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Research Article

Implementing a Type of Block Predictor-corrector Mode for Solving General Second Order Ordinary Differential Equations

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Abstract: The paper is geared towards implementing a type of block predictor-corrector mode capable of integratinggeneral second order ordinary differential equations using variable step size. This technique will be carried out on nonstiff problems. The mode which emanated from Milne's estimate has many computation advantages such as changing and designing a suitable step size, correcting to convergence, error control/minimization with better accuracy compare to other methods with fixed step size. Moreover, the approach will adopt the estimates of the principal local truncation error on a pair of explicit (predictor) and implicit (corrector) Adams family which are implemented in $P(CE)^m$ mode. Numerical examples are given to examine the efficiency of the method and compared with subsisting methods.

Keywords: And phrase block predictor-corrector mode, correcting to convergence, nonstiff problems, principal local truncation error, variable step size technique

INTRODUCTION

Rising from the advent of computing machines and programming languages, the numerical solution of Initial Value Problems (IVPs) for Ordinary Differential Equations (ODEs) has been the topic to explore by numerical analysts, mostly procedures for the numerical solution of the general second-order ODEs of the form as seen in Ken *et al*. (2011):

$$y'''(x) = f\left(x, y, y'\right), \quad y(a) = \alpha, \quad y'(a) = \beta, \quad x \in [a,b]$$
$$\text{and } f : R \times R^m \to R^m \qquad (1)$$

The general solution to (1) can be coded as:

$$\sum_{i=1}\alpha_i y_{n-i} = h^2 \sum_{i=1}\beta_i f_{n+i} \qquad (2)$$

where, the step size is h, $\alpha_j = 1$, α_j, $i = 1, \ldots j$, β_j, are unknown constants which are uniquely specified such that the formula is of order j as discussed in Akinfenwa *et al*. (2013).

We assume that $f \in R$ is sufficiently differentiable on $x \in [a, b]$ and satisfies a global Lipchitz condition, i.e., there is a constant $L \geq 0$ such that:

$$\left|f(x, y) - f\left(x, \bar{y}\right)\right| \leq L\left|y - \bar{y}\right|, \ \forall y, \bar{y} \in R.$$

Under this presumption, Eq. (1) assured the existence and uniqueness defined on $x \in [a, b]$ as discussed in Lambert (1973) and Xie and Tian (2014).
where, a and b are finite and $y^{(I)}[y^{(i)}{}_1, y^{(i)}{}_2, \ldots, y^{(i)}{}_n]^T$ for $i = 0(1)3$ and $f = [f_1, f_1, \ldots, f_n]^T$,

Again, Weierstrass approximation theorem stands as a justification for (1). See (Jain *et al*., 2007) for details.

Eq. (1) arises from many physical phenomena in a broad compass of applications. Largely in the field science and engineering such as in the electric circuits, damped and undamped mass-spring systems, forced oscillations and other areas of practical applications as introduced by Majid *et al*. (2012). Authors such as Anake *et al*. (2012), Ismail *et al*. (2009) and Majid and Suleiman (2009) indicated that (1) can be reduced to an equivalent first-order system of two times the dimension and evaluated utilizing the existing one-step method like Runge-Kutta method or block multistep method. This method has been described to increase the dimension of the problem, computational effect and very difficult. Block multistep methods are one of the numerical methods which have been suggested by several researchers, (Adesanya *et al*., 2012; James *et al*., 2013; Ken *et al*., 2011; Majid *et al*., 2012; Majid and Suleiman, 2009; Zarina *et al*., 2007). The common block methods used to solve the problems can be arranged into categories as one-step block method and

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multistep block method. Nevertheless, scholars have proposed an alternative method to solve at once (1) as discussed in Adesanya *et al*. (2012, 2013), Anake *et al*. (2012), Ehigie *et al*. (2011), Ismail *et al*. (2009), Ken *et al*. (2011) and Majid *et al*. (2012).

However, (Adesanya *et al*., 2012; Ehigie *et al*., 2011; Ismail *et al*., 2009; Ken *et al*., 2011) proposed block multistep methods which were employed in predictor-corrector mode. Block multistep methods have the vantage of evaluating simultaneously at all points with the integration interval, thereby reducing the computational burden when evaluation is required at more than one point within the grid. Again, Taylor series expansion is used to provide the initial values in order to compute the corrector.

Researchers such as Adesanya *et al*. (2012), Ehigie *et al*. (2011), Ismail *et al*. (2009) and Ken *et al*. (2011), innovated block predictor-corrector method in which at each practical application of the method, the method was only intended to predict and correct the results generated. In this study, the motivation is stemmed by the fact that there are very few work been done in solving nonstiff ODEs using block predictor-corrector mode, effort will be geared towards developing a type of block predictor-corrector mode using variable step size technique otherwise called Milne's estimate. This method have several benefits as earlier stated in the abstract.

Definition 1: According to Akinfenwa *et al*. (2013). A block-by-block method is a method for computing vectors Y_0, Y_1, … in sequence. Let the r-vector (r is the number of points within the block) Y_μ, F_μ, and G_μ, for n = mr, m = 0, 1,. . . be given as:

$$Y_w = (y_{n+1},\dots,y_{n+r})^T, F_w = (f_{n+1},\dots,f_{n+r})^T$$

Then the l -block r-point methods for (1) are given by:

$$Y_w = \sum_{i=1}^{j} A^{(i)} Y_{w-i} + h\sum_{i=1}^{j} B^{(i)} F_{w-i}$$

where, $A^{(i)}$, $B^{(i)}$, $i = 0, \dots, j$ are r by r matrices as introduced by Fatunla (1990).

Thus, from the above definition a block method has the vantage that in each application, the solution is approximated at more than one point simultaneously. The number of points depends on the manner of construction of the block method. Therefore applying these methods can give quicker and faster solutions to the problem which can be managed to produce a desired accuracy. See (Majid and Suleiman, 2007; Mehrkanoon *et al*., 2010).

The block algorithm proposed in this study is based on interpolation and collocation. The continuous representation of the algorithm generates a main discrete collocation method to render the approximate solution Y_{n+i} to the solution of (1) at points x_{n+i}, $i = 1, \dots, k$ as in Akinfenwa *et al*. (2013). The main aim of this study is to introduce a type of block predictor-corrector mode using variable steps size technique for mathematically integrating general second order ODEs directly.

DERIVATION OF THE METHOD

Adopting (Akinfenwa *et al*., 2013) in this section, the target is to derive the principal block predictor-corrector mode of the form (2). We move ahead by seeking an estimate of the exact solution y(x) by assuming a continuous solution Y(x) of the form:

$$Y(x) = \sum_{i=0}^{q+k-1} m_i \vartheta_i(x) \tag{3}$$

Such that $x \in [a, b]$, m_i are unknown coefficients and $\vartheta_i(x)$ are polynomial basis functions of degree $q+k$ -1, where q is the number of interpolation points and the collocation points k are respectively chosen to satisfy $q = j$ and $k \geq 1$. The integer $j \geq 1$ denotes the step number of the method. We thus construct a j-step block method block method with $\vartheta_{i(x)} = \left(\frac{x - x_i}{h}\right)^i$ by imposing the following conditions:

$$\sum_{i=0}^{q} m_i \left(\frac{x - x_i}{h}\right)^i = y_{n-i}, \quad i = 0,\dots,q-1 \tag{4}$$

$$\sum_{i=0}^{q} m_i i(i-1)\left(\frac{x - x_i}{h}\right)^{i-2} = f_{n+i}, \quad i \in Z, \tag{5}$$

where, y_{n+i} is he approximation for the exact solution $y(x_{n+i})$, $f_{n+i} = f(x_{n+i}, y_{n+i})$ n is the grid index and $x_{n+i} = x_n^{+ih}$. It should be noted that Eq. (4) and (5) leads to a system of q+1 equations of the AX = B.
where,

$$A = \begin{vmatrix} x_n^0 & 0 & 0 & 0 & 0 & 0 \\ x_{n-1}^0 & -x_{n-1}^1 & x_{n-1}^2 & -x_{n-1}^3 & \cdots & x_{n-1}^q \\ 0 & 0 & k(k-1)x_{n-2}^2 & -k(k-1)x_{n-2}^3 & \cdots & k(k-1)x_{n-2}^q \\ 0 & 0 & k(k-1)x_{n+1}^2 & k(k-1)x_{n+1}^3 & \cdots & k(k-1)x_{n+1}^q \\ 0 & 0 & \cdots & \cdots & & \\ 0 & 0 & k(k-1)x_{n+k}^2 & k(k-1)x_{n+k}^3 & \cdots & k(k-1)x_{n+k}^q \end{vmatrix}$$
$$X = [x_0, x_1, x_2, \dots, x_k]^T$$
$$U = [f_n, f_{n-1}, \dots, f_{n-k-1}, f_{n+1}, f_{n+2}, \dots, f_{n+k-1}, y_{n-k}]^T \tag{6}$$

Solving Eq. (6) using Mathematica, we get the coefficients of m_i and substituting the values of m_i into (4) and after some algebraic computation, the block predictor-corrector mode is obtain as:

$$\sum_{i=0}^{q-1} \alpha_i y_{n-i} = h^2 \left| \sum_{i=0}^{q-1} \beta_i f_{n-i} + \sum_{i=0}^{q-1} \beta_i f_{n+i} \right| \tag{7}$$

where, α_i and β_i are continuous coefficients.

ANALYSIS OF SOME THEORETICAL PROPERTIES

Order of accuracy of the method: Conforming to Akinfenwa *et al*. (2013) and Lambert (1973), we specify the associated linear multistep method (7) and the difference operator as:

$$L[y(x);h] = \sum_{i=0}^{j}\left[\alpha_i y(x+ih) + h^2 \beta_i y''(x+ih)\right] \tag{8}$$

Presuming that $y(x)$ is sufficiently and continuously differentiable on an interval [a, b] and that $y(x)$ has as many higher derivatives as demanded then, we write the conditions in (8) as a Taylor series expression of $y(x_{n+i})$ and $f(x_{n+i}) \equiv y''(x_{n+i})$ as:

$$y(x_{n+i}) = \sum_{k=0}^{\infty}\frac{(ih)}{k!}y^{(k)}(x_n) \text{ and } y'''(x_{n+i}) = \sum_{k=0}^{\infty}\frac{(ih)}{k!}y^{(k+1)}(x_n) \tag{9}$$

Substituting (8) and (9) into (7) we obtain the following expression:

$$L[y(x);h] = c_0 y(x) + c_1 h y^{(1)}(x) + \dots + c_{p+1} h^{p+1} y^{(p+1)}(x) + \dots, \tag{10}$$

Hence, we noticed that the Predictor-Corrector Mode (P(CE)^m) of (7) has order p, if C_{p+1}, p = 0, 1, 2, …, I = 1, 2,…, j, are given as follows:

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_k,$$
$$c_1 = \alpha_0 + 2\alpha_1 + \dots + k\alpha_k,$$
$$c_2 = \frac{1}{2!}(\alpha_0 + \alpha_1 + \alpha_2 + \dots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \dots + \beta_k),$$
$$c_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \dots + k^q\alpha_k) - \frac{1}{(q-2)!}(\beta_0 + 2^{q-2}\beta_1 + \dots + k^{q-2}\beta_k),$$
$$q = 3,2,\dots$$

Therefore, the method (7) has order $p \geq 1$ and error constants given by the vector, $C_{p+2} \neq 0$.

Concurring with Lambert (1973), we say that the method (2) has order p if:

$$L[y(x);h] = O(h^{p+2}), C_0 = C_1 = \dots = C_p = C_{p+1} = 0,$$
$$C_{p+2} \neq 0. \tag{11}$$

Therefore, C_{p+2} is the error constant and $C^{p+2}\ h^{p+2}\ y^{(p+2)}\ (X_n)$ is the principal local truncation error at the point x_n.

Stability analysis of the method: To examine the method for stability, (7) is normalised and composed as a block method given by the matrix finite difference equations as presented in Akinfenwa *et al*. (2013), Ken *et al*. (2011), Mohammed *et al*. (2013) and Awari (2013).

$$A^{(0)}Y_m = A^{(1)}Y_{m-1} + h^3(B^{(0)}F_m + B^{(1)}F_{m-1}), \tag{12}$$

where,

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+1} \\ . \\ . \\ . \\ y_{n+r} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n-r+1} \\ y_{n-r+2} \\ . \\ . \\ . \\ y_n \end{bmatrix}, \quad F_m = \begin{bmatrix} f_{n+1} \\ f_{n+1} \\ . \\ . \\ . \\ f_{n+r} \end{bmatrix}, \quad F_{m-1} = \begin{bmatrix} f_{n-r+1} \\ f_{n-r+2} \\ . \\ . \\ . \\ f_n \end{bmatrix}.$$

The matrices $A^{(0)}$, $A^{(1)}$, $B^{(0)}$, $B^{(1)}$ are r by r matrices with real entries while Y_m, Y_{m-1}, F_m, F_{m-1} are r-vectors specified above.

Following (Ken *et al*., 2011; Lambert, 1973), we stick to the boundary locus method to decide on the region of absolute stability of the block predictor-corrector mode and to obtain the roots of absolute stability. Substituting the test equation $y' = -\lambda y$ and $\bar{h} = h^2\lambda^2$ into the block method (12) to obtain:

$$\rho(r) = \det\left[r\left(A^{(0)} + B^{(0)}h^2\lambda^2\right) - \left(A^{(1)} - B^{(1)}h^2\lambda^2\right)\right] = 0 \tag{13}$$

Replacing $h = 0$ in (13), we obtain all the roots of the derived equation to be $r \leq 1$. Therefore, according to Lambert (1973), the predictor-corrector mode is absolutely stable.

So, as considered in Adesanya *et al*. (2013), Lambert (1973) and Awari (2013), the boundary of the region of absolute stability can be obtained by filling (7) into:

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \tag{14}$$

And permit $r = e^{i\theta} = \cos\theta + i\sin\theta$ then after reduction together with simplifying (14) within [0^0, 180^0]. Accordingly, the boundary of the region of absolute stability rests on the real axis.

Fig. 1 is free hand drawing.

IMPLEMENTATION OF THE METHOD

Embracing (Faires and Burden, 2012; Lambert, 1973), afterward this is implemented in the P(EC)^m mode then it becomes very pertinent if the explicit (predictor) and the implicit (corrector) methods are individually of the same order and this prerequisite makes it essential for the step number of the explicit (predictor) method to be greater than that of the implicit (corrector) method. Consequently, the mode P(EC)^m can be formally decided as follows for m = 1, 2, ….: P(EC)^m:

$$y_{n+j}^{[0]} + \sum_{i=0}^{j-1}\alpha_i^{*} y_{n+i}^{[m]} = h^2\sum_{i=0}^{j-1}\beta_i^{*} f_{n+i}^{[m]},\ f_{n+j}^{[s]} \equiv f(x_{n+j}, y_{n+j}^{[s]}),$$
$$y_{n+j}^{[s+1]} + \sum_{i=0}^{j-1}\alpha_i y_{n+i}^{[m]} = h^2\beta_j f_{n+j}^{[s]} + h^2\sum_{i=0}^{j-1}\beta_i f_{n+i}^{[m-1]},\} s = 0,1,\dots,m-1, \tag{15}$$

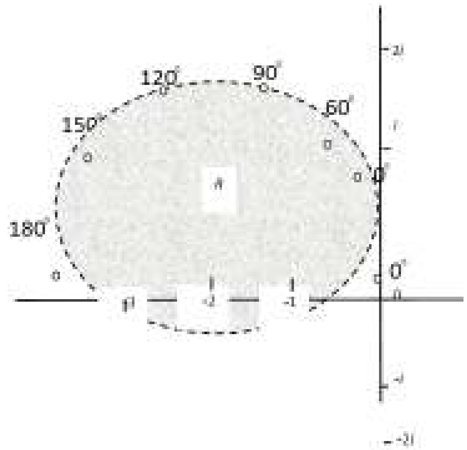

Fig. 1: Showing the region of absolute stability of the block predictor-corrector mode, since the root of the stability polynomial is r≤1

Note that as m→∞, the result of calculating with the above mode will incline to those given by the mode of correcting to convergence.

Moreover, predictor-corrector pair based on (1) can be applied. The mode $P(EC)^m$ specified by (15), where h^2 is the step size. Since the predictor and corrector both have the same order p, Milne's device is applicable and relevant.

According to Dormand (1996) and Lambert (1973), Milne's device suggests that it is possible to estimate the principal local truncation error of the explicit-implicit (predictor-corrector mode) method without estimating higher derivatives of $y(x)$. Assume that $p = p^*$, where p^* and p represents the order of the explicit (predictor) and implicit (corrector) method with the same order. Now for a method of order p, the principal local truncation errors can be well defined as:

$$C^*_{p+2} h^{p+2} y^{(p+2)}(x_n) = y(x_{n+j}) - W_{n+j} + O(h^{p+3}) \tag{16}$$

Also:

$$C_{p+2} h^{p+2} y^{(p+2)}(x_n) = y(x_{n+j}) - C_{n+j} + O(h^{p+3}) \tag{17}$$

where, W_{n+j} and C_{n+j} are called the predicted and corrected approximations given by method of order p while C^*_{p+2} and C_{p+2} are independent of h.

Neglecting terms of degree p+3 and above, it is easy to make estimates of the principal local truncation error of the mode as:

$$C_{p+2} h^{p+2} y^{(p+2)}(x_n) \quad \frac{C_{p+2}}{C^*_{p+2} \quad C_{p+2}} \left| W_{n+j} - C_{n+j} \right| \tag{18}$$

Noting the fact that $C_{p+2} \neq C^*_{p+2}$ and $W_{n+j} \neq C_{n+j}$.

Furthermore, the estimate of the principal local truncation error (18) is used to determine whether to accept the results of the current step or to reconstruct the step with a smaller step size. The step is accepted based on a test as prescribed by (18) as in Ascher and Petzold (1998). Equation (18) is the convergence criteria otherwise called Milne's estimate for correcting to convergence

Furthermore, Eq. (18) ensures the convergence criterion of the mode during the test evaluation.

NUMERICAL EXPERIMENTS

The performance of the block predictor-corrector mode ($P(CE)^m$) was carried out on nonstiff and mildly stiff problems. For problem tested 5.1 and 5.2, the following tolerances (convergence criteria)10^{-2}, 10^{-4}, 10^{-6} , 10^{-8}, $10^{-10}$and 10^{-12} were used to compare the performance of the newly proposed method with other existing methods as in James *et al*. (2013) and Ken *et al.* (2011).

Tested problems:

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$$

The exact solution is given by:

$$y(x) = 1 + \frac{1}{2} In\left(\frac{2 + x}{2 - x}\right)$$

Tested problem: Trigonometry problem (Nonstiff):

$$y'' = -\omega y + (w^2 - 1)\sin x, \quad y(0) = 1,$$
$$y' = 1 + \omega, \quad \omega = 10, \ x \in [0,2]$$

The exact solution is given by:

$$y(x) = \cos(\omega x) + \sin(\omega x) + \sin x$$

The first tested problem 5.1 to be discussed was extracted from James *et al*. (2013). Moreover, four steps continuous method for the solution of $y'' = f(x, y, y')$ was developed and implemented using fixed step size. Thus, the newly proposed method is formulated to solve nonstiff using variable step size technique.

Secondly, tested Problem 5.2 was extracted from Ken *et al*. (2011). However, the block methods for special second order ODEs was designed and executed using fixed step size technique. Moreover, the implementation of the explicit 2-point 1-block method and implementation of the explicit 3-point 1-block method was carried out using linear difference operator as well as their comparison. The newly proposed method belongs to the family of Adams otherwise called Milne's estimate and was created to solve nonstiff. ODEs.

Table 1: Comparing the numerical results of James *et al.* (2013) and newly proposed method (BPC) for solving problem 5.1 (James *et al*., 2013) newly proposed method (BPC)

X	Maximum errors	Tolerance levels	Maximum errors
0.1	9.992 (-15)	10^{-2}	2.68601 (-3)
0.2	8.149 (-14)		
0.3	4.700 (-13)	10^{-4}	2.78372 (-4)
0.4	1.637 (-12)		
0.5	4.664 (-12)	10^{-6}	1.16321 (-6)
0.6	1.116 (-11)		
0.7	2.501 (-11)	10^{-8}	4.72375 (-10)
0.8	5.2157 (-11)		
0.9	1.076 (-11)	10^{-10}	2.95247 (-11)
1.0	2.170 (-10)		

Table 2: Comparing the numerical results of Ken *et al.* (2011) and newly proposed method (BPC) for solving problem 5.2

TOL	MTH	MAXE
10^{-2}	E2PIB	7.64563 (-2)
	E3PIB	2.38177 (-2)
	BPC	5.16037 (-4)
10^{-4}	E3PIB	7.65958 (-4)
	BPC	1.52203 (-5)
10^{-5}	E2PIB	2.48030 (-5)
	BPC	1.94809 (-6)
10^{-6}	E2PIB	7.66004 (-6)
	BPC	8.20591 (-8)
10^{-8}	E2PIB	7.66283 (-8)
	E3PIB	2.48056 (-8)
	BPC	3.25265 (-10)
10^{-10}	E3PIB	5.84226 (-10)
	BPC	2.02565 (-11)

CONCLUSION

The implementation of the type of block predictor-corrector mode ($P(CE)^m$) was carried out on general second order ODEs and executed on nonstiff problems. Tested Problems 5.1 and 5.2 are good examples of nonstiff ODEs. See (Lambert, 1973) for details. The newly proposed method is a type of block predictor-corrector mode ($P(CE)^m$) otherwise called Milne's estimates.

From Table 1, (James *et al*., 2013) was implemented using fixed step size which does not allow for step size changes, correcting to convergence, error control/minimization. Again, from Table 2, (Ken *et al*., 2011) was also executed using fixed step size which as usual make room for step size variation. Nevertheless, this cannot be compared with the result of the newly proposed method (BPC) which yields better accuracy in terms of the maximum error at all tested tolerance levels, since it was implemented using variable step size technique. In addition, this gives a better result at all tested tolerance levels.

Hence, the newly proposed method (BPC) is preferable applying variable step size technique introduced by Milne's. The region of absolute stability for the type of block predictor-corrector mode when $r \leq 1$ explains that the method was implemented on nonstiff ODEs.

The block predictor-corrector mode is written in computer language using Mathematica and implemented on windows operating system using Mathematica 9 Kernel. The computational results for tested problems 5.1-5.2 in Table 1 and 2 are computed using the block predictor-corrector mode as well as the method in James *et al*. (2013) and Ken *et al*. (2011).

NOTATIONS

TOL : Tolerance level
MTD : Method employed
MAXE : Magnitude of the maximum error of the computed solution
BPC : Block Predictor-Corrector Mode ($P(CE)^m$)

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