

## Research Article

### Lebesgue Constant Minimizing Shape Preserving Barycentric Rational Interpolation Optimization algorithm

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**Abstract:** The barycentric rational interpolation possesses various advantages in comparison with other interpolation, such as small calculation quantity, no poles and no unattainable points. It is definite when weights are given, so how to choose optimal weights becomes the key issue. A new optimization algorithm to compute the optimal weights was found by minimizing the Lebesgue constant. The biggest advantage of this algorithm is that the linearity of interpolation process with respect to the interpolated function is preserved. In this paper, we will study the shape control in barycentric rational interpolation under this new optimization algorithm, then numerical examples are given to shown the effectiveness of this algorithm.

**Keywords:** Barycentric rational interpolation, lebesgue constant, optimization algorithm, shape preserving, weight

#### INTRODUCTION

Interpolation is one of the primary algorithm of mathematics, it always the first choice to solve analytical problems. Most books devote a chapter to it. It is well known that polynomial interpolation have small calculation quantity, simple structure, interpolation function exist and only, so it is easy to realize the calculation and theoretical analysis. But when the degree of the polynomial is higher, it will product Runge phenomenon (John and Fei, 2009). And the effect of approach is not very good. Above all, it is a bad choice for practical computations, so it usually is a theoretical tool for proving problems.

Rational function approximation is a typical non-linear approximation. It than polynomial interpolation not only flexible, effective, fast convergence, but also reflect some properties of the function itself. But it also has some drawbacks, such as big calculation quantity, difficult to avoid poles and unattainable points. In 1984, Schneider and Werner have been the first to determine barycentric representations of rational interpolation (Claus and Wilhelm, 1986). The barycentric form is the most stable formula for a rational interpolant for use on a finite interval. Barycentric rational interpolation has good numerical stability, small calculation quantity, no poles, no unattainable points and high approximation orders (Michael and Kai, 2007). Recent years, barycentric rational interpolation is one of the hottest research points among all the interpolation (Zhang *et al.*, 2012). The problem of shape preservation has been discussed by a number of authors (Hoa *et al.*, 2010).

Berrut and Mittelmann (1997) found a new optimization algorithm that minimizes the Lebesgue constant for the given nodes to choose the optimal weighs. In this paper, the barycentric weights can be chosen on the basis of this optimization algorithm so as to deliver shape control and present numerical results to shown the effectiveness of the new algorithm.

#### BARYCENTRIC RATIONAL INTERPOLATION

In 1945, W. Taylor discovered the barycentric formula for evaluating the interpolating polynomial. It has good numerical stability and small amount of calculation (Jean-Paul *et al.*, 2005). In 1984, Schneider and Werner have been first to determine barycentric representations of interpolants. In this paper, it relaxed the requirements for degree of rational function and constructed the barycentric rational interpolation by introducing to different barycentric weights. The barycentric representation of rational interpolants has several advantages over other representation, in particular, it allows for an easier detection of unattainable points and of poles in the interval of interpolation (Berrut and Mittelmann, 1997).

Let  $r(x) \in R_{m,n}$ , where  $R_{m,n}$  is the set of all rational functions with the degrees of numerator at most  $m$  and the degrees of denominator at most  $n$ .

**Lemma 1:** Let  $\{(x_j, f_j)\}$ ,  $j = 0(1) N$ , be  $N+1$  pairs of real numbers with  $x_j \neq x_k$ ,  $j \neq k$  and let  $\{w_j\}$  be  $N+1$  real numbers.

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- If  $w_k \neq 0$ , the rational function:

$$r(x) = \frac{\sum_{j=0}^N \frac{w_j}{x - x_j} f_j}{\sum_{j=0}^n \frac{w_j}{x - x_j}} \quad (1)$$

Interpolation  $f_k$  at  $x_k$  :  $\lim_{x \rightarrow x_k} r(x) = f_k$

- Conversely, every rational interpolant  $r(x) \in R_{m,n}$  of the  $f_j$  may be written as in (1) for some  $w_j$ .

From the above, as long as the weights are not all zero, it will not appear unattainable points the necessary condition of no poles is also given by Schneider and Werner (Claus and Wilhelm, 1986). As shown in the following theorem.

**Theorem 1:** Let  $x_0 < x_1 < \dots < x_N$ ,  $w_i \neq 0$ ,  $i = 0 \dots N$ .

- If  $r$  has no poles in  $[x_0, x_N]$  then  $\text{sing}(w_i) \cdot \text{sing}(w_{i+1}) = -1$ ,  $i = 0, \dots, N-1$ .
- If  $\text{sing}(w_j) = -\text{sing}(w_{j+1})$  for some  $j$ ,  $0 \leq j \leq N-1$ , then  $r$  has an odd number of poles in  $(x_j, x_{j+1})$ .

### LEBESGUE CONSTANT MINIMIZING BARYCENTRIC RATIONAL INTERPOLATION

Let  $f$  be a complex-valued function on some interval  $[\alpha, \beta]$  of the real line, let  $x_0, x_1, \dots, x_n$  be  $n+1$  distinct points in  $[\alpha, \beta]$  and  $f_k = f(x_k)$ ,  $k = 0, \dots, n$ . Then

$$P_n(x) = \sum_{k=0}^n l_k(x) f_k \quad (2)$$

$$l_k(x) = \frac{\prod_{j=0, j \neq k}^n (x - x_j)}{\prod_{j=0, j \neq k}^n (x_k - x_j)} \quad (3)$$

- Is the Lagrangian representation of the unique polynomial of degree  $n$ , which interpolates  $f$  between the points  $x_k$  :  $P(x_k) = f_k$ .

**Theorem 2:** Let  $x_n, x_0, x_1, \dots, x_n$  be  $n+1$  distinct points in  $[\alpha, \beta]$ . The linear projection  $P_n$  which in  $C[\alpha, \beta]$  associates with any function  $f$  the polynomial  $P_n$   $f \in P_n$  interpolating  $f$  between the  $x_k$ 's has the norm:

$$\|P_n\| = \Lambda_n = \sup_{\alpha \leq x \leq \beta} \sum_{k=0}^n |l_k(x)| \quad (4)$$

$\lambda_n(x) = \sum_{k=0}^n |l_k(x)|$  is called the Lebesgue function of the

approximation operator and  $\Lambda_n$  its Lebesgue constant (Simon, 2006).

The Lagrange fundamental polynomials in (3) may be written as  $l_k(x) = w_k(l(x)/(x - x_k))$  with:

$$l(x) = (x - x_0)(x - x_1) \dots (x - x_n) \quad (5)$$

and

$$w_k = 1 / \prod_{i=0, i \neq k}^n (x_k - x_i) \quad (6)$$

so that:

$$P_n(x) = \frac{\sum_{k=0}^n (w_k / (x - x_k)) f_k}{\sum_{k=0}^n (w_k / (x - x_k))} \quad (7)$$

$$\Lambda_n = \sup_{\alpha \leq x \leq \beta} \sum_{k=0}^n |l_k(x)| = \sup_{\alpha \leq x \leq \beta} \left| \frac{\sum_{k=0}^n \frac{w_k}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}} \right| \quad (8)$$

In Jean-Paul *et al.* (2005) Berrut's optimization problem becomes a minimization problem with simple bounds:

$$\min \Lambda_n = \min_{\alpha \leq x \leq \beta} \sup_{\alpha \leq x \leq \beta} \left| \frac{\sum_{k=0}^n \frac{w_k}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}} \right| \quad (9)$$

Let (9) as the objective function without interpolated function, then used the optimization software to search for optimal weights. This optimization algorithm has great practical significance; however, its shape control is much less studied. In this paper we will focus on its shape control.

### SHAPE CONTROL IN BARYCENTRIC RATIONAL INTERPOLATION OPTIMIZATION ALGORITHM

The researchers in the field of the developing interpolation are always interested in shape preserving. Comonotonicity, co convexity and co positivity are well-understood in uniform barycentric rational interpolation (Hoa *et al.*, 2011).

It is well known that the different barycentric rational interpolations are obtained based on different given weights; we take the weights  $w_i (i = 0, \dots, n)$  as the decision variable and take  $\min \Lambda_n$  as the objective function. Meanwhile, in order to ensure the barycentric rational function has no unattained points and no poles, the weights should satisfy the constraint conditions:

$$\text{sing}(w_i) \cdot \text{sing}(w_{i+1}) = -1, i = 0, \dots, n \quad (10)$$

$$w_i \neq 0, i = 0, \dots, n \quad (11)$$

Furthermore, for ensuring the uniqueness of the optimization solution, we take the normative constraint

$$\sum_{i=0}^n |w_i| = 1 \quad (12)$$

On weights additionally.

From the above, the optimization algorithm model for computing the optimal interpolation weights as follows:

$$\min \Lambda_n = \min_{a \leq x \leq b} \sup_{a \leq x \leq b} \frac{\sum_{k=0}^n \frac{|w_k|}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}}$$

$$\text{sing}(w_i) \cdot \text{sing}(w_{i+1}) = -1, i = 0, \dots, n$$

$$w_i \neq 0, i = 0, \dots, n$$

$$\sum_{i=0}^n |w_i| = 1.$$

So we can get a set of optimal weights by LINGO software.

**Controlling monotonicity:** The formula for the first derivative of  $r(x)$ , with  $x$  not being a pole or interpolation point of  $r(x)$ , can be written as:

$$r'(x) = \begin{cases} \frac{\sum_{i=0}^n \frac{w_i}{x - x_i} r[x, x_i]}{\sum_{i=0}^n \frac{w_i}{x - x_i}}, & x \neq x_k, k = 0, \dots, n \\ \frac{-\left(\sum_{i=0, i \neq k}^n w_i r[x_k, x_i]\right)}{w_k}, & x = x_k \end{cases} \quad (13)$$

It is well known that if  $f'(x) > 0$  ( $f'(x) < 0$ ),  $x \in I$ ,  $f$  will be a increasing (decreasing) function. Add

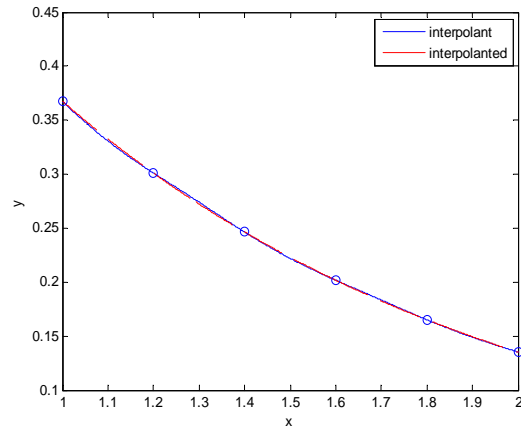


Fig. 1: The function of interpolated and interpolant

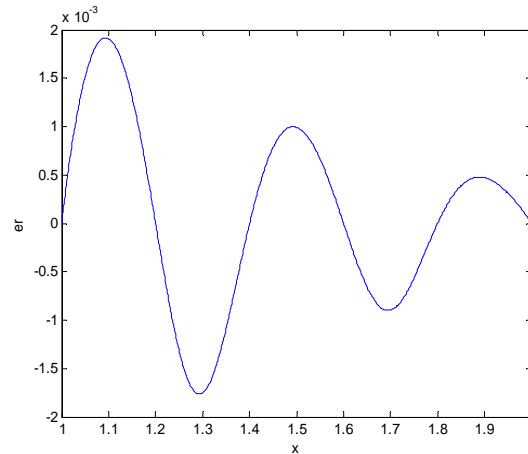


Fig. 2: Interpolation error

$r'(x) > 0$  ( $r'(x) < 0$ ) to the above optimization algorithm model, we can get a set of optimal weights.

**Example 1:** Given  $f(x) = e^{-x}$  as the interpolated function and choose  $x_0 = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6, x_4 = 1.8, x_5 = 2$  as interpolation points.

We can obtain the optimal weights by the software LINGO as follows:

$$w_0 = -0.1164657, w_1 = 0.1365075, w_2 = -0.1581999, w_3 = 0.1762474, w_4 = -0.1945403, w_5 = 0.2180392$$

In order to show the effectiveness of new algorithm, we give figures of function interpolated and interpolant (Fig. 1) and their error (Fig. 2) by using MATLAB as follows:

**Controlling convexity or concavity:** The second derivative  $r''(x)$  in a point  $x$  that is not a pole or an interpolation point, equals:

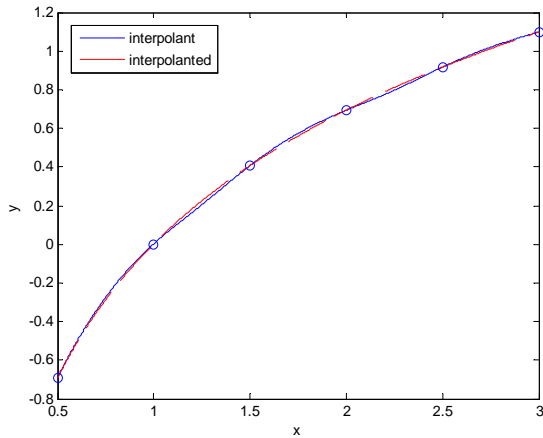


Fig. 3: The function of interpolated and interpolant

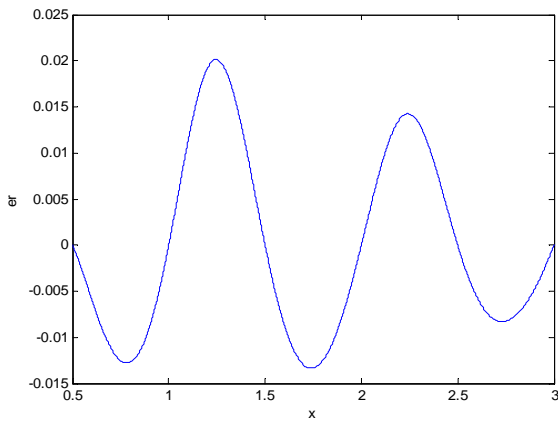


Fig. 4: Interpolation error

$$r''(x) = \begin{cases} \frac{2 \sum_{i=0}^n \frac{w_i}{x-x_i} r[x, x, x_i]}{\sum_{i=0}^n \frac{w_i}{x-x_i}}, & x \neq x_k, k=0, \dots, n \\ \frac{-2 \left( \sum_{i=0, i \neq k}^n w_i r[x, x, x_i] \right)}{w_k}, & x = x_k \end{cases} \quad (14)$$

with  $r[x, x, x_i] = \frac{r'(x) - r[x, x_i]}{x - x_i}$

It is easy to know that  $r''(x) > 0 (r''(x) < 0)$  can keep barycentric rational interpolation function convex (concave). From above, we can get the optimization algorithm model and use the software LINGO to obtain the optimal weights.

**Example 2:** Given  $f(x) = \ln(x)$  as the interpolated function and choose  $x_0 = 0.5, x_1 = 1, x_2 = 1.5, x_3 = 2, x_4 = 2.5, x_5 = 3$ , as interpolation points.

By the LINGO software can get the optimal weights as follows:

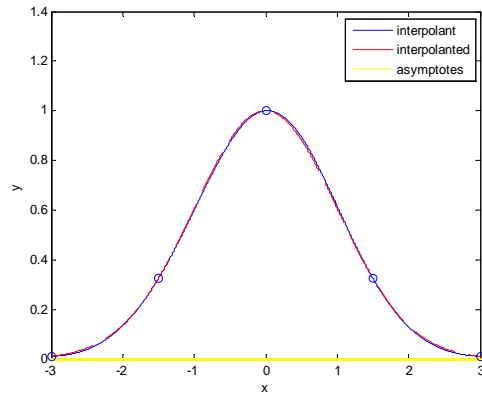


Fig. 5: The function of interpolated and interpolant

$$w_0 = -0.0888195, w_1 = 0.1110244, w_2 = -0.1387805$$

$$w_3 = 0.1734756, w_4 = -0.2168445, w_5 = 0.2710556$$

The interpolating function  $r(x)$  is shown in Fig. 3 and the error (Fig. 4) with the interpolated function  $f(x)$  can be got by the MATLAB software.

**Asymptotes:** From (2.1) we can see that the highest degree coefficients in numerator and denominator of  $r(x)$  are given by:

$$P_n = \sum_{i=0}^n w_i f_i, q_n = \sum_{i=0}^n w_i \quad (15)$$

When  $q_n \neq 0$ , a horizontal asymptote at  $y = \alpha$  ( $\alpha$  is a constant) can be imposed by choosing  $w_i$  such as:

$$\sum_{i=0}^n (f_i - \alpha) w_i = 0 \quad (16)$$

**Example 3:** Given  $f(x) = e^{-x^2/2}$  as the interpolated function and  $x_0 = -3, x_1 = -1.5, x_2 = 0, x_3 = 1.5, x_4 = 3$  as interpolation points.

By the software LONGO we can get the optimal barycentric weights as follows:

$$w_0 = -0.1809454, w_1 = 0.2122513, w_2 = -0.1768761$$

$$w_3 = 0.1734756, w_4 = -0.2210951, w_5 = 0.2088321$$

The function of interpolated and interpolant is show in Fig. 5 and the interpolation error (Fig. 6) can be obtained by the MATLAB software.

**Positivity preservation:** In many physical situations entities only meaningful when their values are positive, such as progress of an irreversible process, volume, density. So positivity is also an important shape property.

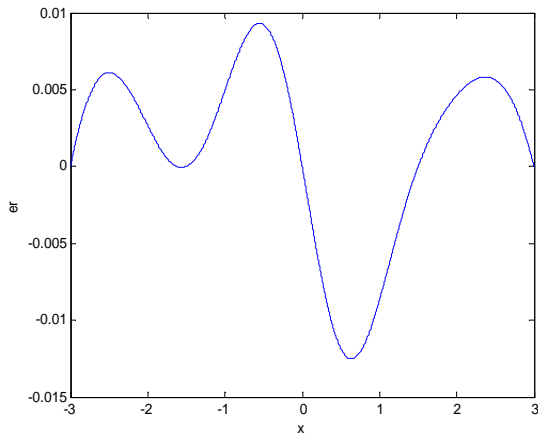


Fig. 6: Interpolation error

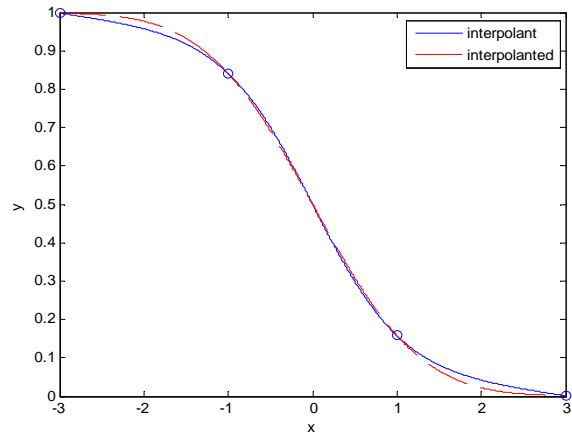


Fig. 9: The function of interpolated and interpolant

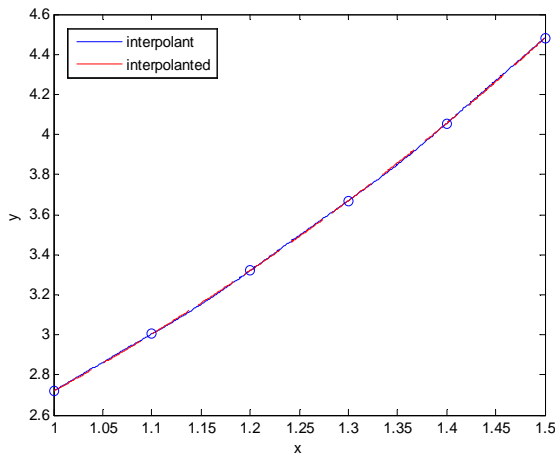


Fig. 7: The function of interpolated and interpolant

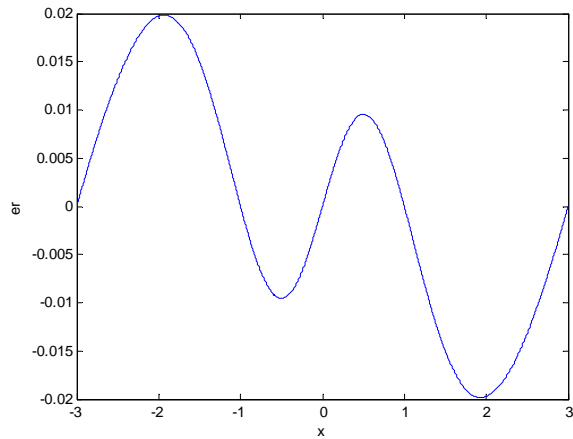


Fig. 10: Interpolation error

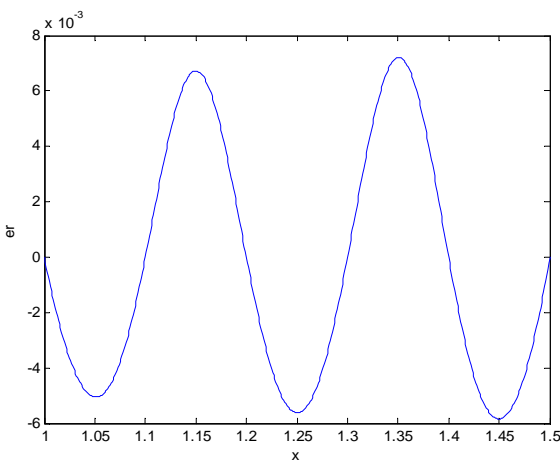


Fig. 8: Interpolation error

**Example 4:** Given  $f(x) = e^x$  as the interpolated function and  $x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5$  as interpolation points.

We can get the optimal barycentric weights by the software LINGO as follows.

$$w_1 = -0.1754505, w_2 = 0.1744595, w_3 = -0.1716674, w_4 = 0.1670000, w_5 = -0.1602543, w_6 = -0.1511685$$

The interpolating function  $r(x)$  is shown in Fig. 7 and the error (Fig. 8) with the interpolated function  $f(x)$  can be obtained by the MATLAB software.

**Controlling in  $[a, b]$ :**

**Example 5:** Given  $f(x) = \frac{1}{\sqrt{2 \cdot \pi}} \int_x^{+\infty} e^{-\frac{t^2}{2}} dt$  as the interpolated function and  $x_0 = -3, x_1 = -1, x_2 = 1, x_3 = 3$  as interpolation points.

We can get the optimal barycentric weights by the software LINGO as follows:

$$w_0 = -0.25, w_1 = 0.25, w_2 = 0.25, w_3 = 0.25$$

The interpolating function is shown in Fig. 9 and the error (Fig. 10) with the interpolated function can be obtained by the MATLAB software.

## CONCLUSION

In this study, the main research is sought for optimization algorithm model for shape preserving barycentric rational interpolation. Let minimizing Lebesgue constant as objective function (weights as the only decision variable) to seek the optimal weights. We can see how the barycentric weights can be chosen to influence the shape of the rational function. The biggest advantage of this optimization algorithm is that the interpolated function is preserved, so it can solve many practical problems.

In future study, shape control in multivariate barycentric rational interpolation base on this method will be studied

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