

## Research Article

### Modeling the Implied Volatility Surface-: A Study for S&P 500 Index Option

<sup>1</sup>Jin Zheng and <sup>2</sup>Yan Cai

<sup>1</sup>Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK

<sup>2</sup>Department of Mathematical Sciences, Xi'an Jiaotong University, Shanxi, Xi'an, 430079, China

**Abstract:** The aim of this study is to demonstrate a framework to model the implied volatilities of S&P 500 index options and estimate the implied volatilities of stock prices using stochastic processes. In this paper, three models are established to estimate whether the implied volatilities are constant during the whole life of options. We mainly concentrate on the Black-Scholes and Dumas' option models and make the empirical comparisons. By observing the daily-recorded data of S&P 500 index, we study the volatility model and volatility surface. Results from numerical experiments show that the stochastic volatilities are determined by moneyness rather than constant. Our research is of vital importance, especially for forecasting stock market shocks and crises, as one of the applications.

**Keywords:** Moneyness, out-of-the money options, parameters estimation, volatility surface

#### INTRODUCTION

The implied volatility of an option is the market's measure of the price of the underlying asset from today to the option's expiration. Classical stock option pricing is largely based on Black-Scholes theory which assumes that the underlying stock price follows what is called Geometric Brownian motion. According to Black and Scholes (1973) model built by Fischer, the implied volatilities of the same underlying asset with the same expiration date keep constant, even if the underlying asset has a different strike price. They made one important assumption of BS model, the constant volatility, or local volatility.

However, a variety of researches have questioned the validity of the constant volatility assumption. Derman *et al.* (1994) have built models which represented that the implied volatility is a function of share price, exercise price, time to maturity and a drift. This formula was developed in 1994 by analyzing the implied binomial tree. Afterwards, they added one parameter to the implied binomial model and built up the implied trinomial tree model, which triggered a further prove that the implied volatility is not constant. Furthermore, Dumas *et al.* (1998) analyzed the price of S&P 500 index options from the period June 1988 to December 1992 and proposed several functions which show the implied volatility is a function of asset price and time to maturity. Similarly, Peña *et al.* (1999) investigated the reasons why the shape of volatility function turned to be smile by recording options on the Spanish IBEX-35 exchange market. While in their experiment, they only regarded moneyness as a determinant of the implied volatility. Cont and Fonseca

(2002) worked at the implied volatility surface, and especially focused on dynamic. The surfaces of S&P 500 index and FTSE100 index, out of the money options were compared in the literatures.

The aim of this paper is to investigate whether the implied volatility surface is flat for S&P 500 index. We use daily-recorded data set of S&P 500 index, out-of-the-money options from 1996 to 2006. To estimate the determinants of implied volatility, the BS model and Dumas' model are used and compared. It can be concluded from the statistical techniques that the implied volatility is a function of moneyness rather than constant. In addition, the figures of volatility surface present a rich structure departs from a flat. The importance of researching the implied volatility model is that it could determine the value of options contracts. Hence, the volatility model can be applied widely in risk management and pricing exotic options.

#### MATERIALS AND METHODS

**Data selection:** The data set used in this paper is the S&P 500 index, out-of-the-money options, which means the strike price is larger than the asset price at the time that the option is written for call options. For put options, out-of-the-money options stand for the options of which strike price is lower than asset price. As it is known, S&P 500 is a stock market index which contains 500 stocks of large companies in US. The S&P 500 index is chosen for the following considerations. First, since its inception in 1957, S&P 500 index has been a barometer for the American economy, which draws interests from many researchers, including Dumas *et al.* (1998) and Cont and Fonseca (2002). The

market trend predicted by the many financial analysts is mostly based on observing the movement of S&P 500 index, although such a trend analysis may not always quickly foresee the potential bubbles existed in the market. Second, since there is a large trading volume of S&P 500 index every day, the data set is abundant and representative for the experiment in this model. Third, the data are maintained continuous and consistent in S&P 500 index, which is critically important for model testing and selection. The data set is composed of trading date, maturity date, call/put, strike price, stock price, option price, maturity days and interest rate.

**Model derivations:** The experiment tests whether the implied volatilities are constant all the time. Hence, three models should be built and compared. BS model demonstrates that the volatilities are constant, while in the model designed by Dumas *et al.* (1998), as mentioned in introduction, implied volatilities are determined by two factors: strike price and time to maturity:

$$\begin{aligned} \text{Model 1 } \sigma(MN, T) &= \beta_0 + \epsilon \\ \text{Model 2 } \sigma(MN, T) &= \beta_0 + \beta_1 MN + \epsilon \\ \text{Model 3 } \sigma(MN, T) &= \beta_0 + \beta_1 MN + \beta_2 MN^2 + \epsilon \end{aligned}$$

Model 1 represents the BS model. Model 2 shows that the implied volatilities have a linear relationship with moneyness. Moneyness is used to describe the relationship between spot price and strike price. Moneyness equals  $S/K$ , which is a measurement of the option's intrinsic value. For call options, in the money option means the strike price is lower than the spot price when then contract is written. Hence, the options have an intrinsic value. If the strike price is higher than the spot price, the call option is out of the money option. At the money option refers that the strike price equals the spot price, which has no intrinsic value. Nevertheless, we define moneyness for put options in a contrary way. In the money option means when the spot price is lower than the strike price while out of the money option refers to the situation that the spot price is higher than the strike price. In this paper, as to Gross and Waltner (1995), moneyness is defined as  $MN = (\log(F/K)) / \sqrt{T}$ . Model 3 captures the quadric

relationship between the implied volatilities and moneyness.

## RESULTS

The results of each model show the period from March 6<sup>th</sup>, 2006 to March 10<sup>th</sup>, 2006, which was picked out from the data set. The maturities of the index options chosen are about 100 days, medium-long maturity. Table 1 shows the constant implied volatility for Model 1. As it presents in the table, the parameters,  $\beta_0$ , are around 0.07 to 0.08. Since the parameters have small fluctuations,  $\beta_0$  can be approximately regarded as constant during these 5 days. The average root mean square error is 0.064. Since the implied volatility is constant and there is no dependent variable in Model 1, both  $R^2$  and adjust  $R^2$  are zero. Figure 1 shows the implied volatility surface of all the index options trading on March 6<sup>th</sup>. As expected, the implied volatility surface is flat, which examines the validity of the assumption of BS model. However, BS model is used a benchmark for comparison purpose.

The estimated parameters for Model 2 are presented in Table 2. The results show that  $\beta_0$  lies in the interval (0.04, 0.05), while the value of the coefficient of  $MN$ ,  $\beta_1$ , is around 0.26. The average root mean square error in Model 2 is half as smaller as model 1, which shows that Model 2 is in preference to Model 1. The average residual sum of square of the implied volatilities is about 0.0313 in Model 2 with respect to the result of 0.1 in Model 1, which is a relatively smaller figure.  $R$  Square and Adjust  $R^2$  are the measurement of goodness of fit. With one additional parameter to obtain the term structure, Model 2 has an average  $R^2$  of 76%, which explains 76% of the variance on independent variable. Adjust  $R^2$ , which takes degree of freedom into consideration, is 75% on average. However,  $R^2$  of Model 1 is zero, which implies the implied volatility is constant. Akaike Information Criterion (AIC) of Model 2 is much less than that in Model 1. Figure 2 describes the surface of implied volatility for Model 2 on the date of March 6<sup>th</sup>, 2006. The surface is not flat as shown in Fig. 1. The implied volatilities vary with the changes of yield to maturity and moneyness. As a result, the figure is in compatible with the Model built.

Table 1: Estimated parameters for model 1

Date	Parameters $\beta_0$	RMSE	$R^2$	Adjust $R^2$	Absolute AIC
March 6	0.086039	0.0626	0	0	0.9543
March 7	0.086336	0.0636	0	0	0.9493
March 8	0.086270	0.0634	0	0	0.9655
March 9	0.071515	0.0650	0	0	1.0567
March 10	0.073693	0.0672	0	0	1.0868

Table 2: Estimated parameters for model 2

Date	Parameters $\beta_0$	Parameters $\beta_1$	RMSE	$R^2$	Adjust $R^2$	Absolute AIC
March 6	0.0589	0.2367	0.0336	0.7121	0.6978	0.3263
March 7	0.0529	0.2612	0.0294	0.7856	0.7749	0.1901
March 8	0.0581	0.2424	0.0323	0.7406	0.7289	0.2933
March 9	0.0451	0.2764	0.0315	0.7641	0.7546	0.3557
March 10	0.0416	0.2919	0.0298	0.8032	0.7953	0.3061

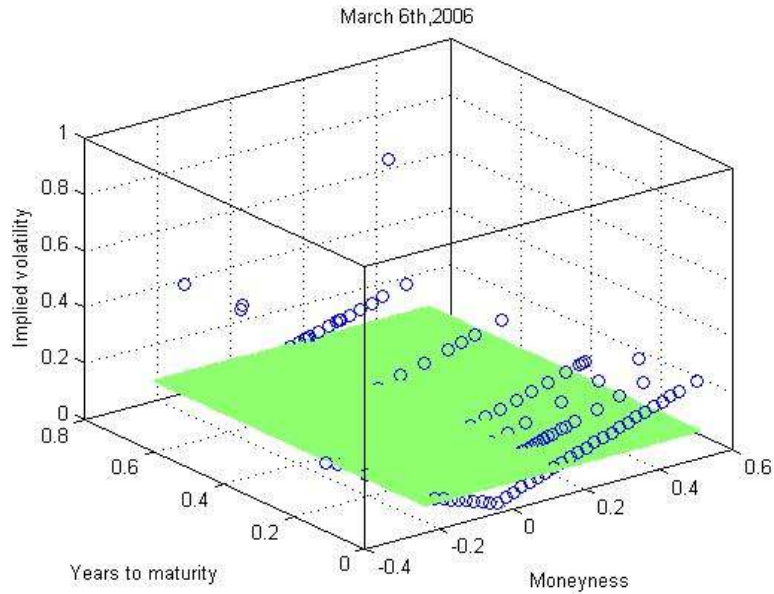


Fig. 1: Estimated implied volatility surface for model 1 on the date of March 6<sup>th</sup>, 2006  
The blue circles refer to the observed implied volatilities of each individual index option

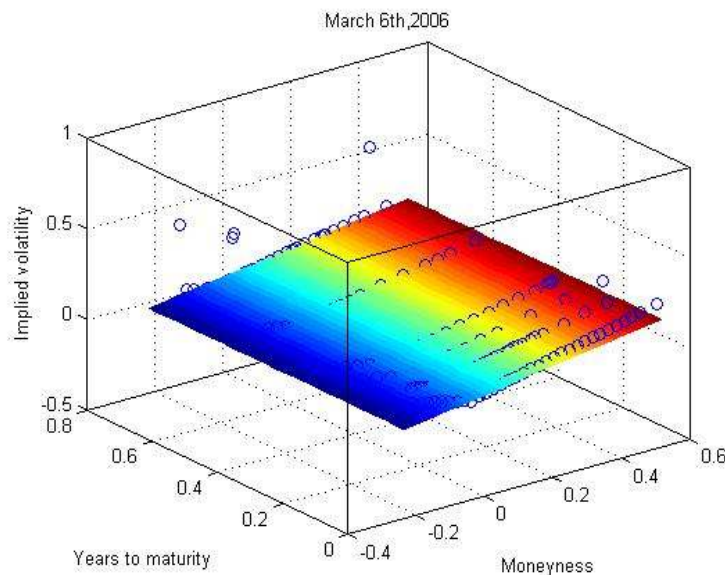


Fig. 2: Estimated implied volatility surface for model 2 on the date of March 6<sup>th</sup>, 2006  
The blue circles refer to the observed implied volatilities of each individual index option

Table 3: Estimated parameters for model 3

Date	Parameters $\beta_0$	Parameters $\beta_1$	Parameters $\beta_2$	RMSE	$R^2$	Adjust $R^2$	Absolute AIC
March 6	0.0368	0.0519	0.6884	0.0162	0.9331	0.9261	0.2218
March 7	0.0368	0.0729	0.6358	0.0157	0.9332	0.9328	0.2676
March 8	0.0369	0.0661	0.6488	0.0156	0.9332	0.9331	0.2518
March 9	0.0287	0.0652	0.7214	0.0177	0.9255	0.9193	0.0705
March 10	0.0293	0.0798	0.6503	0.0176	0.9312	0.9312	0.0757

The estimated parameters for Model 3 are listed in Table 3. The average estimated values of  $\beta_0$ ,  $\beta_1$ ,  $\beta_2$  are 0.0337, 0.0672 and 0.6677, respectively. The average

residual sum of square of the implied volatilities is 0.0166 in Model 3. Compared to the value of average residual sum of square in Model 1 and 2, RMSE

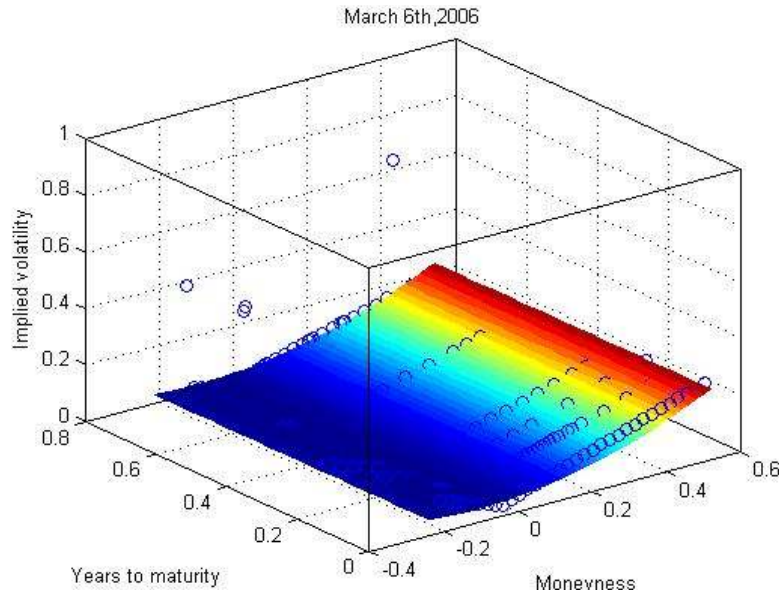


Fig. 3: Estimated implied volatility surface for model 3 on the date of March 6<sup>th</sup>, 2006  
The blue circles refer to the observed implied volatilities of each individual index option

in Model 3 is a significant reduction. The average  $R^2$  is 93.4%. The model 3, with three parameters in a quadric function, results on a 93.4% increase compared to Model 1, and a 17.4% increase with respect to Model 2. The average Adjust  $R^2$  of Model 3 is 93%, which is 18% larger than Model 2. Akaike Information Criterion (AIC) in Model 3 is much less than that in Model 1 and 2. Figure 3 shows the surface of implied volatility for Model 3 on March 6<sup>th</sup>, 2006. As in the model 3, the surface is also not as flat as shown in Fig. 1. The implied volatilities change with yield to maturity and moneyness. In addition, the surface obtained in Fig. 3 is more suitable to fit these individual implied volatilities.

## DISCUSSION AND APPLICATIONS

**Value-at-risk estimation:** In the area of risk management and financial mathematics, Value-at-Risk is a widely used risk measure technique which estimates the risk of loss on a specific portfolio of financial assets. To be more concrete, Value-at-Risk of a given portfolio demonstrates the expected maximum loss over an aimed horizon within a given confidence interval (Zou *et al.*, 2003). Since the implied volatility implies an unbiased and efficient forecast of future, average implied volatility realizes over a specific period of time. As a result, the implied volatility of the past period calculated can be regarded as a forecasting of S&P 500 index implied volatility for this specific period. In this situation, the implied volatility models in Section 2 can be used for computing Value-at-Risk portfolios, which include assets whose payoffs/returns rely on the S&P 500 index.

**Financial crisis prediction:** Stock volatility, which represents massiveness of the change of stock prices, is

a major indicator of future economic activity (Romer, 2011).

It is generally believed that the stock market fluctuation is closely related to the macroeconomic development. The increased stock volatility usually signals higher uncertainty for future economic activity. The investors or the stock holders will tend to spend less money or wealth on consumption and investment, which will lead to decrease in aggregate demand, even economic slack. On the other hand, the decrease in stock volatility encourages the investors to make consumption or investment, which is often the first move in a cycle towards the economic prosperity. The relationship between the stock market volatility and aggregate demand has been studied by many researchers, such as Zhou (2009).

Financial crisis, a symbol of economic downturn, or recession, happens when financial institutions or financial markets lose large part of their value. The stock volatility is a powerful indicator of financial crisis because it is closely related to business or economic cycle. By observing the U.S. stock market's implied volatility and quarterly percentage growth of real GDP, Zhou (2009) carried out an experiment on the relationship between stock volatility and market crash and found a negative relationship between volatility and GDP growth, where the implied volatility is relatively higher in stock crash period than non-crash period.

## CONCLUDING REMARKS

To estimate whether the implied volatilities are constant during the whole life of options as Black-Schole model assumed, we concentrate on the Black-Scholes and Dumas' option models and make the empirical comparisons. The data set in this paper is

S&P 500 index options from 1996 to 2006. According to the statistic techniques, as a result, Model 3 is the most suitable compared to other two models, which captures the quadric relationship between the implied volatilities and moneyness. Furthermore, it can be concluded that the assumption of constant volatilities in BS model cannot exist in the real financial market.

The implied volatilities can be used as an unbiased and efficient forecast of the performance of financial market. The model of implied volatility can be applied in many fields of financial market, including for calculation of the value-at-risk for portfolios and for stock crisis prediction. The results support the idea that the implied volatility tends to be higher in the recession period. In the meantime, the mean-reverting disillusion, which explains the mean-reverting speed is higher after the stock crash period than before, is proved by both the t-statistics and likelihood ratio test in the study (Zheng and Xie, 2012).

The approaches of financial modeling and the relating parameter estimation procedures contained in this study also provide useful supports and facilitate neighboring studies such as characterization of mortgage loans and American put options (Xie, 2009; Xie *et al.*, 2011). A further study should be carried on modifying the minimization problem. The technique in this paper is to use Ordinary Least Square. However, to obtain a more precise estimation of the implied volatilities, the coefficients should be measured based on a weight sum of square errors.

#### REFERENCES

- Black, F. and M. Scholes, 1973. The pricing of options and corporate liabilities. *J. Polit. Econ.*, 81: 637-654.
- Cont, R. and J.D. Fonseca, 2002. Dynamics of implied volatility surfaces. *Quant. Finan.*, 2: 40-56.
- Derman, E., I. Kani and N. Chriss, 1994. The Volatility Smile and its Implied Trees. *Quantitative Strategies Research Notes*, Goldman Sachs.
- Dumas, B., J. Fleming and R.E. Whaley, 1998. Implied volatility functions: Empirical tests. *J. Finan.*, 53(6): 2059-2106.
- Gross, L. and N. Waltner, 1995. S&P 500 Options: Put Volatility Smile and Risk Aversion. *Salomon Brothers*, Mimeo.
- Peña, I., G. Serna and G. Rubio, 1999. Why do we smile? On the determinants of the implied volatility function. *J. Bank. Finan.*, 23: 1151-1179.
- Romer, C., 2011. The great crash and the onset of the great depression. *Quart. J. Econ.*, 105: 597-1179.
- Xie, D., 2009. An integral equation approach to pricing fixed rate mortgages. *Far East J. Appl. Math.*, 35: 233-242.
- Xie, D., D.A. Edwards, G. Schleiniger and Q.G. Zhu, 2011. Characterization of the american put option using convexity. *Appl. Math. Finan.*, 18: 353-365.
- Zheng, J. and D. Xie, 2012. Stochastic modeling and estimation of market volatilities with applications in financial forecasting. *Int. J. Statist. Probabil.*, 1(1): 2-19.
- Zhou, L., 2009. Stock Volatility modeling and the implications in china's market research. *Math. Pract. Theory*, 39(3): 25-34.
- Zou, J., Z. Zhang and Z. Qin, 2003. The application of GARCH model in computing the VaR of chinese stock market. *Theory Pract. Syst. Eng. J.*, 5: 20-25.