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Research Article

On the Nearest Weighted Point Theorem for Ranking Fuzzy Numbers

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Abstract: In this study the author discusses the problem of defuzzification by minimizing the weighted distance between two fuzzy quantities. Also, this study obtains the nearest point with respect to a fuzzy number and shows that this point is unique relative to the weighted distance. By utilizing this point, a method is presented for effectively ranking fuzzy numbers and their images to overcome the deficiencies of the previous techniques. Finally, several numerical examples following the procedure indicate the ranking results to be valid.

Keywords: Defuzzification, fuzzy number, ranking, weighted distance, weighted point

INTRODUCTION

Representing fuzzy numbers by proper intervals is an interesting and important problem. An interval representation of a fuzzy number may have many useful applications. By using such a representation, it is possible to apply some results, in fuzzy number approaches, derived in the field of interval number analysis. For example, it may be applied to a comparison of fuzzy numbers by using the order relations defined on the set of interval numbers. Various authors in (Grzegorzewski, 2002; Saneifard, 2009a) have studied the crisp approximation of fuzzy sets. They proposed a rough theoretic definition of that crisp approximation, called the nearest interval approximation of a fuzzy set. Moreover, quite different approach to crisp approximation of fuzzy sets was applied in Chakrabarty *et al*. (1998). They proposed a rough theoretic definition of that crisp approximation, called the nearest ordinary set of a fuzzy set. And they suggested a construction of such a set. They discussed rather discrete fuzzy sets. Their approximation of the given fuzzy set is not unique. Thus this study will not discuss this method.

Having reviewed the previous interval approximations, this study proposes here a method to find the weighted interval approximation of a fuzzy number that it is fulfills two conditions. In the first, this interval is a continuous interval approximation operator. In the second, the parametric distance between this interval and the approximated number is minimal.

PRELIMINARIES

The basic definitions of fuzzy number are given in Chu *et al*. (1994), Dubois and Prade (1987), Heilpern (1992) and Kauffman and Gupta, 1991) as follows:

Definition 1: A fuzzy number A in parametric form is a pair (L_A, R_A) of functions $L_A(\alpha)$ and $R_A(\alpha)$ that $0 \leq \alpha \leq 1$, which satisfy the following requirements:

- $L_A(\alpha)$ is a bounded monotonic increasing left continuous function
- $R_A(\alpha)$ is a bounded monotonic increasing left continuous function
- $L_A(\alpha)R_A(\alpha)$, $0\leq \alpha \leq 1$

Definition 2: The trapezoidal fuzzy number A= (x_0, y_0, σ, β), with two defuzzifier x_0 , y_0 and left fuzziness σ> 0 and right fuzziness β>0 is a fuzzy set where the membership function is as:

$$A(x)=\begin{cases}\frac{1}{\sigma}(x-x_0+\sigma) & when \quad x_0-\sigma\leq x\leq x_0\\ 1 & when \quad x_0\leq x\leq y_0\\ \frac{1}{\beta}(y_0-x+\beta) & when \quad y_0\leq x\leq y_0+\beta\\ 0 & otherwise\end{cases}$$

If $x_0 = y_0$,then A= (x_0, σ, β) is called trapezoidal fuzzy number. The parametric from of triangular fuzzy number is $L_A(\alpha) = x_0 - \sigma + \sigma r$, $R_A(\alpha) = y_0 + \beta - \beta r$.

Definition 3: For arbitrary fuzzy number $A \in F$ (F denotes the space of fuzzy numbers) and $0\leq \sigma \leq 1$, function f: F × [σ, 1] → F such that f (A, σ) = ($L_{A\sigma}$, $R_{A\sigma}$) is called delta – vicinity of the fuzzy number A. Then there is:

$$L_{A\sigma}(\alpha) = \begin{cases}L_A(\alpha), & when \ \alpha\in[\delta,1],\\ L_A(\delta), & when \ \alpha\in[0,\delta]\end{cases}$$

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and

$$R_{A\delta}(\alpha)=\begin{cases} R_A(\alpha) \ when \ \alpha \in [\delta,1], \\ R_A(\delta) \ when \ \alpha \in [0,\delta], \end{cases} \tag{1}$$

If A = (x_0, y_0, σ, β) is a trapezoidal fuzzy number, the parametric form of it is A_δ = ($L_{A\sigma}$, $R_{A\sigma}$) that is as follows:

$$L_{A\delta}(\alpha)=\begin{cases} x_0-\delta+\sigma\alpha, \ when \ \ \alpha \in (\delta,1] \\ x_0-\delta+\sigma\delta, \ when \ \ \alpha \in [0,\delta] \end{cases}$$

and

$$R_{A\delta}=\begin{cases} y_0+\sigma-\beta\alpha, \ when \ \ \alpha \in (\delta,1] \\ y_0+\beta+\sigma\delta, \ when \ \ \alpha \in [0,\delta] \end{cases} \tag{2}$$

Definition 4: The following values constitute the weighted averaged representative and weighted width, respectively, of the fuzzy number A**:**

I (A) = $\int_0^1 (cL_A(\alpha)+(1-c)R_A(\alpha))p(\alpha)da$

D (A) = $\int_0^1 (R_A(\alpha)-L_A(\alpha))p(\alpha)da$

Here $0 \le c \le 1$.

Definition 5: Saneifard (2010): For two arbitrary fuzzy numbers A = (L_A, R_A) and B = (L_B, R_B) the quantity:

$$d_P(A, B) = \left(\int_0^1 f(\alpha)[L_A(\alpha)-L_B(\alpha)]^2 da+\int_0^1 f(\alpha)[R_A(\alpha)-R_B(\alpha)]^2 da\right)^{\frac{1}{2}} \tag{3}$$

Was f: [0, 1] → [0, 1] is a bi – symmetrical (regular) weighted, is called the bi – symmetrical (regular) weighted distance between A and B based on f.

Definition 6: Grzegorzewski (2002): An operator I: F→ (set of c losed intervals in R) is called an interval approximation operator if for any A ∈ F:

(a') I (A) ⊆ supp (A)
(b') Core (A) ⊆ I (A)
(c')
$\forall(\varepsilon>0)\exists(\sigma>0) \ \ s.t \ \ d(A,B)<\sigma \Rightarrow d(I(A),I(B))<\varepsilon$

where, d : F→ [0, + ∞] denotes a metric defined in the family of all fuzzy numbers.

Definition 7: Grzegorzewski (2002): An interval approximation operator satisfying in condition (c') for any A, B ∈ F is called the continuous interval approximation operator.

NEAREST WEIGHTED INTERVAL OF A FUZZY NUMBER

Various authors in Grzegorzewski (2002) and Saneifard (2009b) have studied the crisp approximation of fuzzy sets. They proposed a rough theoretic definition of that crisp approximation, called the nearest ordinary set and nearest interval approximation of a fuzzy set. In this section, the researchers will propose another approximation called the weighted interval – value approximation. Let A= (L_A, R_A) be an arbitrary fuzzy number. This study will try to find a closed interval C_{dp} (A) = [L_C, R_C], which is the weighted interval to A with respect to metric d_p. So, this study has to minimize:

$$D_p(A,C_{dp}(A))=\left(\int_0^1 f(\alpha)\left[(L_A(\alpha)-L_C)^2+(R_A(\alpha)-R_C)^2\right]da\right)^{\frac{1}{2}} \tag{4}$$

with respect to I_L and I_R . In order to minimize d_p it suffices to minimize:

$$\overline{D}_p(L_C,R_C)=d_p^2(L_C,R_C)$$

It is clear that, the parameters L_C and R_C which minimize Eq. (4) must satisfy in

$$\nabla\overline{D}_p(L_C,R_C)=\left(\frac{\partial\overline{D}_P}{\partial L_C},\frac{\partial\overline{D}_P}{\partial R_C}\right)=0$$

Therefore; this study has the following equations:

$$\begin{cases} \dfrac{\partial\overline{D}_P(L_C,R_C)}{\partial L_C}=-2\int_0^1 f(\alpha)[R_A(\alpha)-L_C)]d\alpha=0 \\ \dfrac{\partial\overline{D}_P(L_C,R_C)}{\partial R_C}=-2\int_0^1 f(\alpha)[L_A(\alpha)-R_C)]d\alpha=0 \end{cases} \tag{5}$$

The parameters I_L associated with the left bound and I_R associated with the right bound of the nearest weighted interval can be found by using Eq. (5) as follows:

$$\begin{cases} L_C=\int_0^1 f(\alpha)L_A(\alpha)d\alpha \\ R_C=\int_0^1 f(\alpha)R_A(\alpha)d\alpha \end{cases} \tag{6}$$

Remark 1: Since det

$$\begin{bmatrix} \dfrac{\partial\overline{D}_P^2(L_C,R_C)}{\partial L_C^2} & \dfrac{\partial\overline{D}_P^2(L_C,R_C)}{\partial R_C\partial L_C} \\ \dfrac{\partial\overline{D}_P^2(L_C,R_C)}{\partial L_C\partial R_C} & \dfrac{\partial\overline{D}_P^2(L_C,R_C)}{\partial R_C^2} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1>0$$

therefore L_C and R_C given by (6), minimize (d_p (A, C_{dp}(A)) Therefore, the interval:

$$C_{d_p}(a)=\left[\int_0^1 f(\alpha)L_A(\alpha)d\alpha, \int_0^1 f(\alpha)R_A(\alpha)d\alpha\right] \tag{7}$$

is the nearest weighted interval approximation of fuzzy number A with respect d_p. Now, suppose that this study wants to approximate a fuzzy number by a crisp interval. Thus the researchers have to use an operator C_{dp} which transforms fuzzy numbers into family of closed intervals on the real line.

Lemma 1: Saneifard (2009a):

Theorem 1: Saneifard (2009a): The operator C_{d_o} is an interval approximation operator, i.e., C_{d_o} is a continuous interval approximation operator.

$$(2\int_0^1 f(\alpha)g(\alpha)d\alpha)^2 \le 2\int_0^1 f(\alpha)g^2(\alpha)d\alpha$$

NEAREST WEIGHTED POINT OF A FUZZY NUMBER

Let A = (L_A, R_A) be an arbitrary fuzzy number and C_{dp} (A) = [L_C, R_C], be the nearest weighted interval of it. If $L_C = R_C$, then C_{dp} (A) = [L_C, R_C] = N_P (A), is the nearest weighted point approximation to fuzzy number A and its unique. The value of N_P (A) is as follows:

$$N_P(A) = \int_0^1 f(\alpha)(L_A(\alpha) + R_A(\alpha))d\alpha$$

The above equation introduces in the following Theorem.

Theorem 1: Let A = (L_A, R_A) be a fuzzy number and f(α) be a bi symmetrical weighted function. Then N_p (A) is nearest weighted point to fuzzy number A.

Proof: For the proof of Theorem it suffices that we replace $L_C = R_C = N_P$ (A) in (4) and then minimize function $\overline{D}_P(A, N_P(A)) = d_P^2(A, N_P(A))$ with respect N_P (A). Thus this study has to minimize

$$\overline{D}_P(A, N_P(A)) = (\int_0^1 f(\alpha)L_A(\alpha) - N_P(A))^2 + (R_A(\alpha) - N_P(A))^2)d\alpha \quad (8)$$

with respect to N_P(A). It is clear that, the parameter N_P(A) which minimizes Eq. (8) must satisfy in.

$$\nabla\overline{D}_P(A, N_P(A)) = (\frac{\partial\overline{D}_P}{\partial N_P(A)}) = 0$$

Therefore, this study has:

$$\frac{\partial\overline{D}_P(A, N_P(A))}{\partial N_P(A)} = -2\int_0^1 f(\alpha) \quad (9)$$
$$[L_A(\alpha) - N_P(A))N_P + R_A(\alpha) - N_P(A)]d\alpha = 0$$

The solution is:

$$N_P(A) = \int_0^1 f(\alpha)(cL_A(\alpha) + (1+c)R_A(\alpha))d\alpha \quad (10)$$

Since $\frac{\partial \overline{D}_P}{\partial N_P^2}$ = 2>0, therefore N_P (A) actually minimize $\overline{D}_P$ (A, N_P (A)) and simultaneously minimize d_p(A, N_P(A)).

Theorem 2: The nearest weighted point approximation to a given fuzzy number A is unique.

Proof: To prove the uniqueness of the operator N_P (A), we show that for any C ∈ R $\overline{D}_P((A, N_P(A)) \le \overline{D}_P(A, C)$ Holds. We can write

$$\overline{D}_P(A, C) =$$
$$\int_0^1 ((I(A) - C)^2) + (D(A) - C)^2)f(\alpha).d\alpha =$$
$$\int_0^1 ((I(A) + N_P(A) - N_P(A) - C)^2$$
$$(D(A) + N_P(A) - C)^2 f(\alpha).da =$$
$$\int_0^1 ((I(A) - N_P(A))^2 + (D(A) - N_P(A))^2.d\alpha +$$
$$\int_0^1 ((N_P(A) - (N_P(A) - C)^2)f(\alpha).d\alpha + 2(N_P(A) - C)$$
$$\int_0^1 ((I(A) - N_P(A)) + (D(A) - N_P(A)))f(\alpha)d\alpha \; .$$

The last sentence of the above diction is zero, hence $\overline{D}_P(A, C) = \overline{D}_P(A, N_P(A)) + 2(N_P(A) - C2$ Consequently *DP A, C= DPA, NPA+2NPA−C2≥0* then we have $\overline{D}_P(A, C) \ge \overline{D}_P(A, N_P(A))$ which completes the proof of theorem.

Remark 1: Let A and B be two fuzzy numbers and λ and μ be positive numbers. Then we have $N_P(\lambda A \pm$ *μB=λNP(A)±μNB(B)*

Proof: Let us suppose for all 0 ≤ α ≤ 1, A = (L_A(α), R_A(α)) and B = (L_C, (α), R_B (α)) and f(α) is a bi-symetrical weighted function. Then,

$$\lambda A \pm \mu B = \lambda(L_A(\alpha), R_A(\alpha)) \pm \mu(L_B(\alpha), R_B(\alpha))$$
$$= (\lambda L_A(\alpha) \pm \mu L_B(\alpha), \lambda R_A(\alpha) \pm \mu R_B(\alpha)$$

Then

$$N_P(\lambda A \pm \mu B) =$$
$$\int_0^1 f(\alpha)[\lambda L_A(\alpha) \pm \mu L_B(\alpha) + \lambda R_A(\alpha) \pm \mu R_B(\alpha)]d\alpha$$

$$= \lambda\int_0^1 f(\alpha)[\lambda L_A(\alpha) + R_A(\alpha)]d\alpha \pm \mu$$
$$\int_0^1 f(\alpha)[L_B(\alpha) + \mu R_B(\alpha)]d\alpha = \lambda N_P(A) \pm \mu N_P(B)$$

ORDERING OF FUZZY NUMBERS BY THE NEAREST WEIGHTED POINT

In this section, the researchers will propose the ranking of fuzzy numbers associated with the nearest weighted point approximation. Ever, the nearest weighted point can be used as a crisp approximation of a fuzzy number, therefore the resulting approximation is used to rank the fuzzy numbers. Thus, N_P (.)s used to rank fuzzy numbers.

Definition 1: Let A, B∈F are two fuzzy numbers and N_P (A) and N_P (B) are the nearest weighted point of them. Define the ranking of A and B by N_P on F, i.e.

- $N_P(A) < N_P(B)$, R_B if and only if $A \prec B$
- $N_P(A) > N_P(B)$, if and only if $A \prec B$
- $N_P(A) = N_P(B)$,

If $N_P(A_\delta) < N_P(B_\delta)$ then $A \prec B$, else if $N_P(A_\delta) > N_P(B_\delta)$ then, $A \prec B$, else A~B

Via Theorem 1, the nearest weighted point approximation of the δ - vicinity A is as follows:

$$N_P(A_\delta) = \int_0^1 f(\alpha)(L_{A\delta}(\alpha) + R_{A\delta}(\alpha))d\alpha$$

Then, this study formulates the order $\succsim$ and $\precsim$ as

A $\precsim$ B if and only if A B or A~B, A < B if and only if A $\prec$ B or A~ B. This study considers the following reasonable axioms that Wang and Kerre (2001) proposed for fuzzy quantities ranking.

Let *I* be an ordering method, *S* the set of fuzzy quantities for which the method *I* can be applied and F a finite subset of *S*. If *I* is applied to the fuzzy quantities then A,B in F satisfy that A has a higher ranking than B, that is A ≻ B by *I* on F.

A~B by *I* on F and A ≥ B by I on F are similarly interpreted. The following proposition shows the reasonable properties of the ordering approach *I*. Let S be the set of fuzzy quantities for which the nearest point method can be applied and F and F' are two arbitrary finite subsets of *S*. Then we have:

A-1: For an arbitrary finite subset A of S and $A \in A$; $A \succsim A$.

A-2: For an arbitrary finite subset A of S and $(A,B) \in A^2; A \succsim B$ and $B \succsim A$ results in A~B.

A-3: For an arbitrary finite subset A of S and $(A,B,C) \in A^3; A \succsim B$ and $B \succsim C$ produces A≻C.

A-4: For an arbitrary finite subset A of S and $(A,B) \in A^2$; inf sup $p(A) \rangle$ sup sup $p(B)$; give A ≻ B.

A′-4: For an arbitrary finite subset A of S and concludes as A≻ B.

A-5: Let S and S' be two arbitrary finite sets of fuzzy quantities in which $C_{dp}(.)$ can be applied, where both A and B belong to $S \cap S'$. Thus, the ranking order A≻ B by $C_{dp}(.)$ on S' iff $A \succsim B$ by $C_{dp}(.)$ on S is achieved.

A-6: Let A, B, A + C and B + C be elements of S. If, A ≻ B then $A + C \succsim B$ by $C_{d_p}(.)$ on {A + C, B + C}.

A′-6: Let A, B, A+C and B+C be elements of S. If A ≻B by $C_{d_p}(.)$ on A and B, then $A + C \succ B + C$ by $C_{d_p}(.)$ on {A+C, B+C}.

Remark 1: Ranking order N_P has the axioms $A_1, A_2, \ldots, A_6$.

Proof: The proof is similar to Saneifard (2010).

Remark 2: If $A \prec B$, then $-A \prec -B$.

Hence, this study can infer ranking order of the images of the fuzzy numbers.

EXAMPLES

In this section, we want compare proposed method with others. Saneifard (2009b):

Example 1: Consider the data used in Saneifard (2009b), i.e., the three fuzzy numbers, A = (5, 6, 7), B = (5.9, 6, 7) and C = (6, 6, 7), as shown in Fig. 1.

According to Eq. (10) the ranking index values are obtained, i.e., N_P (A) = 6, N_P (B) 6.15 and N_P (C) = 6.16.Thus, the ranking order of fuzzy numbers is A≺B ≺C However, by Chu and Tsao's Approach (Chu and Tsao, 2002), the ranking order is B≻C≻A.

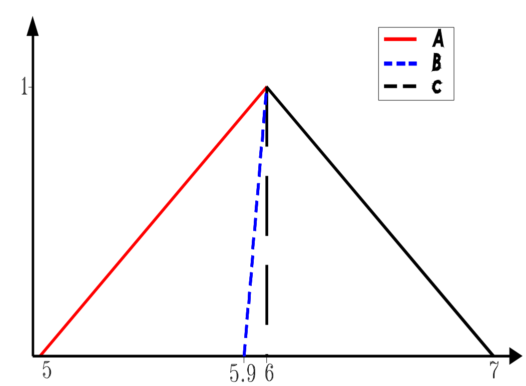

Fig. 1: Fuzzy numbers A, B and C

Table 1: Comparative results of example 1

Fuzzy number	New approach	Sign distance with p =1	Sign distance with p = 2	Chu-Tsao	Cheng distance	Cv Index
A	6	6.12	8.52	3.000	6.021	0.028
B	6.15	12.45	8.82	3.126	6.349	0.009
C	6.16	12.50	8.85	3.085	6.351	0.008
Results	A<B<C	A<B<C	A<B<C	A<C<B	A<B<C	A<C<B

Meanwhile, using the proposed CV index, the ranking order is A>B>C. From Fig. 1, it is obvious that the ranking results obtained by the existing approaches (Cheng, 1999; Chu and Tsao, 2002) are unreasonable and inconsistent. On the other hand, in Abbasbandy and Asady (2006), the ranking result is C>B>A hich the same as the one is obtained by the proposed method. However the authors' approach proves to be simpler in the computation procedure. Based on the analysis results from Abbasbandy and Asady (2006), the ranking results of this effort and other approaches are listed in Table 1.

Example 2: Consider the data used in Saneifard (2009b), i.e., the four fuzzy numbers, A = (-1, 0, 1), B = (2.2, 2.75, 3.5), C = (2, 3, 3.5) and D = (4, 5, 6). According to Eq. (10) the ranking index values are obtained, i.e., N_P (A) = 1, N_P (B) = 1.63, N_P (C)= 2.4 and N_P (D) = 5. The ranking order of fuzzy numbers is $A \prec B \prec C \prec D$. But by the Z.-X. Wang method the ranking order is A< C<B<D. Which is unreasonable and this is another shortcoming of the Z.-X. Wang method. In the real word, we can illustrate many examples like the above example again.

CONCLUSION

In this study, the authors proposed a defuzzification using minimize of the weighted distance between two fuzzy numbers and by using this defuzzification we proposed a method for ranking of fuzzy numbers. Roughly, there not much difference in our method and theirs. The method can effectively rank various fuzzy numbers and their images.

REFERENCES

Abbasbandy, S. and B. Asady, 2006. Ranking of fuzzy numbers by sign distance. Inform. Sci., 176: 2405-2416.

Chakrabarty, K., R. Biswas and S. Nanda, 1998. Nearest ordinary set of a fuzzy set: A rough theoretic construction. Bull. Polish Acad. Sci., 46: 105-114.

Cheng, C.H., 1999. Ranking alternatives with fuzzy weights using maximizing set and minimizing set. Fuzzy Set. Syst., 105: 365-375.

Chu, H., K. Lee and H. Wang, 1994. Ranking fuzzy values with satisfication function. Fuzzy Set. Syst., 64: 295-311.

Chu, T. and C. Tsao, 2002. Ranking fuzzy numbers with an area between the centroid point and original point. Comput. Math. Appl., 43: 111-117.

Dubois, D. and H. Prade, 1987. The mean value of a fuzzy number. Fuzzy Set. Syst., 24: 279-300.

Grzegorzewski, P., 2002. Nearest interval approximation of a fuzzy number. Fuzzy Set. Syst., 130: 321-330.

Heilpern, S., 1992. The expected value of a fuzzy number. Fuzzy Set. Syst., 47: 81-86.

Kauffman, A. and M.M. Gupta, 1991. Introduction to Fuzzy Arithmetic: Theory and Application. Van Nostrand Reinhold, New York.

Saneifard, R., 2009a. Ranking L-R fuzzy numbers with weighted averaging based on levels. Int. J. Ind. Math., 2: 163-173.

Saneifard, R., 2009b. A method for defuzzification by weighted distance. Int. J. Ind. Math., 3: 209-217.

Saneifard, R., 2010. Defuzzification through a bi-symmetrical weighted function. Aust. J. Basic Appl. Sci., 4: 4856-4865.

Wang, X. and E.E. Kerre, 2001. Reasonable properties for the ordering of fuzzy quantities (I). Fuzzy Set. Syst., 118: 375-385.