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Research Article

The Numerical Solution of a Class of Delay Differential Equations Boundary

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Abstract: In this study, we propose a new method for the Cauchy solution of the Common Delay Differential Equations $\dot{x}$ (t) = f(t, x(t), x(t - τ_1 (t)),…, x(t - τ_m (t))). Moreover, they exist only and approximation of equations boundary were especially proposed. From our analyze, we could get the results that Cauchy solution of the Common Delay Differential Equations play an important role in the network information processing.

Keywords: Cauchy, common delay differential equations, equations boundary, network information processing

INTRODUCTION

It is well known that a dynamical system is usually expressed as the form of a calculus equation:

$$\dot{x} = f(t, x),\ x \in R^n \tag{1}$$

the delay of dynamical systems is inevitable, even through information systems based on the speed of light is no exception. In this sense, the above equation is an approximate description of dynamical systems, which delay factor, is omitted. But he necessary precision will not be reached even leads to the wrong system or do not establish the mathematical model system, so as to use the theories and methods of delay Differential Equations to solve a variety of forms. Henry and Penney (2004) study the differential equations and boundary value problems computing and modeling. (Guo, 2001) make a detailed research of the nonlinear functional analysis. Shi *et al*. (2005) analyze the differential equation theory and its application. Kuang (1999) analyze the differential equation theory and its application. Sun (1997) have a research of the measurement of the motor winding temperature rise. Dong (2011) analyzes the measurement and calculation of the motor winding temperature rise.

In this study, we have a research of the numerical solution of a class of delay differential equations boundary. We propose a new method for the Cauchy solution of the Common Delay Differential Equations. Moreover, they exist only and approximation of equations boundary were especially proposed. From our analyze, we could get the results that Cauchy solution of the Common Delay Differential Equations play an important role in the network information processing.

PROBLEM FORMULATION

Common Delay Differential Equations can be summarized in two forms; the 1st form is as follows:

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t))) \tag{2}$$

in this equation, τ_i (t) ≥0, i = 1, 2,…, m which reflects the lagged effect of the limited time at the state of the current system.

Make $\tau(t) = \max_{1\le i\le m} \tau_i(t)$, when $\tau(t) \le r < \infty$, Eq. (2) known as functional differential equations with finite delay. Otherwise, known as functional differential equations with infinite delay. The 2nd form is as follows:

$$\dot{x}(t) = \int_0^{\sigma(t)} g(t, x(t), x(t-\tau))d\tau, \quad \sigma(t) \ge 0. \tag{3}$$

We call Eq. (3) functional differential equations with distributed delay.

When $|\sigma(t)| \le r < \infty$, functional differential Eq. (3) known as functional differential equations with finite delay. Otherwise, known as functional differential equations with infinite delay.

Equation (3) reflects the state of a system affect the current state of the system. The above two methods can be integrated to a very general form within the framework.

As $r \in (0, \infty)$, $C = C([-r, 0], R^n)$, $\forall\ \varphi \in C$, If the norm is defined as:

$$||\varphi|| = \sup_{\theta \in [-r, 0]} |\varphi(\theta)|,\ \varphi \in C$$

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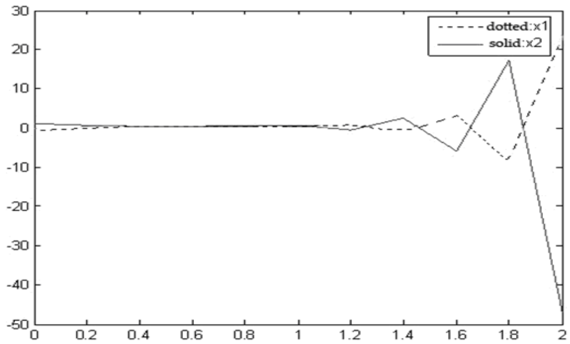

Fig. 1: The solution for the delayed state Eq. (5)

the C was given uniformly convergent topology. The Hutchison initial time is $\sigma \in R$. As $A \in (0, r)$ or ∞, for $x \in C([\sigma - r, \sigma + A], R^n)$, and $\forall\ t \in [\sigma, \sigma + A]$, If def. $x_t = x_t\ (\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, then $x_t \in C$. Assume that $\Omega \subset R\times C$ is an open set, f: $\Omega \to R^n$. Then equation:

$$\dot{x}(t) = f(t, x_t) \tag{4}$$

called Delay Differential Equations on Ω.

Definition 1: High-level overview of a variety of Retarded Functional Differential Equations, just select the appropriate operator $f(t, \varphi)$, Eq. (1.3) can represent a variety of Retarded Functional Differential Equations. For example, if we can take $f(t, \varphi) = f(t, \varphi(0), \varphi(-\tau_1(t)), \ldots \varphi(-\tau_m(t)))$. Then (4) become (2). If we want to determine the solution of Delay Differential Equations, only give the state of the system at a given moment is not enough, we need to give the state of the system at some point before this moment for some time that is, given an initial function. So, Delayed Functional Differential Equations Cauchy problem in the form as follows (Fig. 1):

$$\begin{cases} \dot{x}(t) = f(t, x_t) \\ \dot{x}_\sigma = \varphi \end{cases} \tag{5}$$

where, $\sigma \in R$ is the initial moment and $\varphi \in C$ is the initial function.

Here is the definition of Delayed Functional Differential Equation:

Definition 2: Assume that function $x \in C([\sigma - r, \sigma + A), R^n$, when $t \in [\sigma, \sigma + A)$, $(t, x_t) \in \Omega \sqsubset R\times C$.

- If x(t) satisfies the Eq. (3), then X(t) is called a solution when the Eq. (4) in the interval $[\sigma - r, \sigma + A)$
- If X (t) is called a solution when the Eq. (4) in the interval $[\sigma - r, \sigma + A)$ and $x_\sigma = \varphi$, then x (t) is called a solution through (σ, φ), as know as the solution of Delayed Functional Differential Equations Cauchy problem (5), (2) form of Cauchy problem numerical solution method for solving, especially working on the equation has an important role in the network information processing:

$$\begin{cases} \dot{x}_1(t) = -a_1 x_1(t) + b_1 f_1(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))) + I_1 \\ \dot{x}_2(t) = -a_2 x_2(t) + b_2 f_2(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))) + I_2 \end{cases} \tag{6}$$

Here, $\alpha_i>0$, b_i, I_i (i = 1, 2) are constant, $\tau_{ij}(t) \in C(R, [0, \infty))$ are bounded, $f_i \in C(R^2, R)$ (i, j = 1, 2), $x_i(t) \in C^1([0, \infty), R)$ (i = 1, 2).

THE SOLUTION EXISTENCE AND UNIQUENESS THEOREM

Assume that $\sigma \in R$, $\varphi \in C$, $\Omega \subset R\times C$ is an open set, operator $f \in C(\Omega, R^n)$, hen following conclusions are about Cauchy problems:

Lemma 1: Assume that $\Omega \subset R\times C$ is an open set, operator $f^0 \in C\ (\Omega, R^n)$, The Cauchy Problem (5) for any $(\sigma, \varphi) \in \Omega$, the solution exists.

Lemma 2: Assume that $\Omega \subset R\times C$ is an open set, $W \subset \Omega$ is a compact set, $f^0 \in C\ (\Omega, R^n)$ given. Then there exists a neighborhood of W $V \in \Omega$, Make $f^0 \in C^0\ (V, R^n)$. Move forward a single step, there exists a neighborhood of f^0 $U \subset C^0\ (V, R^n)$, for $\forall\ f \in U$, $(\sigma, \varphi) \in \Omega$, problem (5) existence of the solution $x(\sigma, \varphi, f)$ (t).

Lemma 3: Assume that $\Omega \subset R\times C$ is an open set, $f \in C\ (\Omega, R^n)$ and $f(t, \varphi)$ each compact subset in Ω about φ Satisfies the Lipschitz condition:

$$\| f(t,\varphi) - f(t,\psi) \| \le L \| \varphi - \psi \| \tag{7}$$

Here, $L \in R^+$ is Lipschitz constant. Then problem (5) exists a unique solution.

The following discussion based on (7) is satisfied under the premise that the existence and uniqueness is guaranteed. Numerical method our idea is into long differential equations appropriate qualitative considerations. So that you can facilitate the use of classical numerical methods have been established to solve problems, for example, Runge-Kutta method. The following 2 cases to consider Introduction 1:

The 1st situation:

$$\begin{cases} \dot{x}(t) = f(t, x(t - \tau(t))) \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{8}$$

Here $x(t) = (x_1(t), x_2(t),\ldots, x_n(t))^T$ is the n-dimensional vector function:

$\varphi(t) = (\varphi_1(t), \varphi_2(t),\ldots, \varphi_n(t))^T$
is the Known n-dimensional vector function:

$f \in C(\Omega, R^n)$, $\tau(t) \in C^1(R, [0, \infty))$

monotonically decreasing and:

$\lim_{t\to\infty}\tau(t) = 0$ (usually there is $\tau(t) \sim \lambda e^{-\lambda t}$)

$r > r_0 = \max_{t \geq 0} \tau(t)$

Let $S(t) = t - \tau(t)$, and $\tau(t)$ monotonically decreasing makes $S(t)$ monotonically increasing, therefore, there is a continuous inverse function $t = t(S)$. Substituted into the (8) then we got:

$$\begin{cases} \dot{x}(s) = \tilde{f}(s, x(s)) \\ x(s) = \varphi(s), \quad s \in A = \{s \mid s = t - \tau(t), t \in [-r, 0]\} \bigcap [-r, 0] \end{cases}$$

where, $\dot{x}(S) = \dot{x}(t).\frac{dt}{ds}$, $\tilde{f}(S, x(S)) = f(t(S)).\frac{dt}{ds}$. Because $r>r_0 = \max_{t \geq 0} \tau(t)$, so $A \neq \emptyset$, as a matter of fact, $S_0 = S(0) = -\tau(0) \in [-r, 0]$, the, (8) conversion for the following appropriate qualitative issues:

$$\begin{cases} \dot{x}(s) = \tilde{f}(s, x(s)) \\ x(s_0) = \varphi(s_0) \quad s_0 = -\tau(0) \end{cases} \tag{9}$$

at this point, we can choose the following Runge-Kutta method of Simpson formula to solve the problem.

$$\begin{cases} x_{m+1} = x_m + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 = \tilde{f}(t_m, x_m) \\ k_2 = \tilde{f}(t_m + \frac{h}{2}, x_m + \frac{h}{2}k_1) \\ k_3 = \tilde{f}(t_m + \frac{h}{2}, x_m + \frac{h}{2}k_2) \\ k_4 = \tilde{f}(t_m + h, x_m + \frac{h}{2}k_3) \\ x_0 = x(s_0) \end{cases} \tag{10}$$

of course we can also use other formulas of the Runge-Kutta method.

Now we study (5) form of the Cauchy problem to the following special case of solving:

$$\begin{cases} \dot{x}_1(t) = -a_1 x_1(t) + b_1 f_1(x_1(t-\tau(t)), x_2(t-\tau(t))) \\ \dot{x}_2(t) = -a_2 x_2(t) + b_2 f_2(x_1(t-\tau(t)), x_2(t-\tau(t))) \\ x_1(t) = \varphi_1(t), \quad x_2(t) = \varphi_2(t), \quad t \in [-r, 0], r > r_0 = \max_{t \geq 0} \tau(t) \end{cases} \tag{11}$$

here, $x_1(t), x_2(t) \in C^1([0, \infty), R)$, $f_i \in C(R^2, R)(i, j = 1, 2)$.

Set $x(t) = (x_1(t), x_2(t))^T$, in the transformation $x(t) = A(t)\tilde{x}(t)$ $\tilde{x}(t) = \tilde{x}_1(t), \tilde{x}_2(t))^T$, $A(t)$ as function matrix:

$$\begin{pmatrix} e^{-a_1 t} & 0 \\ 0 & e^{-a_2 t} \end{pmatrix}$$

Then (11) becomes:

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t-\tau(t)))$$

In this equation,

$$\tilde{f} = (\tilde{f}_1, \tilde{f}_2)^T$$

$$\tilde{f}_1 = b_1 f_1(\tilde{x}_1(t-\tau(t))e^{-a_1(t-\tau(t))}, \tilde{x}_2(t-\tau(t))e^{-a_2(t-\tau(t))})$$

$$\tilde{f}_2 = b_2 f_2(\tilde{x}_1(t-\tau(t))e^{-a_1(t-\tau(t))}, \tilde{x}_2(t-\tau(t))e^{-a_2(t-\tau(t))})$$

At this point, the method described above can be solved. The 2nd situation:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_m(t))) \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{12}$$

Here $x(t) = (x_1(t), x_2(t),\ldots, x_n(t))^T$ is the n-dimensional vector function, $\varphi(t) = (\varphi_1(t), \varphi_2(t),\ldots, \varphi_n(t))^T$ is the Known n-dimensional vector function:

$f \in C(\Omega, R^n)$, $\tau_i(t) \in C(R, [0, \infty))$

$(i = 1, 2, \ldots, m)$ monotonically decreasing also:

$\lim_{t\to\infty}\tau_i(t) = 0$, (usually there is $\tau(t) \sim \lambda e^{-\lambda t}$)

$r>r_0 = \max_{t \geq 0, 0 \leq i \leq m} \tau_i(t)$

First, we consider the case when $m = 1$, In this case, (13) becomes:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau(t)),) \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{13}$$

Here $x(t) = (x_1(t), x_2(t),\ldots, x_n(t))^T$ is the n-dimensional vector function, $\varphi(t) = (\varphi_1(t), \varphi_2(t),\ldots, \varphi_n(t))^T$ is the Known n-dimensional vector function:

$f \in C(\Omega, R^n)$, $r > r_0 = \max_{t \geq 0} \tau(t)$

$\tau(t) \in C(R, [0, \infty))$ monotonically decreasing also:

$\lim_{t\to\infty} \tau(t) = 0$, (usually get $\tau(t) \sim \lambda e^{-\lambda t}$)

Numerical solution in the form given below using the Runge-Kutta method for solving the above problems.

Make (a, b, c) to a given Runge-Kutta method, which i.e., $a = (a_{ij})_{s\times s}$, $b = (b_1, b_2, \ldots b_s)^T$, $c = (c_1, c_2, \ldots c_s)^T$, $\sum_{i=1}^{s} b_i = 1$, ci ∈ [0, 1](1≤i ≤s). The specific form of the corresponding Runge-Kutta method as follows:

$$\begin{cases} X_i = x_{n-1} + h\sum_{j=1}^{s} a_{ij} f(t_{n-1} + c_j h, X_j) \\ x_i = x_{n-1} + h\sum_{j=1}^{s} b_j f(t_{n-1} + c_j h, X_j) \end{cases} \quad (i = 1, 2, \cdots, s) \tag{14}$$

Thus, the specific form of the corresponding Runge-Kutta method in Delay Differential:

$$\begin{cases} X_i = x_{n-1} + h\sum_{j=1}^{s} a_{ij} f(t_{n-1} + c_j h, X_j, Z_j^{(n)}) \\ x_i = x_{n-1} + h\sum_{j=1}^{s} b_j f(t_{n-1} + c_j h, X_j, Z_j^{(n)}) \end{cases} \quad (i = 1, 2, \cdots, s) \tag{15}$$

On the time delay options $x(t - \tau(t))$ processing is the key to solving problems. If you use the symbol "~" exact and numerical solutions of the approximate relations, then $x_n \sim x(t_n)$, $Z_j^n \sim x(t_n + c_j h - \tau(t_n + c_j h))$ It said the n-th iteration approximation Delay Term which $t_n = t_0 + n.h$, $t_0 = 0$ especially, if $\tau(t)$ is a constant, when we take a step stance: $h = \tau/k$(where k is a positive integer), get $t_n + c_j h - \tau = t_{n-k} + c_j h$, then $Z_j^n \sim x(t_{n-k} + c_j h)$.

when $t_n + c_j h - \tau(t_n + c_j h) \leq 0$, we get:

$$Z_j^{(n)} = \varphi(t_n + c_j h - \tau(t_n + c_j h)) \tag{16}$$

(13) and (14) as a whole constitute solving (11) Runge-Kutta method. Matlab or C++ programming for solving the above numeric format can be.

For the case when m≥2. At this point, (10) becomes:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_m(t))) \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{17}$$

Here $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is n-dimensional vector function:

$\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$

is the Known n-dimensional, $f \in C(\Omega, R^n)$

$r > r_0 = \max_{t \geq 0, 1 \leq i \leq m} \tau_i(t)$

$\tau_i(t) \in C(R, [0, \infty))$

(i = 1, 2,…, m) monotonically decreasing and

$\lim_{t\to\infty} \tau_i(t) = 0$ (i = 1, 2,…, m)

(usually get $\tau_i(t) \sim \lambda e^{-\lambda t}$)

At this point, you can get the corresponding Runge-Kutta iteration scheme as follows:

$$\begin{cases} X_i = x_{n-1} + h\sum_{j=1}^{s} a_{ij} f(t_{n-1} + c_j h, X_j, Z_{1j}^{(n)}, Z_{2j}^{(n)}, \cdots, Z_{mj}^{(n)}) \\ x_i = x_{n-1} + h\sum_{j=1}^{s} b_j f(t_{n-1} + c_j h, X_j, Z_{1j}^{(n)}, Z_{2j}^{(n)}, \cdots, Z_{mj}^{(n)}) \end{cases} \quad (i = 1, 2, \cdots, s) \tag{18}$$

$$x_n \sim x(t_n),\ Z_{ij}^n \sim x(t_n + c_j h - \tau_i(t_n + c_j h)) \tag{19}$$

when $t_n + c_j h - \tau_i(t_n + c_j h) \leq 0$, we get:

$$Z_{ij}^{(n)} = \varphi(t_n + c_j h - \tau_i(t_n + c_j h))(i = 1, 2, \cdots, m) \tag{20}$$

Parametric c_i, b_i, a_{ij} can be expressed as the Butcher array form, is:

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \vdots & \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Determine the coefficients implicit Runge-Kutta formula, follow these steps:

- Find the S-order polynomial S-zeros $c_1, c_2, \ldots c_s$ of $P_s(2c - 1)$ (This can look-up table), here $P_s(x)$ is the s-order Legendre polynomial
- For each i(1≤i≤s), Solution of equations $\sum_{j-1}^{s} a_{ij} c_j^{k-1} = 1/k\ c_i^k$, k = 1, 2, …, s
- From the current equations $\sum_{j-1}^{s} b_j c_j^{k-1} = 1/k$, k = 1, 2, …, s

Determine the elements of $\{b_i\}$ is here $c_1 \neq 0$.

Commonly used formula for the Butcher is:

$$
\begin{array}{c|ccc}
\frac{5-\sqrt{15}}{10} & \frac{5}{36} & \frac{10-3\sqrt{15}}{45} & \frac{25-6\sqrt{15}}{180} \\
\frac{1}{2} & \frac{10+3\sqrt{15}}{72} & \frac{2}{9} & \frac{10-3\sqrt{15}}{72} \\
\frac{5+\sqrt{15}}{10} & \frac{25+6\sqrt{15}}{180} & \frac{10+3\sqrt{15}}{45} & \frac{5}{36} \\
\hline
 & \frac{5}{18} & \frac{4}{9} & \frac{5}{18}
\end{array}
$$

NUMERICAL EXAMPLE

Now we work on the introduction of (5). Specifically consider the problems:

$$\dot{x}_1(t) = -4x_1(t) + \frac{1}{20} f_1(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))) + 2$$

$$\dot{x}_2(t) = -4x_2(t) + \frac{1}{5} f_2(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))) + 1$$

here,

$$f_1(x_1, x_2) = \cos(\frac{1}{4}x_1) + \frac{1}{8}x_1 + \sin(\frac{1}{3}x_2) - \frac{1}{6}|x_2| + 1$$

$$f_2(x_1, x_2) = \sin(\frac{1}{2}x_1) - \frac{1}{6}|x_1| + \cos(\frac{1}{4}x_2) + \frac{1}{8}x_2 + 2$$

$\tau_{ij}(t) = \lambda_{ij}e^{-\lambda_{ij}t}$). (i, j = 1, 2) and $\lambda_{11} = 0.8$, $\lambda_{12} = 1.3$, $\lambda_{21} = 4.7$, $\lambda_{22} = 1.0$.

Can get $r_0 = \max_{t \geq 0} \tau_{ij}(t) = 4.7$

Known initial conditions as follows:

$$(x_1(t), x_2(t))^T = (\sin t, \cos t)^T, t \in [-5, 0]$$

Make $\dot{x}(t) = (x_1(t), x_2(t))^T$,

$$f(t, x(t), x(t-\tau_{11}(t)), x(t-\tau_{12}(t)), x(t-\tau_{21}(t)), x(t-\tau_{22}(t))) = (f_1, f_2)^T$$

here,

$$f_1 = -4x_1(t) + \frac{1}{20} f_1(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))) + 2$$

$$f_2 = -4x_2(t) + \frac{1}{5} f_2(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))) + 1$$

Based on the above analysis, we can use the following format to solve the problem:

$$\begin{cases} X_i = x_{n-1} + h\sum_{j=1}^{s} a_{ij} f(t_{n-1} + c_j h, X_j, Z_{1j}^{(n)}, Z_{2j}^{(n)}, \cdots, Z_{mj}^{(n)}) \\ x_i = x_{n-1} + h\sum_{j=1}^{s} b_j f(t_{n-1} + c_j h, X_j, Z_{1j}^{(n)}, Z_{2j}^{(n)}, \cdots, Z_{mj}^{(n)}) \end{cases} \quad (i = 1, 2, \cdots, s)$$

$x_n \sim x(t_n)$, $Z_{ij}^{n} \sim x(t_n + c_j h - \tau_i(t_n + c_j h))$

When $t_n + c_j h - \tau_i(t_n + c_j h) \leq 0$, we get:

$$Z_{ij}^{(n)} = \varphi(t_n + c_j h - \tau_i(t_n + c_j h))(i = 1, 2, \cdots, m)$$

CONCLUSION

The Cauchy solution was given a new method to the key, especially science and numeral of the exist and only and approximation and equations boundary was belong to itself, which has an important role in the network information processing.

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