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Research Article

Neighborhoods of Certain Classes of Generalized Ruscheweyh Type Analytic Functions of Complex Order

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Abstract: In this study, we introduce and study the (n, δ) neighborhoods of subclasses $S_n(\gamma, \lambda, \beta, \eta)$ $R_n(\gamma, \lambda, \beta, \mu, \eta)$ $S^{\alpha}_n(\gamma, \lambda, \beta, \mu, \eta)$ and $R^{\alpha}_n(\gamma, \lambda, \beta, \mu, \eta)$ of the class A(n) of normalized analytic functions in $U = \{z \in \mathbb{C} : |z|<1\}$ with negative coefficients, which are defined by using of the generalized Ruscheweyh derivative operator.

Keywords: Analytic functions, convex functions, neighborhoods generalized ruscheweyh derivative, Star like functions, univalent functions

INTRODUCTION

Let $A(n)$ denote the class of functions f of the form:

$$f(z) = -\sum_{k=n+1}^{\infty} \alpha_k z^k \ (\alpha_k \geq 0, k \in \mathbb{N} = \{1, 2, ...\} =) \quad (1)$$

which are analytic in the open unit disk:

$$U = \{z \in \mathbb{C} : |z|<1\}$$

For any function f ∈ A(n) and δ ≥0, we define:

$$N_{n,\delta}(f) = \left\{g \in A(n) : g(z) = z - \sum_{k=k+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|\alpha_k - b_k| \leq \delta\right\} \quad (2)$$

which is the (n, δ) neighborhood of f.
For the identity function $e(z) = z$, we have:

$$N_{n,\delta}(f) = \left\{g \in A(n) : g(z) = z - \sum_{k=k+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leq \delta\right\} \quad (3)$$

The concept of neighborhoods was first introduced by Goodman (1983) and then generalized by Ruscheweyh (1981).

A function $f \in A$ is star like of complex order $\gamma(\gamma \in \mathbb{C}/\{0\})$that is, $f \in C_n(\gamma)$ if:

$$R\left\{1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} > 0 (z \in U, \gamma \in \mathbb{C}/\{0\}) \quad (4)$$

A function $f \in A(n)$ is said to be convex of complex order $\gamma(\gamma \in/\{0\})$that is, $f \in C_n(\gamma)$ if:

$$R\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0 (z \in U, \gamma \in \mathbb{C}/\{0\}) \quad (5)$$

The classes $S^*_n(\gamma)$ and $C_n(\gamma)$ stem essentially form the classes of starlike and convex functions of complex orders which were introduced earlier by Nasr and Aouf (1985). The Hadamard product of two power series:

$$f(z) = z + \sum_{k=2}^{\infty} \alpha_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is defined as $(f * g)(z) = z + \sum_{k=2}^{\infty} \alpha_k b_k z^k$.

The main objective of this study is to use a recent generalization of the Ruscheweyh derivative operator (Al-Shaqsi and Darus, 2007) denoted D^{λ}_{η}, define as follows:

$$D^{\lambda}_{\eta} f(z) = \frac{z}{(1-z)^{1+\lambda}} * D\eta f(z) \left(f \in A(n), z \in U\right)$$

where, * stands for the Hadamard product of two power series. Further, we have:

$$D_{\eta} f(z) = (1-\eta) f(z) + \eta z f'(z), \lambda > -1, \eta \geq 0, z \in U.$$

We can easily see that D^{λ}_{η} admits a representation of the form:

$$D^{\lambda}_{\eta} f(z) = z - \sum_{k=k+1}^{\infty} \left[1 + (k-1)\eta\right] \binom{\lambda + k - 1}{k - 1} \alpha_k z^k \left(\lambda > -1, f \in A(n), \eta \geq 0\right) \quad (6)$$

Make use of the following standard notation:

$$\binom{K}{k} = \frac{K(K-1)(K-2)...(K-k+1)}{k!} (K \in \mathbb{C}, k \in \mathbb{N})$$

A function $f \in A(n)$ is said to belong to the class $S_n(\gamma, \lambda, \beta, \eta)$ if:

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$$\left|\frac{1}{\gamma}\left(\frac{z\left(D_\eta^\lambda f(z)\right)'}{D_\eta^\lambda f}-1\right)\right|<\beta, \tag{7}$$

where, $\lambda \in \mathbb{C}/\{0\}, \lambda>-1, 0<\beta\le 1, \eta\ge 0$ and $z \in U$.

A function $f \in A(n)$ is said to belong to the class $R_n(\gamma, \lambda, \beta, \mu, \eta)$ if:

$$\left|\frac{1}{\gamma}\left((1-\mu)\frac{D_\eta^\lambda f(z)}{z}+\mu\left(D_\eta^\lambda f(z)\right)'-1\right)\right|<\beta, \tag{8}$$

where, $\lambda \in \mathbb{C}/\{0\}, 0<\beta\le 1, \eta\ge 0, \lambda>-1$ and $z\in U$.

Neighborhoods for classes: $S_n(\gamma, \lambda, \beta, \eta)$ and $R_n(\gamma, \lambda, \beta, \mu, \eta)$

Lemma 1: A *function* $f \in S^\alpha_n(\gamma, \lambda, \beta, \eta)$ if and only if:

$$\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}\left(\beta|\gamma|+k-1\right)\alpha_k \le \beta|\gamma|. \tag{9}$$

Proof: Let $f \in S_n(\gamma, \lambda, \beta, \eta)$ Then, by (6) we can write:

$$R\left\{\frac{z\left[\left(D_\eta^\lambda f(z)\right)'\right]}{D_\eta^\lambda f(z)}-1\right\}>-\beta|\gamma| \ (z\in U). \tag{10}$$

Equivalently:

$$R\left[-\frac{\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}(k-1)\alpha_k z^k}{z-\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}\alpha_k z^k}\right]>-\beta|\gamma| \ (z\in U). \tag{11}$$

We choose values of Z on the real axis and let $Z \to 1-$, through real values, the inequality (11) yields the desired condition (9). Conversely, by applying the hypothesis (2.1) and letting $|Z|=1$ we have:

$$\left|\frac{z\left(D_\eta^\lambda f(z)\right)'}{D_\eta^\lambda f(z)}-1\right|=\left|\frac{\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}(k-1)\alpha_k z^k}{z-\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}\alpha_k z^k}\right|'$$

$$\le\frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}\alpha_k\right)}{1-\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}\alpha_k}$$

$$\le\beta|\gamma|$$

Hence, by the maximum modulus Theorem, we have $f \in R_n(\gamma, \lambda, \beta, \eta)$ Similarly, we can prove the following Lemma.

Lemma 2: A *function* $f \in R_n(\gamma, \lambda, \beta, \mu, \eta)$ if and only if:

$$\sum_{k=n+1}^{\infty}\left[1+(k-1)\eta\right]\binom{\lambda+k-1}{k-1}\left[\mu(k-1)+1\right]\alpha_k \le \beta|\gamma|. \tag{12}$$

Theorem 1: *if*:

$$\delta=\frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}},\left(|\gamma|<1\right), \tag{13}$$

Then, $S_n(\gamma, \lambda, \beta, \eta) \cap N_{n,\delta}(e)$

Proof: Let $f \in S_n(\gamma, \lambda, \beta, \eta)$. By lemma 2, we have,

$$(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty}\alpha_k\le\beta|\gamma|$$

So,

$$\sum_{k=n+1}^{\infty}\alpha_k\le\frac{\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}} \tag{14}$$

Using (9) and (14), we have:

$$(1+n\eta)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty}k\alpha_k\le\beta|\gamma|+(1-\beta|\gamma|)(1+n\eta)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty}\alpha_k$$

$$\le\beta|\gamma|+(1-\beta|\gamma|)(1+n\eta)\binom{\lambda+n}{n}$$

$$\frac{\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}}$$

$$\le\frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)}\left(|\gamma|<1\right),$$

That is:

$$\sum_{k=n+1}^{\infty}k\alpha_k\le\frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}}=\delta.$$

Thus, by the definition given by (3), $f \in N_{n,\delta}(e)$.

Theorem 2: *if*:

$$\delta=\frac{(n+1)\beta|\gamma|}{(\mu n+1)(1+n\eta)\binom{\lambda+n}{n}} \tag{15}$$

Then, $R_n(\gamma, \lambda, \beta, \mu, \eta) \subset N_{n,\delta}(e)$.

Proof: Let $f \in R_n(\gamma, \lambda, \beta, \mu, \eta)$. Then, by lemma 2, we have:

$$(\mu n+1)(1+n\eta)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty}\alpha_k\le\beta|\gamma|,$$

Which yields the following coefficient inequality:

$$\sum_{k=n+1}^{\infty} \alpha_k \leq \frac{\beta|\gamma|}{(\mu n+1)(1+n\eta)\binom{\lambda+n}{n}} \tag{16}$$

Using (12) and (16), we also have:

$$\mu(1+n\eta)\binom{\lambda+n}{n}\sum_{k=n+1}^{\infty} k\alpha_k \leq \beta|\gamma| + (\mu-1)(1+n\eta)\binom{\eta+n}{n}\sum_{k=n+1}^{\infty}\alpha_k$$
$$\leq \beta|\gamma| + (\mu-1)(1+n\eta)\binom{\eta+n}{n}$$
$$\frac{\beta|\gamma|}{(\mu n+1)(1+n\eta)\binom{\lambda+n}{n}},$$

That is:

$$\sum_{k=n+1}^{\infty} k\alpha_k \leq \frac{(n+1)\beta|\gamma|}{(\mu n+1)(1+n\eta)\binom{\lambda+n}{n}} = \delta.$$

Thus, by the definition given by (3), f ∈$N_{n,\delta}$(e)

Neighborhoods for classes: $S^{\alpha}_{n}(\gamma, \lambda, \beta, \eta)$ and $R^{\alpha}_{n}(\gamma, \lambda, \beta, \mu, \eta)$

In this section, we define the neighborhood for each of the classes $S^{\alpha}_{n}(\gamma, \lambda, \beta, \eta)$ and $S^{\alpha}_{n}(\gamma, \lambda, \beta, \mu, \eta)$ A function f ∈A(n) is said to be in the class $S^{\alpha}_{n}(\gamma, \lambda, \beta, \eta)$ if there exists a function g ∈$S_n(\gamma, \lambda, \beta, \eta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha \quad (0 \leq \alpha \leq 1, z \in U). \tag{17}$$

Similarly, a function f ∈A(n) is said to be in the class $R^{\alpha}_{n}(\gamma, \lambda, \beta, \mu, \eta)$ if there exists a function g∈$R^{\alpha}_{n}(\gamma, \lambda, \beta, \mu, \eta)$ such that the inequality (17) holds true.

Theorem 3: If g ∈$S_n(\gamma, \lambda, \beta, \mu, \eta)$ and:

$$\alpha = 1 - \frac{(\beta|\gamma|+n)\delta(1+n\eta)\binom{\lambda+n}{n}}{(n+1)\left[(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n} - \beta|\gamma|\right]} \tag{18}$$

Then $N_{n,\delta}(g) \subset S^{\alpha}_{n}(\gamma, \lambda, \beta, \mu, \eta)$.

Proof: Let f ∈$N_{n,s}$(g) then, by (2), we have:

$$\sum_{k=n+1}^{\infty} k|\alpha_k - b_k| \leq \delta, \tag{19}$$

Which yields that coefficient inequality:

$$\sum_{k=n+1}^{\infty} |\alpha_k - b_k| \leq \frac{\delta}{n+1} (n \in \mathbb{N}) \tag{20}$$

Since g ∈$R^{\alpha}_{n}(\gamma, \lambda, \beta, \mu, \eta)$ by (14), we have:

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}} \tag{21}$$

So that:

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=n+1}^{\infty}|\alpha_k - b_k|}{1-\sum_{k=n+1}^{\infty} b_k}$$
$$\leq \frac{\delta}{(n+1)} \frac{(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n}}{\left[(\beta|\gamma|+n)(1+n\eta)\binom{\lambda+n}{n} - \beta|\gamma|\right]}$$
$$= 1-\alpha.$$

Thus, by definition, f ∈$S^{\alpha}_{n}(\gamma, \lambda, \beta, \eta)$ for α given by (18) Thus, the proof is complete.

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