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Research Article

Some Identities and Recurrences of Compositions

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Abstract: An identity between partitions and compositions was obtained by Agarwal (2003). Subsequently, other identities between partitions and compositions were obtained. In this study, some recurrences and new identities between partitions and compositions are obtained, by using combination method.

Keywords: Composition, identity, partition, recurrence

INTRODUCTION

In the theory of partitions, Euler gave the first partition identity, namely "the number of partitions of positive integer n into odd parts is equal to the number of partitions of positive integer n into different parts" in Macmahon (1894). Since then, many different identities have been studied in connection with positive integer partition in Alladi (1999) and Andrews (1976, 1984).

However, in the study of partition identities, identities between the positive integer partition and ordered partition were discussed relatively less. In 2003, the 1st identity of such problem was obtained by Agarwal (2003) and that identity was proved by using combination method and the theory of partitions. Using the same method like Agarwal's, in 2007, some new identities between the positive integer partition and ordered partition were obtained by Guo, in addition, several new combinatorial properties of the composition of n in with no part 1 appear by the Fibonacci number were given by Guo (2007). In 2008, 2 new identities between partitions and compositions were obtained and proven by using combination method in (Xiaofang *et al.*, 2008). In 2009, the compositions of n with all parts greater than k (or lower than k) were studied and some new identities and recurrences between partitions and compositions were obtained in Huang and Liu (2009). In 2010, some new identities between partitions and compositions were studied and the identical relation between "odd-even" composition and "odd-odd-even-even" partition and the identical relation between "even-odd" composition and "even-odd-odd-even" partition were obtained by using combination method in You and Xing (2010). In 2011, some lacking and errors in corresponding literatures are indicated and corrected some new identities for partitions and compositions of positive integers were given in Pang (2011).

In this study, we first give some new identities and recurrences between partitions and compositions by using Agarwal's combination method. Then, some new recurrences of some compositions are given.

PRELIMINARIES

The following two definitions were given in Guo (2007):

Definition 1: An "odd" partition of a positive integer n is a partition in which the parts (arranged in ascending order) are all odd integer number.

Definition 2: An "even" partition of a positive integer n is a partition in which the parts (arranged in ascending order) are all even integer number.

The following definition and lemmas were given in Bi *et al.* (2008):

Definition 3: An "odd-even" partition of a positive integer n is a partition in which the parts (arranged in ascending order) alternate in parity starting with the smallest part even.

Lemma 4: Let n be an odd integer, m be a positive integer where $2 \le m < n$. Let c (e, o, n) denote the number of compositions of positive integer n into two big parts: the first big part is an "even" composition, the last big part is an odd integer number; Let c^m (e, o, n) denote the number of compositions of positive integer n into two big parts: the 1st big part is an "even" composition with m-1 parts, the last big part is an odd integer number; Let O_n denote the number of "odd" compositions in which contains more than 2 distributions, where the maximum part is n; Let O_n^m denote the number of "odd" partitions

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with m distributions, where the maximum part is n. Then:

$$c^m(e,o,n) = O_n^m,\ c(e,o,n) = O_n$$

Lemma 5: Let n, m be two positive integers, where $2 \le m < n$. Let c (e, o, n) denote the number of compositions of positive integer n into 2 big parts: the 1st big part is an "odd" composition, the last big part is an even integer number; Let c^m (e, o, n) denote the number of compositions of positive integer n into two big parts: the 1st big part is an "odd" composition with m-1 parts, the last big part is an even integer number; Let OE_n denote the number of "odd-even" partitions where the maximum part is n; Let OE_n^m denote the number of "odd-even" partitions with m distributions, where the maximum part is n. Then:

$$c^m(o,e,n) = OE_n^m,\ c(o,e,n) = OE_n$$

MAIN CONCLUSION

Theorem 1: Let n be an even integer, m be a positive integer where $3 \le m < n$. Let c (o, e, o, n) denote the number of compositions of positive integer n into 3 big parts: the first big part is an odd integer number, the second big part is a "even" composition, the last big part is an odd integer number; Let c^m (o, e, o, n) denote the number of compositions of positive integer n into 3 big parts: the first big part is an odd integer number, the second big part is an "even" composition with $m-2$ parts, the last big part is an odd integer number; Let OE_n^1 denote the number of partitions with more than 3 parts where the maximum part is the even number n and the other parts are odd number; Let $OE_n^{1,m}$ denote the number of partitions with m parts where the maximum part is the even number n and the other parts are odd number. Then:

1. $c^m(o,e,o,n) = OE_n^{1,m}$
2. $c(o,e,o,n) = OE_n^1$

Proof: Obviously 2 can be obtained by 1. Then, we only prove 1.

Let π be a partition with m parts where the maximum part is the even number n and the other parts are odd numbers. Then, $\pi = x_1 + x_2 + \ldots + x_m$ where $x_1 < x_2 < \ldots < x_m = n$ (x_i is an odd number i = 1, 2,…, m - 1). So, $n = x_1 + (x_2 + -x_1) + (x_3 - x_2) + \ldots + (x_m - x_{m-1})$ is a partition of positive integer n into three big parts: the first big part is an odd integer number, the second big part is an "even" composition with $m-2$ parts and the last big part is an odd integer number.

On the other hand, Let $n_1 + n_2 + \ldots + n_m$ be a composition of positive integer n where n_1 and n_m are 2 odd numbers, n_i (i = 2, 3, …, m - 1) is an even number. Let $x_1 = n_1$, $x_2 = n_1 + n_2$, $x_3 = n_1 + n_2 + n_3, \ldots, x_m = n_1 + n_2, \ldots, n_m = n$. It is obviously those $x_1, x_2, \ldots, x_{m-1}$ are all odd numbers and x_m is equal to the even number n, then $\pi = x_1, x_2, \ldots, x_m$ is a partition with m parts where the maximum part is the even number n and the other parts are odd numbers.

Therefore, a bijection is constructed between the composition of positive integer n into three big parts (the 1st big part is an odd integer number, the 2nd big part is a "even" composition, the last big part is an odd integer number) and the partition with m parts where the maximum part is the even number n and the other parts are odd number. Thus, we can prove 1.

Example 1: For n = 8, m = 3. There are 6 compositions of 8 into 3 big parts (the 1st big part is an odd integer number, the 2nd big part is an even integer number, the last big part is an odd integer number, viz.: 1 + 6 + 1, 1 + 4 + 3, 1 + 2 + 5, 3 + 4 + 1, 3 + 2 + 3, 5 + 2 + 1. That is c^3 (o, e, o, 8). On the other hand, there are 6 partitions with 3 parts where the maximum part is the even number 8 and the other parts are odd number, viz.: 1 + 7 + 8, 1 + 5 + 8, 1 + 3 + 8, 3 + 7 + 8, 3 + 5 + 8 and 5 + 7 + 8. That is $OE_8^{1,3} = 6$.

There are 11 compositions of 8 into three big parts (the first big part is an odd integer number, the second big part is an "even" composition, the last big part is an odd integer number, viz.: 1 + 6 + 1, 1 + 4 + 3, 1 + 2 + 5, 3 + 4 + 1, 3 + 2 + 3, 5 + 2 + 1, 1 + 2 + 4 + 1, 1 + 4 + 2 + 1, 1 + 2 + 2 + 3, 3 + 2 + 2 + 1, 1 + 2 + 2 + 2 + 1. That is c(o, e, 8) = 11. On the other hand, there are 11 partitions with more than 3 parts where the maximum part is the even number 8 and the other parts are odd number, viz.: 1 + 7 + 8, 1 + 5 + 8, 1 + 3 + 8, 3 + 7 + 8, 3 + 5 + 8, 5 + 7 + 8, 1 + 3 + 7 + 8, 1 + 5 + 7 + 8, 1 + 3 + 5 + 8, 3 + 5 + 7 + 8, 1 + 3 + 5 + 7 + 8. That is $OE_8^1 = 11$.

Theorem 2: Let n be an even integer, m be a positive integer where, $3 \le m < n$. Then. C^m (o, e, o, n) = O^m_{n-1}

Proof: Let π be a partition with m parts where the maximum part is the even number n and the other parts are odd numbers. Then, $\pi = x_1, x_2, \ldots, x_m$ where, $x_1 < x_2 < \ldots < = x_m$ n (x_i is an odd number, i = 1, 2,…, m - 1). So, $\pi' = x_1 + x_2 + \ldots + (x_m - 1)$ is an "odd" partition with m parts where the maximum part is the odd integer number $n-1$. And this correspondence is one to one.

Thus, we have $OE_n^{1,m} = O_{n-1}^m$. By using the theorem 1, we can prove that c^m (o, e, o, n) = $O^m{}_{n-1}$.

By the lemma 4 and the theorem 2, the following theorem can be proved easy.

Theorem 3: Let n be an even integer, m be a positive integer where, 3≤m<n. Then c^m (o, e, o, n) = c^m (e, o, n - 1).

Theorem 4: Let n be an even integer, m be a positive integer where, 3≤m<n. Let c^m (o, n) denote the number of compositions of positive integer n into m parts where the parts are all odd numbers. Then:

1. When m = 3, we have c^3 (o, e, o, n) = c^3 (o, e, o, n - 2) + c^2 (o, n - 2)
2. When 3<m<n, we have c^m (o, e, o, n) = c^m (o, e, o, n - 2) + c^{m-1} (o, e, o, n - 2);

Proof 1: When m = 3, let π = n_1 + n_2 + n_3 be a composition of integer n into 3 parts where n_1 and n_3 are odd integer numbers, n_2 is an even integer number. According to the relationship between n_1 and 1, we consider the following two cases:

1. If n_1>1, then π' = (n_1 - 2) + n_2 + n_3 is a composition of positive integer n - 2 into 3 parts where n_1 - 2 and n_3 are odd integer numbers, n_2 is an even integer number. Then, the number of such compositions is c^3(o, e, o, n - 2)
2. If n_1 = 1, then π' = (n_1 - 1) + (n_2 - 1) + n_3 = (n_2 - 1) + n_3 is a composition of positive integer n - 2 into 2 parts where, n_2 - 1 and n_3 are odd integer numbers. Then, the number of such compositions is c^2 (o, n - 2)

According to 1 and 2, we have c^3 (o, e, o, n) = c^3 (o, e, o, n - 2) + c^2 (o, n - 2).

Proof 2: When 3<m<n, let π = n_1 + n_2 +…+ n_m be a composition of integer n into m parts where n_1 and n_2 are two odd integer number, each n_i (i = 2, 3, … m - 1) is an even integer number. According to the relationship between n_1 and 1, we consider the following two cases:

3. If n_1>1, then π' = (n_1 - 2) + n_2 +…+ n_m is a composition of positive integer n - 2 into m parts where, n_1 - 2 and n_m are 2 odd integer number, each n_i (i = 2, 3,… m - 1) is an even integer number. Then, the number of such compositions is c^m (o, e, o, n - 2)
4. If n_1 = 1, then π' = (n_1 - 1) + (n_2 - 1) +…+ n_m = (n_2 - 1) + n_3 +…+ n_m is a composition of positive integer n - 2 into m - 1 parts where n_2 - 1and n_m are 2 odd integer number, each n_i (i = 3,…, m - 1) is an even integer number. Then, the number of such compositions is c^{m-1} (o, e, o, n - 2)

According to 3 and 4, we complete the proof of 2.

Theorem 5: Let n, m be two positive integers, where, 3≤m<n. Let c (e, o, e, n) denote the number of compositions of positive integer n into 3 big parts: the first big part is an even integer number, the 2nd big part is an "odd" composition with m-2 parts, the last big part is an even integer number. Let c^m (o, e, o, n) be the number of compositions of positive integer n into 3 big parts: the 1st big part is an even integer number, the second big part is an "odd" composition with m - 2 parts, the last big part is an even integer number; Let OE_{n-1} denote the number of "odd-even" partitions where the maximum part is the integer number n - 1; Let OE_{n-1}^m denote the number of "odd-even" partitions with m parts where the maximum part is the integer number n - 1. Then:

1. $c^m(e,o,e,n) = OE_{n-1}^m$
2. $c(e,o,e,n) = OE_{n-1}$

Proof: Obviously 2 can be obtained by 1. Then, we only prove 1.

Let π be an "odd-even" partition with m parts where the maximum part is the integer number n - 1. Then, π = x_1 + x_2 +…+ x_m where, x_1<x_2<…<x_m = n - 1 and x_1 is an even integer number. Thus, n = x_1 + (x_2 - x_1) + (x_3 - x_2) +…+ (x_m - x_{m-1} + 1) is a composition of positive integer n into 3 big parts: the first big part is an even integer number, the second big part is an "odd" composition with m - 2 parts, the last big part is an even integer number.

On the other hand, Let n = n_1 + n_2 +…+ n_m be a composition of positive integer n into m parts where n_1 and n_m are two even integer number, each n_i (i = 2, 3,…, m - 1) is an odd integer number. Let:

x_1 = n_1, x_2, = n_1 + n_2
x_3 = n_1 + n_2 + n_3,…
x_m = n_1 + n_2 +…+ n_m - 1 = n - 1

It is obviously that x_1, x_2, …, x_m alternate in parity starting with the smallest part even and x_1 is an even integer number. Thus, π = x_1, x_2,…, x_m be an "odd-even" partition with m parts where the maximum part is the integer number n - 1.

Therefore, a bijection is constructed between a composition of positive integer n into 3 big parts (where the 1st big part is an even integer number, the 2nd big part is an "odd" composition with m - 2 parts, the last big part is an even integer number) and an "odd-even" partition with m parts where the maximum part is the integer number n - 1. Thus, we can prove 1.

By the lemma 5 and the theorem 5, the following theorem can be proved easy.

Theorem 6: Let n, m be two positive integers, where, $3 \leq m < n$. Then:

$$c^m (e, o, e, n) = c^m (o, e, o, n - 1)$$

Theorem 7: Let n, m be 2 positive integers, where, $3 \leq m < n$. Let c^m (e, n) denote the number of compositions of positive integer n into m parts where the parts are all even integer numbers. Then:

1. When m = 3, we have $c^3 (o, e, o, n) = c^3 (o, e, o, n - 2) + c^2 (e, n - 1)$
2. When $3 < m < n$, we have $c^m (o, e, o, n) = c^m (o, e, o, n - 2) + c^{m-1} (e, o, e, n - 1)$

Proof: 1 When m = 3, let $\pi = n_1 + n_2 + n_3$ be a composition of positive integer n into 3 parts where n_1 and n_3 are even integer numbers, n_2 is an odd integer number. According to the relationship between n_1 and 2, we consider the following two cases:

1. If $n_1 > 2$, then $\pi' = (n_1 - 2) + n_2 + n_3$ is a composition of positive integer n - 2 into 3 parts where $n_1 - 2$ and n_3 are even integer numbers, n_2 is an odd integer number. Then, the number of such compositions is $c^3 (e, o, e, n - 2)$
2. If $n_1 = 2$, then $\pi' = (n_1 - 2) + (n_2 + 1) + n_3 = (n_2 + 1) + n_3$ is a composition of positive integer n - 1 into 2 parts where $n_2 + 1$ and n_3 are all even integer number. Then, the number of such compositions is $c^2 (e, n - 1)$

According to 1 and 2, we have:

$$c^3 (e, o, e, n) = c^3 (e, o, e, n - 2) + c^2 (e, n - 1)$$

Proof 2: When $3 < m < n$, Let $\pi = n_1 + n_2 + \ldots + n_m$ be a composition of positive integer n into m parts where n_1 and n_m are two even integer numbers, each n_i (i = 2, 3,…, m - 1) is an odd integer number. According to the relationship between n_1 and 2, we consider the following two cases:

3. If $n_1 > 2$, then $\pi' = (n_1 - 2) + n_2 + \ldots + n_m$ be a composition of positive integer n - 2 into m parts where $n_1 - 2$ and n_m are two even integer number, each n_i (i = 2, 3,…, m - 1) is an odd integer number. Then, the number of such compositions is $c^m (e, o, e, n - 2)$
4. If n_1 - 2, then $\pi' = (n_1 - 2) + (n_2 + 2) + \ldots + n_m$ $(n_2 + 1)$ $n_3 + \ldots + n_m$ be a composition positive integer n - 1 into m - 1 parts where $n_2 + 1$ and n_m are two even integer number, each n_i (i = 3,…, m - 1) is an odd integer number. Then, the number of such compositions is $c^{m-1} (e, o, e, n - 1)$

According to 3 and 4, we complete the proof of 2.

CONCLUSION

In this study, we study two kinds of identities between partitions and compositions by using Agarwal's combination method. Then, some new recurrences of such compositions are given. Another direction for research in this topic is to find some new identities between partitions and compositions.

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REFERENCES

Agarwal, A.K., 2003. An analogue of Euler's identity and new combinatorial properties of n-color compositions. J. Comput. Appl. Math., 160: 9-15.

Alladi, K., 1999. A Variation on a theme of Sylvester-a smoother road to Gollniz (Big) theorem. Discrete Math., 196: 1-11.

Andrews, G.E., 1976. The Theory of Partitions: Encyclopedia of Mathematics and its Applications. Addison-Wesley Publ. Co., London-Amsterdam-Don Mills-Sydney-Tokyo.

Andrews, G.E., 1984. Ramanujan's lost notebook IV: Stacks and alternating partitions. Adv. Math., 53: 55- 74.

Bi, X., Z. Yan and T. Zhao 2008. Two new positive integer partition identities. J. Yangtze Univ. Natural Sci. Edn. Sci. Eng. V, 5(1): 28-29.

Guo, Y., 2007. Some identities between partitions and compositions. Acta Mathematica Sinica (Chinese Series), 50(3): 707-710.

Huang, F. and B. Liu. 2009. Some identities of compositions. Acta Math. Sinica Chinese Series, 52(2): 403-408.

Macmahon, P.A., 1894. Memoir on the compositions of numbers. Philos, Trans. Roy. Soc. London A, 184: 835-901.

Pang, R., 2011. A discuss of partition and composition. J. Sichuan Normal Univ. Natural Sci., 34(3): 321-324.

You, L. and L. Xing, 2010. Some new identities between partitions and compositions. J. South China Normal Univ. Natural Sci. Edn., 1: 20-23.