

Research Article

Active Vibration Control of Satellite Flexible Structures during Attitude Maneuvers

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Abstract: The purpose of this study is controlling active vibration of satellite flexible structures during attitude maneuvers. A smart structure is a structure which is able to sense and control active reaction to any external factors and stimulation. As it comes from the definition of smart structures, development of this knowledge depends on the materials science development, theories and strategies for control. In materials science, smart materials are developed in such a way that they are able to sense and react in a controllable mode effectively. The smart materials are combined with conventional structures and they are used as sensors and actuators. Furthermore, smart structures subject include design, implementation methods and control systems over structures so that the control system receives signals from sensors after processing. Then, it sends signals to the actuator in order to achieve a desirable response to incoming stimuli. An issue taken into account in this monograph is active vibration control of satellite flexible structures in attitude maneuvers, which was done by the use of smart materials in satellite structure. For this purpose, a mathematical model of a satellite flexible panel was derived first. Then, a computer code was generated. Finally, by comparing the results of non-smart control and smart control, we came to this conclusion that the use of smart materials led to a decrease in the amplitude of the vibration and the reduction of the time required for damping vibrations.

Keywords: Active vibration control, smart materials, smart structures

INTRODUCTION

A smart structure is a structure which is able to sense and control active reaction to any external factors and stimulation. In recent years, according to Gandhi and Thompson (1992), much research has been done in the field of smart structures application. One of the most widely used smart structures is in space and aerial structures. One of the most serious challenges in satellites designing is the reduction of structure vibrations during attitude maneuver. In this research, the flexible structure of the satellite and its performance are simulated during the attitude maneuver. Then, by designing and modeling the structures in smart mode, the results of non-smart control and smart control were compared and thus, the use of smart materials in the satellite structure was shown.

Smart structures and materials: Smart materials and smart structures often are known as smart structures which have created the rapid growth in various branches of technology in areas of materials, structures, sensor-actuator systems, signal and information processing, electronic and control in the modern world (Gandhi and Thompson, 1992). These structures include smart structures interact with sensors which patched to smart structures surface which are collecting

surface information and data. The information is then processed by a special controller to send special control commands to actuators to create and/or perform an appropriate action. A smart structure is a system of continuous parameters that apply sensors and actuators in distinct finite elements position on the structure surface and one or more microprocessor is used to analyze obtained responses from the sensors and uses various logic control to control actuators in order to achieve optimal response and to reach balance of structures. In addition, it can cause significant local strains. This causes the system's ability to respond to stimuli in the environment whether it is internal or external stimuli such as changes in load and temperature change. Smart actuators are used to modify the system's inherent properties (e.g. stiffness or damping) so that the system response is might be strained or deformed in a controllable way.

One way used to describe a smart structure system is a combination of sensing, processing, practice, feedback, self-evaluation and subsystems automatic correction as shown in Fig. 1. Sensors and actuators are patched into a structure, which is called smart structure.

Active vibration control: Active vibration control of structures is one of the most important issues. One way to control vibrations is making smart, adaptive and a

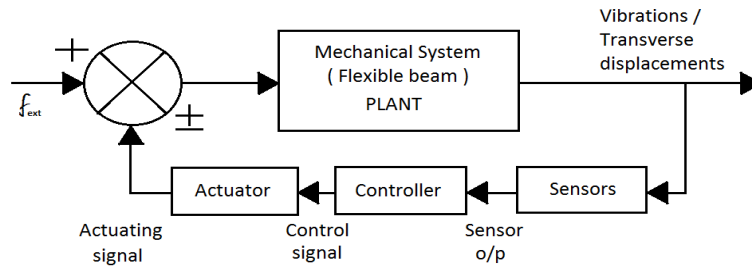


Fig. 1: Block diagram of a smart structure system

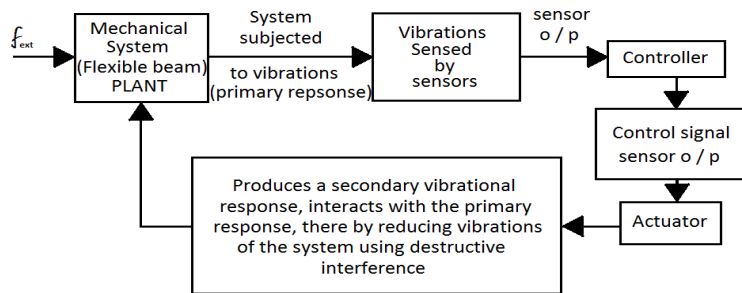


Fig. 2: How to reduce vibration by AVC



Fig. 3: Nano-satellite Structure

self-control structure using of smart materials. The most original concept of Active Vibration Control is vibration reduction system by auto-correction of system response as it is shown in Fig. 2. In many cases, structure vibration reduction is very important because the vibrations will affect the performance and system stability. It is also necessary for increasing the structure's resistance against forces and dynamic disturbances, which causes vibrations reduction and increases the ability to have lighter and efficient structure.

According to Culshaw (1992), such structures are called smart structures because of their high efficiency in many structural applications have been increasing.

Bandyopadhyay *et al.* (2007) states that these smart structures, which are made of smart materials, called sensors and actuators based on the piezoelectric, fluid Magneto-Rheological, Piezo Ceramic, Fluid Electro-Rheological, memory alloys, PVDF, fiber optics, etc. and they have structural advantages (Hermen, 1994).

Structure: The first function of spatial structure is providing a mechanical barrier for all the internal parts of the spacecraft. The structure must provide initial needs of subdivisions including co-axial sensors, actuators, antennas, needs for common borders and integration and testing. A spacecraft during immersion tolerates the mechanical loads. Therefore, the spatial structure should be designed to tolerate all loads during launch and also after launching to all subdivisions. Figure 3 shows a Nano-satellite structure. By considering the above mentioned issues, in this research, the behavior of the panel is studied during attitude maneuver in both smart and non-smart modes.

Attitude control system: Overall analysis of the situation can be divided into three categories including determination, prediction and control of attitude. Attitude determination is the calculation of satellite direction in space, relative to a reference or some celestial objects such as the earth.

Attitude prediction is estimated from satellite direction in the future through using dynamic models and recorded information status. According to Bolandi, *et al* (1999), attitude control is consisted of two issues: attitude stability (the process of keeping current direction) and control of attitude maneuver (satellite direction control shift from one stance to another). In this research, the attitude stability and the process of vibration damping and control caused by attitude maneuver are taken into account.

MODELING

Mathematical modeling of any system is the art and skill of creating a set of equations that are complex and yet simple enough to give a detailed insight into the issue. Therefore, it is necessary to have the knowledge of mathematics and knowledge related to the field to modify an existing system, develop a better sample and predict a specific system and its behavior in different situations. In this research, one of satellite panels assumed as a flexible beam with a head stuck (one end fixed) is shown in Fig. 4.

As Singiresus (2007) believes, the beam is covered fully with a layer of sensors and piezoelectric actuators components as is shown in Fig. 5.

Mathematical model of beam is extracted as a Single-Input Single-Output System (SISO) and Multiple Input Multiple Output Systems (MIMO) which is based on Euler-Bernoulli beam theory. Compulsory vibrations of a flexible beam element have been analyzed by the following partial differential equation of order four (Singiresus, 2007):

$$c^2 \frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\partial^2 w(x,t)}{\partial t^2} = f_{ext} \tag{1}$$

Finite element modeling of smart structure: Fig. 4 and 5 shows the flexible finite element model of a smart beam with a head stuck (one end fixed) and Fig. 6 shows a smart beam element.

In the finite element space, response of Eq. (1) is assumed as a third-degree polynomial a function of x to have the following form Petyt (1990):

$$w(x, t) = a_1 + a_2x + a_3x^2 + a_4x^3 \tag{2}$$

where, w (x, t) is the transfer function which is satisfying the partial differential equation of order four (1). Constants a_1 to a_4 using boundary conditions are obtained in each of the nodes as follows: (Node 1 and Node 2 of beam element):

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \frac{1}{l_b^3} \begin{bmatrix} l_b^3 & 0 & 0 & 0 \\ 0 & l_b^3 & 0 & 0 \\ -3l_b & -2l_b^2 & 3l_b & -2l_b^2 \\ 2 & l_b & -2 & l_b \end{bmatrix} \tag{3}$$

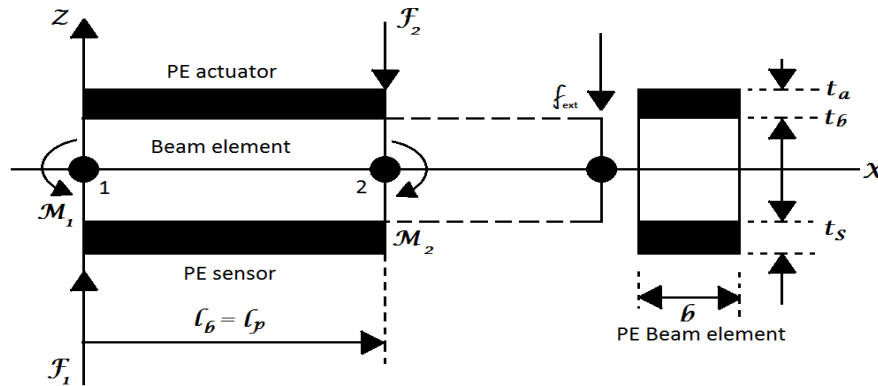


Fig. 4: Finite element model

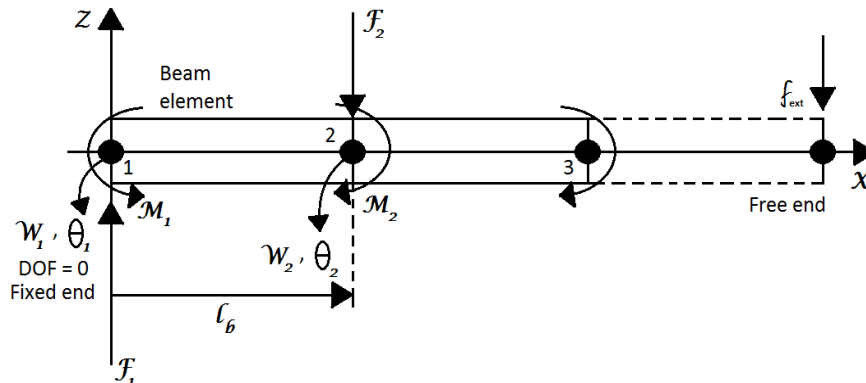


Fig. 5: smart beam model

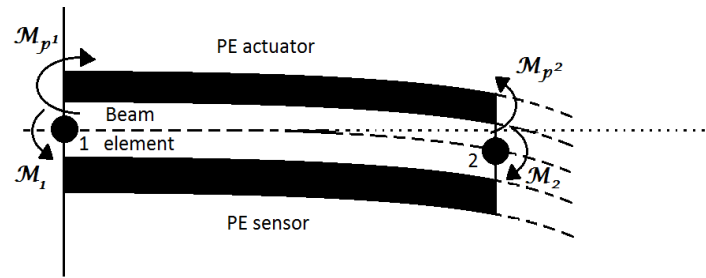


Fig. 6: A smart beam element

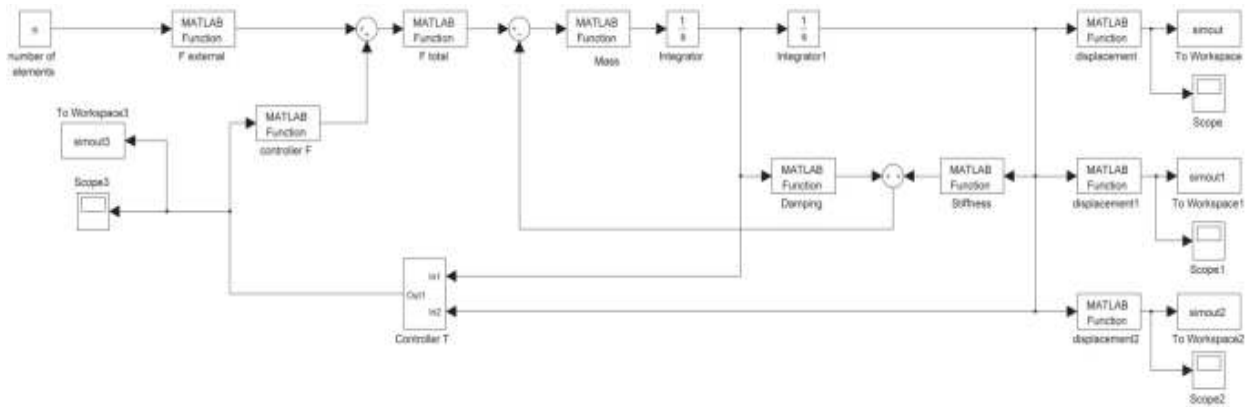


Fig. 7: Block diagram of the smart control system using a single control input

By replacing the constants obtained from Eq. (3) with equation (2), sorting the final shape of $w(x, t)$ is obtained as follows:

$$w(x, t) = [f_1(x) \ f_2(x) \ f_3(x) \ f_4(x)] \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} \quad (4)$$

where, $f_1(x)$ to $f_4(x)$ is considered as the shaped functions of beam element.

Simple (none smart) beam element: Strain energy (U) and kinetic energy (T) of a bended beam element with uniform crossover is obtained from the following relations:

$$U = \frac{E_b I_b}{2} \int \left[\frac{\partial^2 w}{\partial x^2} \right]^2 dx \quad (5)$$

$$T = \frac{\rho_b A_b}{2} \int \left[\frac{\partial w}{\partial t} \right]^2 dt \quad (6)$$

The equation of the motion of a basic beam two-node element with replacement Eq. (5) and (6) in the Lagrange equation:

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_i} \right] + \left[\frac{\partial U}{\partial q_i} \right] = [Z_i] \quad (7)$$

is obtained as follows:

$$M^b \ddot{q} + K^b = f^b(t) \quad (8)$$

where, \ddot{q} is acceleration vector, Z_i is the vector of moments and forces and M^b , K^b and f^b are the mass, stiffness and force vector matrices, respectively. Mass and stiffness matrices for each element were calculated by using the following relations:

$$[M^b] = \rho_b A_b \int [n]^T [n] dx \quad (9)$$

$$[K^b] = E_b I_b \int [n_1]^T [n_1] dx \quad (10)$$

Piezoelectric element: Piezoelectric elements can be used for sensing and actuating purposes and function as sensors and actuators in flexible structures on the main structure. According to Moheimani and Fleming (2005), the assumed piezoelectric element used in the structure has two structural degrees of freedom in each node; one is the transverse displacement and the other is rotation angle or slope and one electrical degree of freedom as voltage. Note that the electrode voltage is a constant value; therefore there is only one electrical degrees of freedom for each piezoelectric element. If mounted piezoelectric material on a structure behaves as an actuator or sensor, respectively, the electrical

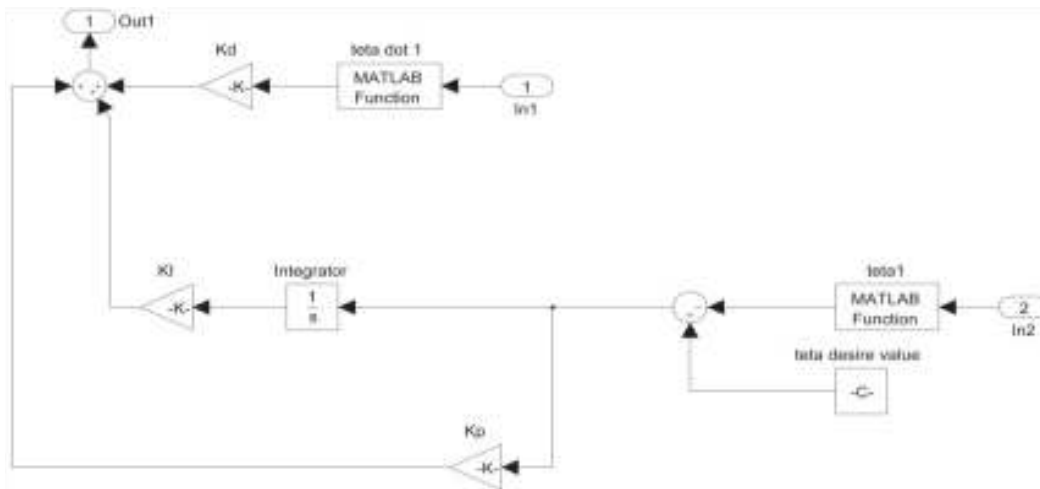


Fig. 8: Block diagram of the T-shaped controller in Fig. 7

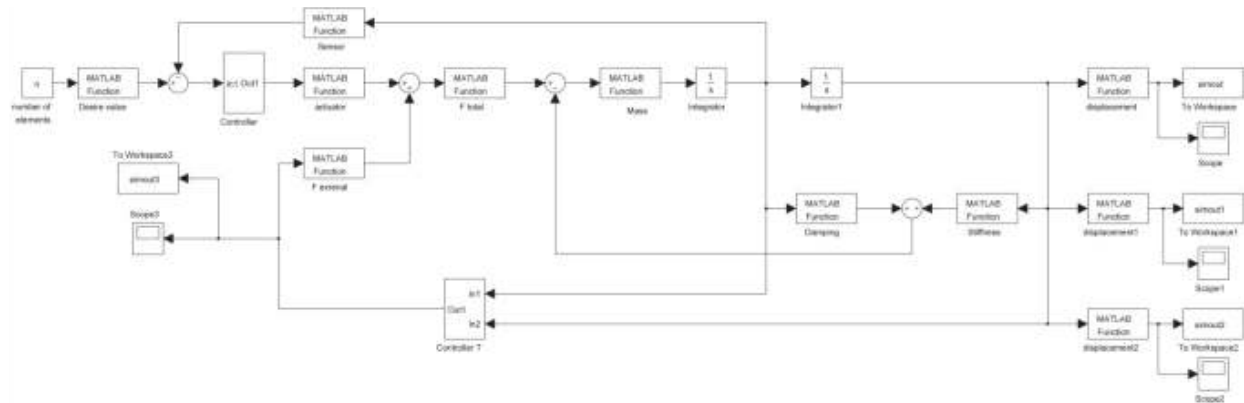


Fig. 9: Block diagram of smart control system using multi hub and structure control input

degree of freedom is used as actuator voltage or sensor voltage. Displacement functions of piezoelectric elements are simple and similar to the beam element displacement function. As Eq. (8) was extracted from simple beam element, the language equation for piezoelectric element motion obtained as follows:

$$M^p \ddot{q} + K^p = f^p(t) \tag{11}$$

where,

$$[M^p] = \rho_p A_p \int [n]^T [n] dx \tag{12}$$

$$[K^p] = E_p I_p \int [n_1]^T [n_1] dx \tag{13}$$

Smart beam element: Piezoelectric beam element (top layer Piezo+beam element+bottom layer Piezo) is obtained by a simple beam sandwiched by two thin layer piezoelectric elements with t_a or t_s thickness and is attached to the original structure. The bottom layer

acts as a piezoelectric sensor and upper layer is piezoelectric actuator. Mass and stiffness smart beam element is obtained through using the following equations (Moheimani and Fleming, 2005):

$$[M] = \rho A \int [n]^T [n] dx \tag{14}$$

$$[K] = EI \int [n_1]^T [n_1] dx \tag{15}$$

In which $EI = E_b I_b + 2E_p I_p$ is the smart beam element bending stiffness, $\rho A = b(\rho_b t_b + 2\rho_p t_p)$ is mass unit length of Smart beam, t_p is Piezo layer thickness, which is equal to t_a actuator thickness or t_s sensor thickness.

Strain rate of sensors and actuators: Piezoelectric materials can be used as sensors and actuators. When used as an actuator, the input is voltage and output is mechanical strain. When used as a sensor, a mechanical strain is played a role as an input and output is turned

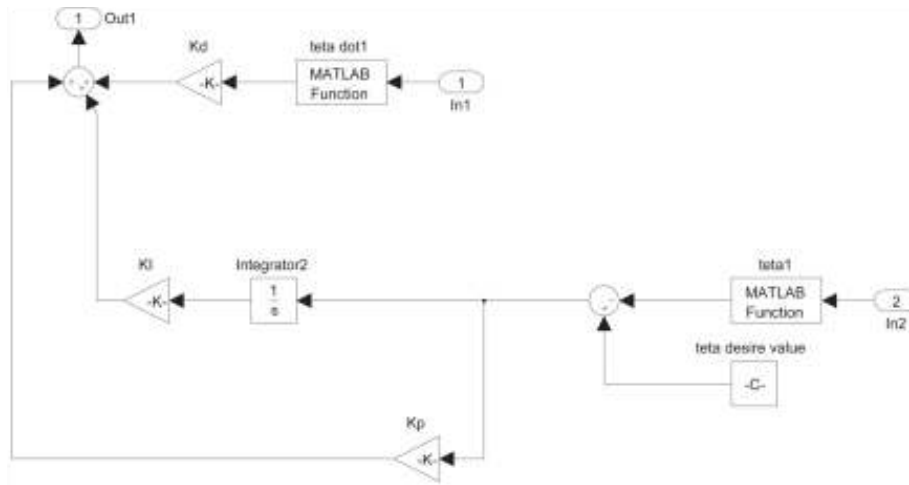


Fig. 10: Block diagram of the T-shaped controller in Fig. 9

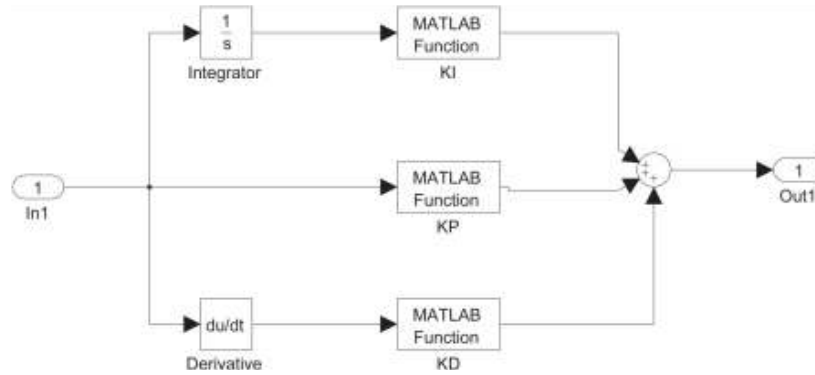


Fig. 11: Block diagram of Fig. 9 controller

out to be the electrical voltage. This voltage is then corrected to the desired shape which is used as controller input. Controller output as a control signal affects the actuator. Actuator produces a mechanical strain and this mechanical strain is used to prevent the vibrations caused by damaging interventions. Linear piezoelectric coupling between the elastic field and electric field can be expressed by direct and reverse piezoelectric structural equations as following (Yang, 1997):

$$D_z = e_{31}\sigma + e^\sigma E_f, \varepsilon = d_{31}E_f + s^E\sigma \quad (16)$$

where, D_z is the electric displacement, d_{31} piezoelectric constant, σ stress, e mean permittivity (dielectric constant), E_f electric field, ε the strain and s^E are piezoelectric medium softness.

SENSOR EQUATION

Sensor equation is extracted from piezoelectric direct equation which is used for calculating the charge

which is produced by structure strain. Total charge $Q(t)$ generated on the surface of the sensor (caused by the strain) is equal to the total charges generated on the sensor layer. Piezoelectric materials were used as sensors of strain rate. In this form of application, output charge can be converted to electrical current of sensor.

$$i(t) = \frac{dQ(t)}{dt} = \frac{d}{dt} \int D_z dA = \frac{d}{dt} \int e_{31} \varepsilon_x dA \quad (17)$$

Strain ε_x of beam at a specific point in terms of the second location derivative of displacement function $w''(x, t)$ is as $\varepsilon_x = z \frac{d^2w}{dx^2}$. Thus, Eq. (17) can be written as follows:

$$i(t) = ze_{31}b \int_0^{l_p} [n_1]^T \dot{q} dx \quad (18)$$

Electrical current output of piezoelectric sensors determines the strain rate of flexible beam. This electrical current is turned out to sensor open circuit voltage V^s , using a modified signal parameter G_c , which is electrical resistance of the sensor):

$$V^s(t) = G_c e_{31} z b \int_0^{l_p} n_1^T \dot{q} dx \quad (19)$$

After integration, we have a more abstract form:

$$V^s(t) = S_1 [0 \quad -1 \quad 0 \quad 1] \dot{q} \quad (20)$$

where,

$$S_1 = G_c e_{31} z \quad (21)$$

which, is called sensor constant.

Controller equation: Sensor output voltage is given as input to the controller, the controller output is equal to controller gain multiplied by sensor voltage $V^s(t)$. Thus the actuator input voltage $V^a(t)$ is in fact the controller output signal which is equal to (Kwon, 2004):

$$V^a(t) = \text{controller gain} \times V^s(t) \quad (22)$$

If so, $S_2 = \text{controller gain} \times S_1$ then we'll have:

$$V^a(t) = S_2 [0 \quad -1 \quad 0 \quad 1] \dot{q} \quad (23)$$

Actuator equation: Actuator equation is derived from piezoelectric Inverse equation. Strain produced ϵ_a by the electric field E_f in actuator layer is equal:

$$\epsilon_a = d_{31} E_f \quad (24)$$

where, d_{31} is piezoelectric constant.

When the input voltage $V^a(t)$ in t_a thickness direction is applied to piezoelectric actuator, electric field E_f and tension σ_a are created as follows:

$$E_f = \frac{V^a(t)}{t_a}, \quad \sigma_a = E_p \epsilon_a \quad (25)$$

$$\Rightarrow \sigma_a = E_p d_{31} E_f = E_p d_{31} \frac{V^a(t)}{t_a}$$

where, E_p is Young's modulus of piezoelectric layer.

The bending stress σ_a creates bending moment M_a in the piezoelectric actuator element that can be achieved with the integration of bending stresses σ_a along the actuator thickness so:

$$M_a = E_p d_{31} \bar{z} b V^a(t) \quad (26)$$

$$M_a = S_3 V^a(t), \quad S_3 = E_p d_{31} \bar{z} b$$

The torque control as a wide torque is distributed throughout actuator element l_p . Therefore, force control f_{ctrl} created by actuator element apply to beam element is equal (Yang, 1997; Kwon, 2004):

$$f_{ctrl} = E_p d_{31} b \bar{z} \int [n]' dx V^a(t) \quad (27)$$

Smart structure equation motion: At first, mass and stiffness matrices can be calculated for each element which are (4×4) order matrices and then force vector which is (1×4) order matrices. By assembling the matrices, equation motion of smart beam finite element were obtained by the following equation:

$$M \ddot{q} + K q = f_{ext} + f_{ctrl} = f^t \quad (28)$$

Solution of smart structure system: After deriving the equation motion of smart structure system in the previous section, MATLAB Software was used for solving the smart structure system. At first, the system is simulated in "Simulink" software environment, the block diagram is drawn for each section and then the codes are written for each of the blocks.

Smart control system using a single hub control input: It is assumed that the beam is at angle zero relative to the horizon, all rises and dips are zero and the beam is under the effect of a torque at angle θ_{des} .

For this purpose, a specific torque should be applied into the beam through the hub. The amount of torque is specified through a signal received from a controller; in general case, it is a PID controller. Therefore, the torque control is applied to the hub as follows:

$$T_{ctrl} = K_p(\theta_2 - \theta_1) + K_D(\dot{\theta}_2 - \dot{\theta}_1) + K_I \int (\theta_2 - \theta_1) dt \quad (29)$$

where, K_p , K_D and K_I are proportional to ratio controller, derivative controller and integral controller. Also, $\theta_2 = \theta_{des}$ and is constant. Therefore, consistently $\dot{\theta}_2 = 0$ and at first interval time $\theta_1 = \dot{\theta}_1 = 0$ (Fig 7 and 8).

Smart control systems using multi-hub and structure control input: It is assumed that the beam is at angle zero relative to the horizon, all rises and dips are zero and the beam is under the effect of a torque at angle θ_{des} .

For this purpose, a specific torque should be used into the beam through the hub. The amount of torque is specified through a signal received from a controller. In general, it is a PID controller. Therefore, the torque control is applied to the hub which is calculated with following Eq. (29). The optimum value block is open circuit voltage vector input to the controller. Other input controls are the same as actuator input voltages. Actuator input voltage $V^a(t)$ is in fact equal to controller output signal (Fig. 9, 10 and 11).

$$V^a(t) = K_p V^s(t) + K_D \frac{dV^s(t)}{dt} + K_I \int V^s(t) dt \quad (30)$$

Table 1: The properties of the flexible aluminum beam stuck with and piezoelectric elements attached to it (sensors and actuators)

Physical properties	Beams with head stuck	Piezoelectric sensor and actuator (PZT)
Length	$L_b = 0.3m$	$L_p = 0.3m$
Width	$b = 0.03m$	$b = 0.03m$
Thickness	$t_b = 0.5mm$	$t_s = t_a = 0.5mm$
Density	8030 Kg/m^3	7700 Kg/m^3
Young's modulus	$E_b = \text{GPa } 193.06$	$E_p = \text{GPa } 193.06$
Piezoelectric stress constant		$d_{31} = m/V125 \times 10^{-12}$
Piezoelectric strain constant		$e_{31} = \text{VmN}^{-1} 10.5 \times 10^{-3}$

NUMERICAL SIMULATION

As it was already mentioned, simulations are based on a specific nano-satellite. Therefore, a flexible aluminum beam with a head stuck (one end fixed) by using piezoelectric actuators and sensors patched at both surfaces were taken into account, Table 1 shows the beam physical properties. In this study, a number of

elements is considered as n and $\theta_{des} = \frac{\pi}{6}$. Thus, the tilt angle of the first node, or in other words angle of baseline (fixed point end) beam relative to horizon after a short time to be $\frac{\pi}{6}$ and also beam end shift ought to be equal to:

$$y_{tip} = L_b \times \theta_{des} = 0.3 \times \frac{\pi}{6} = 0.157m$$

Smart control system results using a single hub control input: By considering the results in Table 2, the followings were obtained

- Mode 5 is not suitable due to charts divergence
- Mode 6 is not appropriate because of high ultra swing torque control
- Mode 4 is not suitable because of the high landing time
- Modes 1, 2 and 3 are more favorable than in other modes

Table 2: Comparison of smart control system by using a single control input hub

Mode	Coefficients of proportional controller K-P	Controller integrator coefficient K-I	Derivative controller coefficients K-D	Control torque graphs changes		Graphs changes of the beam free end displacement [swing]	
				Ultra swing or displacement	Settling time	Ultra swing or displacement	Settling time
Model 1	0.1	0.1	0.01	0.05	3.5	0.25	3.5
Mode 2	0.5	0.5	0.05	0.27	0.7	0.18	3.5
Mode 3	1	1	0.1	0.52	0.5	0.17	3.5
Mode 4	0.05	0.05	0.005	0.03	10	0.28	10
Mode 5	0.01	0.01	0.001	Divergent			
Mode 6	10	10	1	5.2	3.5	0.18	3.5

Table 3: Comparison of smart control system by using multi-control input of hub and structure

Modes	Hub Torque controller			Piezoelectric actuators controller			Control torque graph changes		Graphs changes of the beam free end displacement[swing]	
	Coefficients of proportional controller K-P	Controller integrator coefficient K-I	Derivative controller coefficients K-D	Coefficients of proportional controller K-P	Controller integrator coefficient K-I	Derivative controller coefficients K-D	Ultra swing or displacement	Settling time	Ultra swing or displacement	Settling time
Model1	0.1	0.1	0.01	100	100	10	0.052	3.3	0.24	3.2
Mode2	0.1	0.1	0.01	10	10	1	0.052	3.3	0.24	3.3
Mode3	0.1	0.1	0.01	50	50	5	0.053	3.3	0.24	3.2
Mode4	1	1	0.1	100	100	10	0.38	0.4	0.16	3.2
Mode5	0.5	0.5	0.05	100	100	10	0.23	0.6	0.17	3.2

Table 4: The percentage of result changes in single-control input mode and multi- control input mode

Modes	Percentage changes								
	Control torque graph changes				Beam free end displacement				
	Ultra-swing		settling time		Ultra-swing		settling time		
A	Single input mode 1	0.057	3.5	0.25	3.5	-	-	-	-
	Multi input mode 1	0.052	3.3	0.24	3.2	8.77	5.71	4	8.57
	Multi input mode 2	0.052	3.3	0.24	3.3	8.77	5.71	4	5.71
	Multi input mode 3	0.053	3.3	0.24	3.2	7.01	5.71	4	8.57
B	Single input mode 3	0.52	0.5	0.17	3.5	-	-	-	-
	Multi input mode 4	0.38	0.4	0.16	3.2	26.92	20	5.88	8.57
C	Single input mode 2	0.27	0.7	0.18	3.5	-	-	-	-
	Multi input mode 5	0.23	0.6	0.17	3.2	14.81	14.28	5.55	8.57

In mode 1, swing torque control compared to other two cases was less but landing time torque control is more than other two cases. Also, ultra swing at the free end of beam in modes 2 and 3 are less than 1 and yet swing landing time at free end of beam are equal in all three modes.

Smart control systems results using multi-control input of hub and structure: According to Table 3, the following results were obtained:

- In modes 1, 2, 3 ultra swings [displacement] torques is less than modes 4 and 5, but settling time torque is more.
- In modes 4 and 5 ultra swing [displacement] of free end beam is less than other modes and yet settling time at end of the beam, in almost all modes is equal.

THE COMPARISON RESULTS OF MULTI-CONTROL INPUT MODE WITH SINGLE-CONTROL INPUT MODE

Table 4 shows that both the use of smart materials and the application of structure in all cases lead to a decrease in ultra swing torque control peak of rotation part and the landing time. Furthermore, the changes and the effect of smart materials application in beam displacement and vibration is tangible in so far as this process has reduced the maximum of ultra-swing amplitude of vibration and displacements of beam free end as well as the landing time. It should be noted that the vibrations are created in structure due to rotation or during attitude maneuver. In this research, the structure rotation was simulated by considering a torque control applied to the beam fixed end from the hub side.

CONCLUSION

Table review shows that there was a reduction in the amplitude of torque control, which was used for the

structure rotation during attitude maneuvers. This means that the use of smart structures led to a reduction of force necessary for changing satellite status. Landing time of torque control of rotation simulation was also reduced. Further, there was a decrease in the range of vibrations due to attitude maneuvers in satellite structure. Finally, the landing time vibrations caused by attitude maneuvers reduced in satellite structure.

By the application of smart structures in satellites and spacecraft, we can also indirectly reduce the weight, volume structure, energy consumption and cost and increase the structure's life and the energy received from the sun.

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