $S_C(DG_{FT3})$ is closed under scalar multiplication. Moreover, the conditions of the scalar multiplication are verified for all $C^n_{e_k}, C^m_{e_l}, C^t_{e_j}$ in $S_C(DG_{FT3})$ and α, β any scalars in {0, 1}:

- $\alpha \boxdot (C^n_{e_k} \boxplus C^m_{e_l}) = (\alpha \boxdot C^n_{e_k}) \boxplus (\alpha \boxdot C^n_{e_k})$. It is obvious that $1 \boxdot (C^n_{e_k} \boxplus C^m_{e_l}) = C^n_{e_k} \boxplus C^m_{e_l} = (1 \boxdot C^n_{e_k}) \boxplus (1 \boxdot C^n_{e_k})$ and $0 \boxdot (C^n_{e_k} \boxplus C^m_{e_l}) = \emptyset = \emptyset \boxplus \emptyset = (0 \boxdot C^n_{e_k}) \boxplus (0 \boxdot C^n_{e_k})$.
- $(\alpha +_2 \beta) \boxdot C^n_{e_k} = (\alpha \boxdot C^n_{e_k}) \boxplus (\beta \boxdot C^n_{e_k})$. Note that $(1 +_2 1) \boxdot C^n_{e_k} = 0 \boxdot C^n_{e_k} = \emptyset = C^n_{e_k} \boxplus C^n_{e_k} = (1 \boxdot C^n_{e_k}) \boxplus (1 \boxdot C^n_{e_k})$; $(1 +_2 0) \boxdot C^n_{e_k} = 1 \boxdot C^n_{e_k} = C^n_{e_k} = C^n_{e_k} \boxplus \emptyset = (1 \boxdot C^n_{e_k}) \boxplus (0 \boxdot C^n_{e_k})$; $(0 +_2 1) \boxdot C^n_{e_k} = 1 \boxdot C^n_{e_k} = C^n_{e_k} = \emptyset \boxplus C^n_{e_k} = (0 \boxdot C^n_{e_k}) \boxplus (1 \boxdot C^n_{e_k})$; $(0 +_2 0) \boxdot C^n_{e_k} = 0 \boxdot C^n_{e_k} = \emptyset = \emptyset \boxplus \emptyset = (0 \boxdot C^n_{e_k}) \boxplus (0 \boxdot C^n_{e_k})$.
- $(\alpha \cdot_2 \beta) \boxdot C^n_{e_k} = \alpha \boxdot (\beta \boxdot C^n_{e_k})$. Observe that $(1 \cdot_2 \beta) \boxdot C^n_{e_k} = \beta \boxdot C^n_{e_k} = 1 \boxdot (\beta \boxdot C^n_{e_k})$; $(0 \cdot_2 \beta) \boxdot C^n_{e_k} = 0 \boxdot C^n_{e_k} = \emptyset = 0 \boxdot (\beta \boxdot C^n_{e_k})$.
- By definition of the scalar multiplication Eq. (2), the multiplication of vector by the scalar 1 is the identity operation, $1 \boxdot C^n_{e_k} = C^n_{e_k}$ for each $C^n_{e_k} \in S_C(DG_{FT3})$.

Then, $S_C(DG_{FT3})$ is a vector space over {0, 1}, hence every element in $S_C(DG_{FT3})$ has a vector representation in $\{0,1\}^{|E|}$.

Thus, one can consider $S_C(DG_{FT3})$ as a vector space over $\{0,1\}^{|E|}$. This show that the element $C^n_{e_k} \boxplus C^m_{e_l} = E_{C^n_{e_k} \boxplus C^m_{e_l}}$ can be represented by an element $\vec{C}^n_{e_k} +_2 \vec{C}^m_{e_l} = (c_{e_1}, c_{e_2}, \cdots, c_{e_k}, \cdots, c_{e_l}, \cdots, c_{e_{|E|}})$ in $\{0,1\}^{|E|}$ with addition modulo 2, for all vector representations $\vec{C}^n_{e_k}, \vec{C}^m_{e_l}$ of the FT3-fuzzy detour cycles $C^n_{e_k}, C^m_{e_l}$ respectively and the element $C^n_{e_k} \boxplus C^n_{e_k} = \emptyset$ has also a vector representation $(c_{e_1}, c_{e_2}, \cdots, c_{e_k}, \cdots, c_{e_l}, \cdots, c_{e_{|E|}}) = (0, 0, \cdots, 0, \cdots, 0, \cdots, 0)$.

At this position, it is evident that the notion of the length of the element $C^n_{e_k} \boxplus C^m_{e_l} = E_{C^n_{e_k} \boxplus C^m_{e_l}}$ in the FT3-cycle space of $DG_{FT3}(V, E)$, $S_C(DG_{FT3})$, is implicitly defined as follows:

Definition 4: Suppose $S_C(DG_{FT3})$ is a vector space over $\{0,1\}^{|E|}$. Then, the length of the element $C^n_{e_k} \boxplus C^m_{e_l} = E_{C^n_{e_k} \boxplus C^m_{e_l}}$ in $S_C(DG_{FT3})$ is computed by $|C^n_{e_k} \boxplus C^m_{e_l}| = |E_{C^n_{e_k} \boxplus C^m_{e_l}}| = \sum_{(e_i \in E_{C^n_{e_k} \boxplus C^m_{e_l}})} \frac{1}{\mu(e_i)}$. The summation signifies that the sum of the length of the element $C^n_{e_k} \boxplus C^m_{e_l}$ is the sum of length of unions of edge-disjoint of FT3-fuzzy detour cycles $C^n_{e_k}, C^m_{e_l}$.

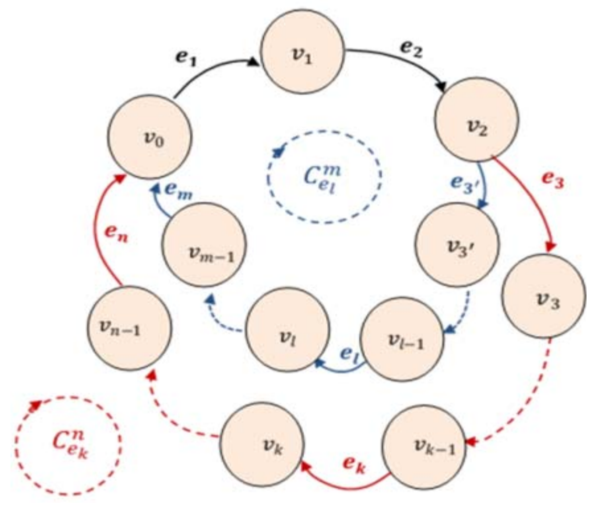


Fig. 7: Length of edge-disjoint unions $C^n_{e_k} \boxplus C^m_{e_l} = E_{C^n_{e_k} \boxplus C^m_{e_l}}$ of FT3-fuzzy detour cycles $C^n_{e_k}, C^m_{e_l}$

Figure 7 illustrates the length of the element $C^n_{e_k} \boxplus C^m_{e_l} = E_{C^n_{e_k} \boxplus C^m_{e_l}}$ in $S_C(DG_{FT3})$.

In Fig. 7, it can be noted that $C^n_{e_k} = (e_1, e_2, e_3, \cdots, e_{k-1}, e_k, \cdots, e_{n-1}, e_n)$ and $C^m_{e_l} = (e_1, e_2, e_{3'}, \cdots, e_{l-1}, e_l, \cdots, e_{m-1}, e_m)$, then: $C^n_{e_k} \boxplus C^m_{e_l} = E_{C^n_{e_k} \boxplus C^m_{e_l}} (e_3, \cdots, e_{k-1}, e_k, \cdots, e_{n-1}, e_n, e_{3'}, \cdots, e_{l-1}, e_l, \cdots, e_{m-1}, e_m)$.

Therefore, by Definition 4:

$$|C^n_{e_k} \boxplus C^m_{e_l}| = |E_{C^n_{e_k} \boxplus C^m_{e_l}}| = \frac{1}{\mu(e_3)} + \cdots + \frac{1}{\mu(e_{k-1})} + \frac{1}{\mu(e_k)} + \cdots + \frac{1}{\mu(e_{n-1})} + \frac{1}{\mu(e_n)} + \frac{1}{\mu(e_{3'})} + \cdots + \frac{1}{\mu(e_{l-1})} + \frac{1}{\mu(e_l)} + \cdots + \frac{1}{\mu(e_{m-1})} + \frac{1}{\mu(e_m)}. = \sum_{(e_i \in E_{C^n_{e_k} \boxplus C^m_{e_l}})} \frac{1}{\mu(e_i)}$$

Normed space and FACS of fuzzy graph type-3: Let $DG_{FT3}(V, E)$ be a FACS of fuzzy graph Type-3. Suppose that $C^n_{e_k}$ is FT3-fuzzy detour cycle in FT3-cycle space $S_C(DG_{FT3})$ which is a vector space over $\{0,1\}^{|E|}$. Then, from Definition 2, $C^n_{e_k}$ has a length $|C^n_{e_k}|$ such that the cycle $C^n_{e_k}$ containing the edge e_k with n is the number of all edges in this cycle. Now, take another cycle $C^m_{e_k}$ containing the edge e_k with m and hence, this cycle has a length $|C^m_{e_k}|$. By continuing the search for FT3-fuzzy detour cycles containing the edge e_k but with different lengths, one can provide a useful method for constructing novel normed space from a given vector space in Theorem 1 with respect to the following norm:

$$\| C_{e_k} \| = \max_{n=2,3,\cdots,|V|}\{ |C_{e_k}^n| \} = \max \{|C_{e_k}^2|, |C_{e_k}^3|, \cdots, |C_{e_k}^{|V|}|\} \quad (3)$$

It is clearly observed that the right side of Eq. (3) is well defined as every FT3-fuzzy detour cycle has a length and a norm of C_{e_k} is one of the maximum length of cycles $C_{e_k}^2, C_{e_k}^3, \cdots, C_{e_k}^{|V|}$. It means that C_{e_k} is one of these cycles $C_{e_k}^2, C_{e_k}^3, \cdots, C_{e_k}^{|V|}$, regardless of writing the number of all edges in this cycle C_{e_k}.

This shows that the concept of the norm of a vector in the FT3-cycle space of $DG_{FT3}(V,E)$, $S_C(DG_{FT3})$ is a generalization of the concept of length in $S_C(DG_{FT3})$. Consequently, from Definition 4, the norm of the element $C_{e_k} \boxplus C_{e_l} = E_{C_{e_k} \boxplus C_{e_l}}$ in $S_C(DG_{FT3})$ is given as:

$$\| C_{e_k} \boxplus C_{e_l} \| = \| E_{C_{e_k} \boxplus C_{e_l}} \| = \sum_{\left(e_i \in E_{C_{e_k} \boxplus C_{e_l}} \right)} \frac{1}{\mu(e_i)} \quad (4)$$

The following interesting theorem reveals the relationship between a normed space and a graph $DG_{FT3}(V,E)$. In other words, a FT3-cycle space of $DG_{FT3}(V,E)$ forms a normed space with the function as presented by Eq. (3).

Theorem 2: Let $S_C(DG_{FT3})$ be a FT3-cycle space of $DG_{FT3}(V,E)$ over $\{0,1\}^{|E|}$ and $\| C_{e_k} \| = \max_{n=2,3,\cdots,|V|}\{ |C_{e_k}^n| \} = \max \{|C_{e_k}^2|, |C_{e_k}^3|, \cdots, |C_{e_k}^{|V|}|\}$ be a real-valued function on $S_C(DG_{FT3})$. Then, $S_C(DG_{FT3})$ is a normed space.

Proof: By Theorem 1, $S_C(DG_{FT3})$ is a vector space over $\{0,1\}^{|E|}$. It requires to verify the following four conditions of a norm:

- $\| C_{e_k} \| \geq 0$ for all element C_{e_k} in $S_C(DG_{FT3})$, since it is the maximum of nonnegative numbers.
- $\| C_{e_k} \| = 0$ if and only if $C_{e_k} = \emptyset$, since $\emptyset$ is the empty graph which as the zero vector element in $S_C(DG_{FT3})$.
- $\| \alpha \boxdot C_{e_k} \| = |\alpha| \, \| C_{e_k} \|$ for all element C_{e_k} in $S_C(DG_{FT3})$ and all scalars {0, 1} (with addition and multiplication modulo 2) and note that $\| 1 \boxdot C_{e_k} \| = \| C_{e_k} \| = |1| \, \| C_{e_k} \|$; $\| 0 \boxdot C_{e_k} \| = \| \emptyset \| = |0| \, \| C_{e_k} \|$.

It is also noticed that the two congruence classes modulo 2 are as follows:

$\cdots \equiv -5 \equiv -3 \equiv -1 \equiv 1 \equiv 3 \equiv 5 \equiv 7 \equiv \cdots$ (mod2) and $\cdots \equiv -6 \equiv -4 \equiv -2 \equiv 0 \equiv 2 \equiv 4 \equiv 6 \equiv \cdots$ (mod2)

- $\| C_{e_k} \boxplus C_{e_l} \| \leq \| C_{e_k} \| + \| C_{e_l} \|$ for all element C_{e_k}, C_{e_l} in $S_C(DG_{FT3})$.

Assume that $C_{e_k} \boxplus C_{e_l}$ is an edge-disjoint unions of these cycles C_{e_k}, C_{e_l} in FACS and denoted by $E_{C_{e_k} \boxplus C_{e_l}}$. Then, by the function given as in Eq. (4), the norm of this element $E_{C_{e_k} \boxplus C_{e_l}}$ in $S_C(DG_{FT3})$ is given by:

$\| C_{e_k} \boxplus C_{e_l} \| = \| E_{C_{e_k} \boxplus C_{e_l}} \| = \sum_{\left(e_i \in E_{C_{e_k} \boxplus C_{e_l}} \right)} \frac{1}{\mu(e_i)} \leq \sum_{(e_i \in C_{e_k})} \frac{1}{\mu(e_i)} + \sum_{(e_i \in C_{e_l})} \frac{1}{\mu(e_i)}$ (since [$\sum_{\left(e_i \in E_{C_{e_k} \boxplus C_{e_l}} \right)} \frac{1}{\mu(e_i)}$] is the sum of the norm of just unions of edge-disjoint of cycles C_{e_k}, C_{e_l}). $=$ max $\{|C_{e_k}^2|, |C_{e_k}^3|, \cdots, |C_{e_k}^{|V|}|\}$ + max $\{|C_{e_l}^2|, |C_{e_l}^3|, \cdots, |C_{e_l}^{|V|}|\}$ (by Definition 2 of FT3-fuzzy detour cycle $C_{e_k}^n$). $= \| C_{e_k} \| + \| C_{e_l} \|$. This complete the proof.

In fact, Theorem 2 entails that the notion of a normed space with FACS can be constructed by this FT3-cycle space $S_C(DG_{FT3})$ which is a vector space over $\{0,1\}^{|E|}$. Furthermore, we can determine the basic FT3-fuzzy detour cycles for each edge $e_k \in E$ in FACS but with its norm $\| C_{e_k} \|$ as in the following section.

IMPLEMENTATION

Implementation of FT3-cycles space of FACS to the clinical incineration process: In this section, the graphical representation of the incineration process in Fig. 3 is used as an example. It was shown in Sabariah (2005) that this graph is a FACS and not a cycle. It means that there exist a cycle that included at most five vertices but could not be contained in a path that passes through all six vertices as in Fig. 3. Since n is the number of all edges in the FT3-cycle containing an edge e_k in FACS, then n = {2, 3, 4, 5} for each FT3-cycle containing an edge e_k in FACS of the incineration process. Now, the study of this incineration process in terms of the FT3-cycle space of $DG_{FT3}(V,E)$ of the incineration process is presented through determination of all the FT3-fuzzy detour cycles of FACS. Therefore, the existing definitions of the previous sections are used to identify FT3-cycles and FT3-fuzzy detour cycles in the following details:

A length of FT3-cycle containing the edge $e_1 \in E$ with each n = {2, 3, 4, 5}, denoted by $|C_{e_1}^n|$, in FACS of fuzzy graph Type-3 is called FT3-fuzzy detour cycle $C_{e_1}^n$ such that $|C_{e_1}^n| = \sum_{(e_i \in C_{e_k}^n)} \frac{1}{\mu(e_i)}$ (by Definition 2). Now, for all n = {2, 3, 4, 5}, these cycles are as follows:

- There exists no FT3-cycle containing the edge e_1 with $n = 2$, then $|C_{e_1}^2| = 0$.
- There exists FT3-cycle containing the edge e_1 with n = 3 which is $C_{e_1,e_8,e_{15}}^3$, then $|C_{e_1}^3| = \frac{1}{\mu(e_1)} + \frac{1}{\mu(e_8)} + \frac{1}{\mu(e_{15})} = \frac{1}{0.00001} + \frac{1}{0.31995} + \frac{1}{0.13401} = 110.58760$.

Thus, $C_{e_1}^3$ is called FT3-fuzzy detour cycle in FACS

- There exists FT3-cycle containing the edge e_1 with $n = 4$ which is $C^4_{e_1,e_6,e_{13},e_{15}}$, then $\left|C^4_{e_1}\right| = \frac{1}{\mu(e_1)} + \frac{1}{\mu(e_6)} + \frac{1}{\mu(e_{13})} + \frac{1}{\mu(e_{15})} = \frac{1}{0.00001} + \frac{1}{0.68004} + \frac{1}{0.00001} + \frac{1}{0.13401} = 208.93262.$

Thus, $C^4_{e_1}$ is called FT3-fuzzy detour cycle in FACS.

- There exists FT3-cycle containing the edge e_1 with $n = 5$ which is $C^5_{e_1,e_7,e_{14},e_{13},e_{15}}$, then $\left|C^5_{e_1}\right| = \frac{1}{\mu(e_1)} + \frac{1}{\mu(e_7)} + \frac{1}{\mu(e_{14})} + \frac{1}{\mu(e_{13})} + \frac{1}{\mu(e_{15})} = \frac{1}{0.00001} + \frac{1}{0.00001} + \frac{1}{0.99999} + \frac{1}{0.00001} + \frac{1}{0.13401} = 308.46213.$

Thus, $C^5_{e_1}$ is called FT3-fuzzy detour cycle in FACS.

As well as a length of FT3-cycle containing the edge $e_{15} \in E$ with each $n = \{2, 3, 4, 5\}$, denoted by $\left|C^n_{e_{15}}\right|$, in FACS is called FT3-fuzzy detour cycle $C^n_{e_{15}}$ in FACS. For all $n = \{2, 3, 4, 5\}$, these cycles are as follows:

- There exists FT3-cycle containing the edge e_{15} with $n = 2$ which is $C^2_{e_{15},e_5}$, then $\left|C^2_{e_{15}}\right| = \frac{1}{\mu(e_{15})} + \frac{1}{\mu(e_5)} = \frac{1}{0.13401} + \frac{1}{0.32752} = 10.51536.$

Thus, $C^2_{e_{15}}$ is called FT3-fuzzy detour cycle in FACS.

- There are three FT3-cycles containing the edge e_{15} with $n = 3$ which are $C^3_{e_{15},e_1,e_8}$, $C^3_{e_{15},e_2,e_{12}}$ and $C^3_{e_{15},e_3,e_{13}}$, consequently, the lengths of FT3-cycles containing the edge e_{15} with $n = 3$ are 110.58760 , 17.21002 and 109.39890, respectively. Then, by Definition 2, a length of FT3-cycle containing the edge e_{15} with $n = 3$ is $\left|C^3_{e_{15},e_1,e_8}\right| = \left|C^3_{e_{15}}\right| = 110.58760.$

Thus, $C^3_{e_{15}}$ is called the FT3-fuzzy detour cycle in FACS.

- There are three FT3-cycles containing the edge e_{15} with $n = 4$ which are $C^4_{e_{15},e_1,e_6,e_{13}}$, $C^4_{e_{15},e_2,e_{10},e_{13}}$ and $C^4_{e_{15},e_4,e_{14},e_{13}}$, then, the lengths of FT3-cycles containing the edge e_{15} with $n = 4$ are 208.93262, 115.43945 and 208.46213, respectively. Therefore, by Definition 2, a length of FT3-cycle containing the edge e_{15} with $n = 4$ is $\left|C^4_{e_{15},e_1,e_6,e_{13}}\right| = \left|C^4_{e_{15}}\right| = 208.93262.$

Thus, $C^4_{e_{15}}$ is called the FT3-fuzzy detour cycle in FACS.

- There are two FT3-cycles containing the edge e_{15} with $n = 5$ which are $C^5_{e_{15},e_1,e_7,e_{14},e_{13},}$ and $C^5_{e_{15},e_2,e_{11},e_{14},e_{13}}$, consequently, the lengths of FT3-cycles containing the edge e_{15} with $n = 5$ are 308.46213 and 164.86622, respectively. Therefore, by Definition 2, a length of FT3-cycle containing the edge e_{15} with $n = 5$ is $\left|C^5_{e_{15},e_1,e_7,e_{14},e_{13},}\right| = \left|C^5_{e_{15}}\right| = 308.46213.$

Thus, $C^5_{e_{15}}$ is called the FT3-fuzzy detour cycle in FACS.

Continuing in this way to other edges e_i, $i = 1,2,3,\cdots,15$, we obtain a sequence of FT3-fuzzy detour cycles (refer to Fig. 3) in FACS for the incineration process. These sequence are given as follows:

- The FT3-fuzzy detour cycles that contained the edge e_1 are: $C^3_{e_1} = (e_1, e_8, e_{15})$ with length $\left|C^3_{e_1}\right| = 110.58760$; $C^4_{e_1} = (e_1, e_6, e_{13}, e_{15})$ with length $\left|C^4_{e_1}\right| = 208.93262$; $C^5_{e_1} = (e_1, e_7, e_{14}, e_{13}, e_{15})$ with length $\left|C^5_{e_1}\right| = 308.46213$.
- The FT3-fuzzy detour cycles that contained the edge e_2 are: $C^2_{e_2} = (e_2, e_9)$ with length $\left|C^2_{e_2}\right| = 21.72037$; $C^3_{e_2} = (e_2, e_{12}, e_{15})$ with length $\left|C^3_{e_2}\right| = 17.21002$; $C^4_{e_2} = (e_2, e_{10}, e_{13}, e_{15})$ with length $\left|C^4_{e_2}\right| = 115.43945$; $C^5_{e_2} = (e_2, e_{11}, e_{14}, e_{13}, e_{15})$ with length $\left|C^5_{e_2}\right| = 164.86622$.
- The FT3-fuzzy detour cycle that contained the edge e_3 is: $C^3_{e_3} = (e_3, e_{13}, e_{15})$ with length $\left|C^3_{e_3}\right| = 109.39890$.
- The FT3-fuzzy detour cycle that contained the edge e_4 is: $C^4_{e_4} = (e_4, e_{14}, e_{13}, e_{15})$ with length $\left|C^4_{e_4}\right| = 208.46213$.
- The FT3-fuzzy detour cycle that contained the edge e_5 is: $C^2_{e_5} = (e_5, e_{15})$ with length $\left|C^2_{e_5}\right| = 10.51536$.
- The FT3-fuzzy detour cycle that contained the edge e_6 is: $C^4_{e_6} = (e_6, e_{13}, e_{15}, e_1)$ with length $\left|C^4_{e_6}\right| = 208.93262$.
- The FT3-fuzzy detour cycle that contained the edge e_7 is: $C^5_{e_7} = (e_7, e_{14}, e_{13}, e_{15}, e_1)$ with length $\left|C^5_{e_7}\right| = 308.46213$.
- The FT3-fuzzy detour cycle that contained the edge e_8 is: $C^3_{e_8} = (e_8, e_{15}, e_1)$ with length $\left|C^3_{e_8}\right| = 110.58760$.
- The FT3-fuzzy detour cycle that contained the edge e_9 is: $C^2_{e_9} = (e_9, e_2)$ with length $\left|C^2_{e_9}\right| = 21.72037$.
- The FT3-fuzzy detour cycle that contained the edge e_{10} is: $C^4_{e_{10}} = (e_{10}, e_{13}, e_{15}, e_2)$ with length $\left|C^4_{e_6}\right| = 115.43945$.
- The FT3-fuzzy detour cycle that contained the edge e_{11} is: $C^5_{e_{11}} = (e_{11}, e_{14}, e_{13}, e_{15}, e_2)$ with length $\left|C^5_{e_{11}}\right| = 164.86622$.

- The FT3-fuzzy detour cycle that contained the edge e_{12} is $C_{e_{12}}^3 = (e_{12}, e_{15}, e_2)$ with length $|C_{e_{12}}^3| = 17.21002$.
- The FT3-fuzzy detour cycles that contained the edge e_{13} are: $C_{e_{13}}^3 = (e_{13}, e_{15}, e_3)$ with length $|C_{e_{13}}^3| = 109.39890$; $C_{e_{13}}^4 = (e_{13}, e_{15}, e_1, e_6)$ with length $|C_{e_{13}}^4| = 208.93262$; $C_{e_{13}}^5 = (e_{13}, e_{15}, e_1, e_7, e_{14})$ with length $|C_{e_{13}}^5| = 308.46213$.
- The FT3-fuzzy detour cycles that contained the edge e_{14} are: $C_{e_{14}}^4 = (e_{14}, e_{13}, e_{15}, e_4)$ with length $|C_{e_{14}}^4| = 208.46213$; $C_{e_{14}}^5 = (e_{14}, e_{13}, e_{15}, e_1, e_7)$ with length $|C_{e_{14}}^5| = 308.46213$.
- The FT3-fuzzy detour cycles that contained the edge e_{15} are: $C_{e_{15}}^2 = (e_{15}, e_5)$ with length $|C_{e_{15}}^2| = 10.51536$; $C_{e_{15}}^3 = (e_{15}, e_1, e_8)$ with length $|C_{e_{15}}^3| = 110.58760$; $C_{e_{15}}^4 = (e_{15}, e_1, e_6, e_{13})$ with length $|C_{e_{15}}^4| = 208.93262$; $C_{e_{15}}^5 = (e_{15}, e_1, e_7, e_{14}, e_{13})$ with length $|C_{e_{15}}^5| = 308.46213$.

The preceding FT3-fuzzy detour cycles in FACS for the incineration process with the edge-disjoint unions of these cycles and an empty graph $\emptyset$ is considered as a FT3-cycle space in FACS of the incineration process. Then, it is clearly seen that the above FT3-fuzzy detour cycles in FACS for the incineration process is easily verified for all conditions of the vector addition and the scalar multiplication as in Theorem 1. Hence, The FT3-cycle space of FACS for the incineration process, $S_C(DG_{FT3})$ is a vector space over {0, 1} (with addition and multiplication modulo 2). Thus, the FT3-cycle space of FACS for the incineration process is considered as a vector space over $\{0,1\}^{|E|}$.

Then, each FT3-fuzzy detour cycle $C_{e_k}^n$ in FACS for the incineration process can be represented as vector representation $\vec{C}_{e_k}^n$ in $\{0,1\}^{|E|}$, where $|E| = 15$ and $\vec{C}_{e_k}^n = \begin{pmatrix} c_{e_1}, c_{e_2}, c_{e_3}, c_{e_4}, c_{e_5}, c_{e_6}, c_{e_7}, c_{e_8}, c_{e_9}, c_{e_{10}}, c_{e_{11}}, \\ c_{e_{12}}, c_{e_{13}}, c_{e_{14}}, c_{e_{15}} \end{pmatrix}$ as follows:

- The vector representations of FT3-fuzzy detour cycles that contain edge e_1 are:

  $C_{e_1}^3 = (e_1, e_8, e_{15}) \Rightarrow \vec{C}_{e_1}^3 = (1, 0,0,0,0,0,0, 1,0,0,0,0, 0,0,1)$
  $C_{e_1}^4 = (e_1, e_6, e_{13}, e_{15}) \Rightarrow \vec{C}_{e_1}^4 = (1,0,0,0,0,1,0, 0,0,0,0,0,1,0,1)$
  $C_{e_1}^5 = (e_1, e_7, e_{14}, e_{13}, e_{15}) \Rightarrow \vec{C}_{e_1}^5 = (1,0,0,0,0,0,1,0, 0,0,0,0,0,1,1,1)$

- The vector representations of FT3-fuzzy detour cycles that contain edge e_2 are:

  $C_{e_2}^2 = (e_2, e_9) \Rightarrow \vec{C}_{e_2}^2 = (0, 1,0,0,0,0,0, 0,1,0,0,0,0,0,0)$;
  $C_{e_2}^3 = (e_2, e_{12}, e_{15}) \Rightarrow \vec{C}_{e_2}^3 = (0, 1,0,0,0,0,0, 0,0,0,0,1, 0,0,1)$;
  $C_{e_2}^4 = (e_2, e_{10}, e_{13}, e_{15}) \Rightarrow \vec{C}_{e_2}^4 = (0, 1,0,0,0,0,0, 0,0,1,0,0, 1,0,1)$;
  $C_{e_2}^5 = (e_2, e_{11}, e_{14}, e_{13}, e_{15}) \Rightarrow \vec{C}_{e_2}^5 = (0, 1,0,0,0,0,0, 0,0,0,1,0, 1,1,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_3 is:

  $C_{e_3}^3 = (e_3, e_{13}, e_{15}) \Rightarrow \vec{C}_{e_3}^3 = (0, 0,1,0,0,0,0, 0,0,0,0,0,0, 1,0,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_4 is:

  $C_{e_4}^4 = (e_4, e_{14}, e_{13}, e_{15}) \Rightarrow \vec{C}_{e_4}^4 = (0, 0,0,1,0,0,0, 0,0,0,0,0,0, 1,1,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_5 is:

  $C_{e_5}^2 = (e_5, e_{15}) \Rightarrow \vec{C}_{e_5}^4 = (0,0,0,0,1,0,0, 0,0,0,0,0,0,0,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_6 is:

  $C_{e_6}^4 = (e_6, e_{13}, e_{15}, e_1) \Rightarrow \vec{C}_{e_6}^4 = (1,0,0,0,0,1,0, 0,0,0,0,0,1,0,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_7 is
  $C_{e_7}^5 = (e_7, e_{14}, e_{13}, e_{15}, e_1) \Rightarrow \vec{C}_{e_7}^4 = (1,0,0,0,0,0,1, 0,0,0,0,0,1,1,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_8 is:

  $C_{e_8}^3 = (e_8, e_{15}, e_1) \Rightarrow \vec{C}_{e_8}^4 = (1,0,0,0,0,0,0, 1,0,0,0,0,0,0,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain the edge e_9 is:

  $C_{e_9}^2 = (e_9, e_2) \Rightarrow \vec{C}_{e_9}^4 = (0,1,0,0,0,0,0, 0,1,0,0,0,0,0,0)$.

- The vector representation of FT3-fuzzy detour cycle that contain edge e_{10} is:
  $C_{e_{10}}^4 = (e_{10}, e_{13}, e_{15}, e_2) \Rightarrow \vec{C}_{e_{10}}^4 = (0,1,0,0,0,0,0, 0,0,1,0,0,1,0,1)$.

- The vector representation of FT3-fuzzy detour cycle that contain edge e_{11} is: $C_{e_{11}}^{5} = (e_{11}, e_{14}, e_{13}, e_{15}, e_{2}) \Rightarrow \vec{C}_{e_{11}}^{4} =$ (0,1,0,0,0,0,0, 0,0,0,1,0,1,1,1)

- The vector representation of FT3-fuzzy detour cycle that contain edge e_{12} is:

  $C_{e_{12}}^{3} = (e_{12}, e_{15}, e_{2}) \Rightarrow \vec{C}_{e_{12}}^{4} =$ (0,1,0,0,0,0,0, 0,0,0,0,1,0,0,1).

- The vector representations of FT3-fuzzy detour cycles that contain edge e_{13} are:

  $C_{e_{13}}^{3} = (e_{13}, e_{15}, e_{3}) \Rightarrow \vec{C}_{e_{13}}^{3} =$ (0,0,1,0,0,0,0, 0,0,0,0,0,1,0,1);
  $C_{e_{13}}^{4} = (e_{13}, e_{15}, e_{1}, e_{6}) \Rightarrow \vec{C}_{e_{13}}^{4} =$ (1,0,0,0,0,1,0, 0,0,0,0,0,1,0,1);
  $C_{e_{13}}^{5} = (e_{13}, e_{15}, e_{1}, e_{7}, e_{14}) \Rightarrow \vec{C}_{e_{13}}^{5} =$ (1,0,0,0,0,0,1,0, 0,0,0,0,0,1,1,1).

- The vector representations of FT3-fuzzy detour cycles that contain edge e_{14} are:

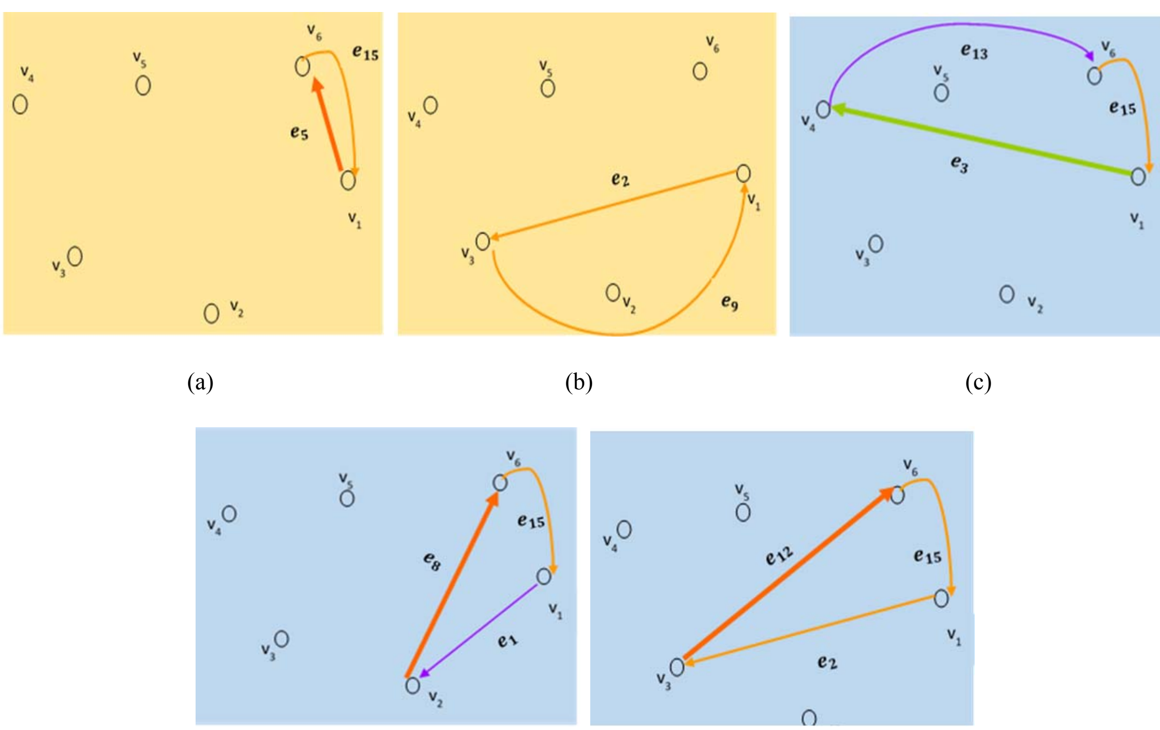
  $C_{e_{14}}^{4} = (e_{14}, e_{13}, e_{15}, e_{4}) \Rightarrow \vec{C}_{e_{14}}^{4} =$ (0,0,0,1,0,0,0, 0,0,0,0,0,1,1,1);
  $C_{e_{14}}^{5} = (e_{14}, e_{13}, e_{15}, e_{1}, e_{7}) \Rightarrow \vec{C}_{e_{14}}^{5} =$ (1,0,0,0,0,0,1,0, 0,0,0,0,0,1,1,1).

- The vector representations of FT3-fuzzy detour cycles that contain edge e_{15} are:

  $C_{e_{15}}^{2} = (e_{15}, e_{5}) \Rightarrow \vec{C}_{e_{15}}^{2} =$ (0,0,0,0,1,0,0, 0, ,0,0,0,0,0,1);
  $C_{e_{15}}^{3} = (e_{15}, e_{1}, e_{8}) \Rightarrow \vec{C}_{e_{15}}^{3} =$ (1,0,0,0,0,0,0, 1,0,0,0,0,0,0,1);
  $C_{e_{15}}^{4} = (e_{15}, e_{1}, e_{6}, e_{13}) \Rightarrow \vec{C}_{e_{15}}^{4} =$ (1,0,0,0,0,1,0, 0,0, ,0,0,1,0,1);
  $C_{e_{15}}^{5} = (e_{15}, e_{1}, e_{7}, e_{14}, e_{13}) \Rightarrow \vec{C}_{e_{15}}^{5} =$ (1,0,0,0,0,0,1, 0,0,0,0,0,1,1,1).

Implementation of normed space of FACS to the clinical incineration process: It is easily checked by Theorem 2 that the FT3-cycle space of FACS for the incineration process satisfies all the four axioms of the norm given in Eq. (3). It is worth to notice that the basic FT3-fuzzy detour cycles with respect to this norm which can be seen in Fig. 8 and 9 are as follows:

- The FT3-fuzzy detour cycle that contained the edge e_1 is:



(a) (b) (c)

(d) (e)

Fig. 8: (a): FT3-fuzzy detour cycle $C_{e_5} = (e_5, e_{15})$; (b): FT3-fuzzy detour cycle $C_{e_9} = (e_9, e_2)$; (c): FT3-fuzzy detour cycle $C_{e_3} = (e_3, e_{13}, e_{15})$; (d): FT3-fuzzy detour cycle $C_{e_8} = (e_8, e_{15}, e_1)$; (e): FT3-fuzzy detour cycle $C_{e_{12}} = (e_{12}, e_{15}, e_2)$

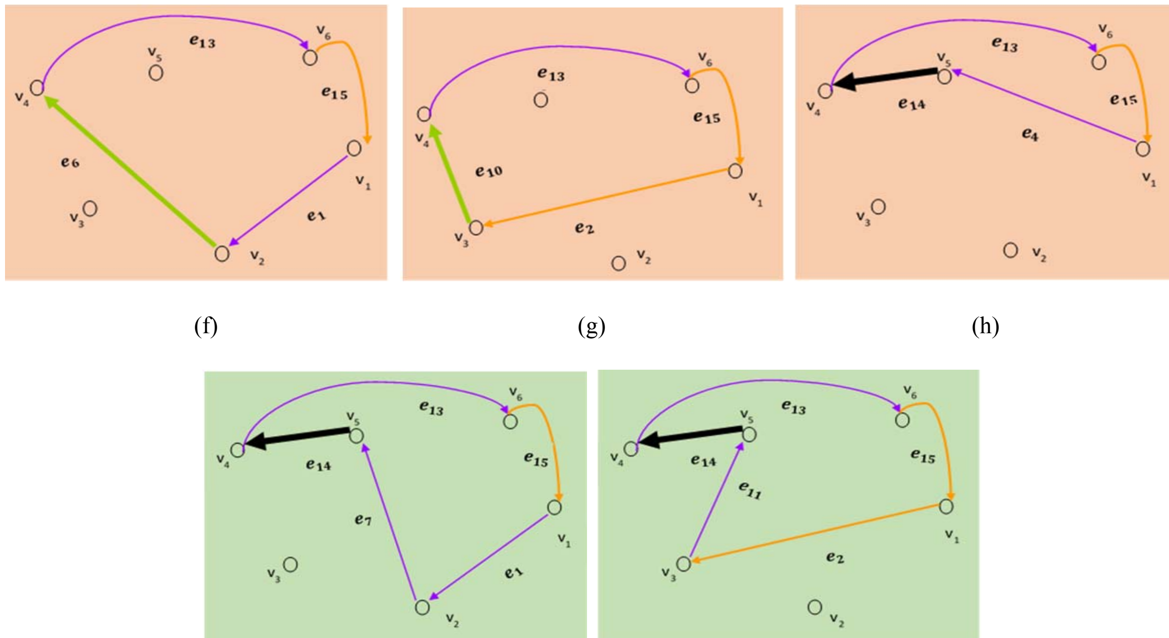


Fig. 9: (f): FT3-fuzzy detour cycle $C_{e_6} = (e_6, e_{13}, e_{15}, e_1)$; (g): FT3-fuzzy detour cycle $C_{e_{10}} = (e_{10}, e_{13}, e_{15}, e_2)$; (h): FT3-fuzzy detour cycle$C_{e_4} = (e_4, e_{14}, e_{13}, e_{15})$; (i): FT3-fuzzy detour cycles $C_{e_1} = C_{e_7} = C_{e_{13}} = C_{e_{14}} = C_{e_{15}} = (e_{15}, e_1, e_7, e_{14}, e_{13})$; (j): FT3-fuzzy detour cycles $C_{e_2} = C_{e_{11}} = (e_{11}, e_{14}, e_{13}, e_{15}, e_2)$

$C_{e_1} = (e_1, e_7, e_{14}, e_{13}, e_{15})$ with norm $\| C_{e_1} \| =$ 308.46213.

- The FT3-fuzzy detour cycle that contained the edge e_2 is:

$C_{e_2} = (e_2, e_{11}, e_{14}, e_{13}, e_{15})$ with norm $\| C_{e_2} \| =$ 164.86622.

- The FT3-fuzzy detour cycle that contained the edge e_3 is:

$C_{e_3} = (e_3, e_{13}, e_{15})$ with norm $\| C_{e_3} \| =$ 109.39890.

- The FT3-fuzzy detour cycle that contained the edge e_4 is:

$C_{e_4} = (e_4, e_{14}, e_{13}, e_{15})$ with norm $\| C_{e_4} \| =$ 208.46213.

- The FT3-fuzzy detour cycle that contained the edge e_5 is:

$C_{e_5} = (e_5, e_{15})$ with norm $\| C_{e_5} \| = 10.51536$.

- The FT3-fuzzy detour cycle that contained the edge e_6 is:

$C_{e_6} = (e_6, e_{13}, e_{15}, e_1)$ with norm $\| C_{e_6} \| =$ 208.93262.

- The FT3-fuzzy detour cycle that contained the edge e_7 is:

$C_{e_7} = (e_7, e_{14}, e_{13}, e_{15}, e_1)$ with norm $\| C_{e_7} \| =$ 308.46213.

- The FT3-fuzzy detour cycle that contained the edge e_8 is:

$C_{e_8} = (e_8, e_{15}, e_1)$ with norm $\| C_{e_8} \| =$ 110.58760.

- The FT3-fuzzy detour cycle that contained the edge e_9 is:

$C_{e_9} = (e_9, e_2)$ with norm $\| C_{e_9} \| = 21.72037$.

- The FT3-fuzzy detour cycle that contained the edge e_{10} is:

$C_{e_{10}} = (e_{10}, e_{13}, e_{15}, e_2)$ with norm $\| C_{e_{10}} \| =$ 115.43945.

- The FT3-fuzzy detour cycle that contained the edge e_{11} is:

$C_{e_{11}} = (e_{11}, e_{14}, e_{13}, e_{15}, e_2)$ with norm $\| C_{e_{11}} \| = 164.86622$.

- The FT3-fuzzy detour cycle that contained the edge e_{12} is:

$C_{e_{12}} = (e_{12}, e_{15}, e_2)$ with norm $\| C_{e_{12}} \| = 17.21002$.

- The FT3-fuzzy detour cycles that contained the edge e_{13} is:

$C_{e_{13}} = (e_{13}, e_{15}, e_1, e_7, e_{14})$ with norm $\| C_{e_{13}} \| = 308.46213$.

- The FT3-fuzzy detour cycles that contained the edge e_{14} is:

$C_{e_{14}} = (e_{14}, e_{13}, e_{15}, e_1, e_7)$ with norm $\| C_{e_{14}} \| = 308.46213$.

- The FT3-fuzzy detour cycles that contained the edge e_{15} is:

$C_{e_{15}} = (e_{15}, e_1, e_7, e_{14}, e_{13})$ with norm $\| C_{e_{14}} \| = 308.46213$.

It is necessary to observe that $C_{e_1} = C_{e_7} = C_{e_{13}} = C_{e_{14}} = C_{e_{15}}$ and $C_{e_2} = C_{e_{11}}$. Consequently, the basic FT3-fuzzy detour cycles with respect to the norm given in Eq. (3) when interpret physically means that each fuzzy connectivity of edge e_i in a FT3-fuzzy detour cycle C_{e_i} has a certain proportion of the chemical interaction with other edges to the greatest extent norm $\| C_{e_i} \|$.

Hence, the basic FT3-fuzzy detour cycles with respect to the norm given in Eq. (3) are ten cycles (Fig. 8 and 9).

CONCLUSION

This study shows that the new concept of the norm of a vector in the FT3-cycle space of FACS is a generalization of the concept of length in the FT3-cycle space of FACS. Then, a new type of normed space which is the normed space of FACS of fuzzy graph Type-3 (see Theorem 2) is presented. This normed space is constructed using the notion of the FT3-cycle space of FACS as a vector space over $\{0,1\}^{|E|}$ (see Theorem1) together with this norm. The basic FT3-fuzzy detour cycles with respect to this norm are determined in the incineration process represented by the FT3-cycle space associated with a graph of FACS of the incineration process.

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