

Research Article

Mathematical Modeling and Algebraic Technique for Resolving a Single-Producer Multi-Retailer Integrated Inventory System with Scrap

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Abstract: This study uses mathematical modeling along with an algebraic technique to resolve the production-distribution policy for a single-producer multi-retailer integrated inventory system with scrap in production. We assume that a product is manufactured through an imperfect production process where all nonconforming items will be picked up and scrapped in each production cycle. After the entire lot is quality assured, multiple shipments will be delivered synchronously to m different retailers in each cycle. The objective is to determine the optimal replenishment lot size and optimal number of shipments that minimizes total expected costs for such a specific supply chains system. Conventional method is by the use of differential calculus on system cost function to derive the optimal policy (Chiu *et al.*, 2012c), whereas the proposed algebraic approach is a straightforward method that enables practitioners who may not have sufficient knowledge of calculus to understand and manage more effectively the real-life systems.

Keywords: Algebraic approach, inventory, multiple retailers, multiple shipments, production lot size, scrap, supply chains

INTRODUCTION

The mathematical techniques for determining the most economical production lot for an inventory model was first proposed by Taft (1918). It is also known as Economic Production Quantity (EPQ) model (Hillier and Lieberman, 2001; Nahmias, 2009). EPQ model assumes a continuous inventory issuing policy for meeting customer's product demand. However, in real world vendor-buyer business environment, the multiple or periodic delivery are often used. Hence, one of the critical decisions for effectively managing such a vendor-buyer integrated system is to accurately determine the optimal number of delivery so that the total production-delivery costs can be minimized. Schwarz (1973) first investigated a one-warehouse N-retailer inventory system with the objective of determining the stock refilling policy that minimizes the expected cost per unit time for the proposed model. Schwarz *et al.* (1985) examined the system fill-rate of a one-warehouse N-identical retailer distribution system as a function of warehouse and retailer safety stock. They used an approximation model from a prior study to maximize system fill-rate subject to a constraint on system safety stock. As results, properties of fill-rate policy lines are suggested and these may be used to provide managerial insight into system optimization and as the basis for heuristics. Many studies on

different aspects of the supply chain optimization have since been extensively conducted (Goyal, 1977; Banerjee, 1986; Aderohunmu *et al.*, 1995; Parija and Sarker, 1999; Abdul-Jalbar *et al.*, 2005; Hoque, 2008; Sarker and Diponegoro, 2009; Chiu *et al.*, 2011).

Product quality issue in the vendor side is another special focus of the present study. One notes that the classic EPQ model implicitly assumes all items produced are of perfect quality. However, in a real life manufacturing environment, due to unpredictable factors it is inevitable to generate defective items. Shih (1980) considered two inventory models to the case where the proportion of defective units in the accepted lot is a random variable with known probability distributions. Optimal solutions to these amended systems were developed and comparisons with the traditional models were also presented via numerical examples. Many studies have since been conducted to address the issues of defectiveness and quality assurance in production systems (Bielecki and Kumar, 1988; Chern and Yang, 1999; Teunter and Flapper, 2003; Ojha, *et al.*, 2007; Chiu *et al.*, 2009a, b; Sana, 2010; Lee *et al.*, 2011; Chiu *et al.*, 2012b). In this study, we consider that all defective items will be picked up at the screening and scrapped in each production cycle. After the entire lot is quality assured, multiple shipments are delivered synchronously to m different retailers in each cycle. The objective of this

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study is to determine the optimal replenishment lot size and optimal number of shipments that minimizes total expected costs for such a specific supply chains system.

Conventional approach uses the differential calculus on system cost function to derive the optimal policy. Recently, Grubbström and Erdem (1999) proposed an algebraic derivation to solve the Economic Order Quantity (EOQ) model with backlogging. Their algebraic approach does not reference to the first- or second-order derivatives. Similar approaches were applied to solve various aspects of production and supply chain optimization (Chiu, 2008; Lin *et al.*, 2008; Chiu *et al.*, 2010; Chen *et al.*, 2012). This study extends such an algebraic approach to resolve a single-producer multi-retailer integrated inventory model with scrap in production (Chiu *et al.*, 2012c).

MODEL DESCRIPTION AND MODELLING

In this study, we proposed an algebraic technique to resolve a single-producer multi-retailer integrated inventory model with scrap in production (Chiu *et al.*, 2012c). The specific supply chains model studied here is described as follows. Consider that a product can be made at an annual production rate P , all items produced are screened and the inspection expense is included in the unit production cost C . The process may randomly generate an x portion of defective items at a rate d and all defective items picked up are assumed to be scrap, they are discarded in each cycle. Under the normal operation, to prevent shortages from occurring, the constant production rate P must satisfies $(P-d-\lambda)>0$, where λ is the sum of annual demands of retailers and $d = Px$.

This study considers a multi-shipment policy and the finished items can only be delivered to the retailers when the entire lot is quality assured in the end of regular production. Each retailer has its own annual demand rate λ_i . Fixed quantity n installments of the finished batch are delivered to multiple retailers synchronously at a fixed interval of time during the downtime t_2 (Fig. 1).

The cost related parameters used in this study include: the unit production cost C , unit disposal cost C_s , production setup cost K , unit holding cost h , the fixed delivery cost K_{1i} per shipment delivered to retailer i , unit shipping cost C_i for item shipped to retailer i and unit holding cost h_{2i} for item kept by retailer i . Other notation also includes:

- n = Number of fixed quantity installments of the finished batch to be delivered to retailers for each cycle, a decision variable (to be determined)
- Q = Production lot size per cycle, a decision variable (to be determined)
- m = Number of retailers

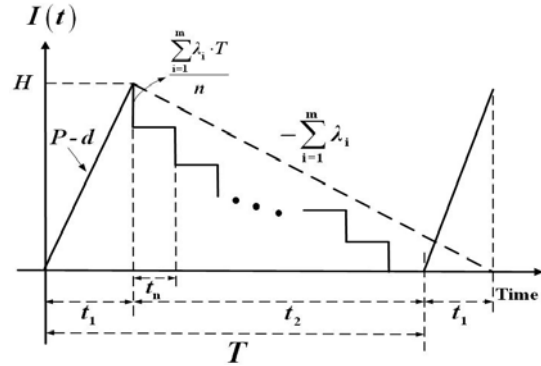


Fig. 1: On-hand inventory of perfect quality products in vendor side

- H = Maximum level of on-hand inventory in units when the production process ends
- t_1 = The production uptime for the proposed system
- t_2 = Time required for delivering all quality assured finished products to retailers
- t_n = A fixed interval of time between each installment of finished products delivered during production downtime t_2
- T = Production cycle length
- $I(t)$ = On-hand inventory of perfect quality items at time t
- $I_c(t)$ = On-hand inventory at the retailers at time t
- $TC(Q, n)$ = Total production-inventory-delivery costs per cycle for the proposed system
- $E[TCU(Q, n)]$ = Total expected production-inventory-delivery costs per unit time for the proposed system

Figure 1 and with reference to the work of Chiu *et al.* (2012c), one notes that the total production-inventory-delivery cost per cycle $TC(Q, n)$ consists of the following (as shown in Eq. (1)): $TC(Q, n)$ consists of the variable production cost, setup cost, disposal cost, fixed and variable delivery cost, holding cost during production uptime t_1 and holding cost for finished goods kept by both the manufacturer and the customer during the delivery time t_2 :

$$TC(Q, n) = CQ + K + C_s[xQ] + n \sum_{i=1}^m K_{1i} + \sum_{i=1}^m C_i \lambda_i T + h \left[\frac{H + dt_1}{2} (t_1) + \left(\frac{n-1}{2n} \right) H t_2 \right] + \frac{1}{2} \sum_{i=1}^m h_{2i} \lambda_i \left[\frac{T t_2}{n} + T t_1 \right] \tag{1}$$

Taking the randomness of scrap rate into account, the expected values of x is used in the cost analysis. Substituting all parameters (Chiu *et al.*, 2012c) and with further derivations, $E[TCU(Q, n)]$ can be obtained:

$$\begin{aligned}
 E[TCU(Q, n)] &= \frac{E[TC(Q, n)]}{E[T]} = \frac{C \sum_{i=1}^m \lambda_i}{(1-E[x])} + \frac{\left(K + n \sum_{i=1}^m K_{li}\right) \sum_{i=1}^m \lambda_i}{Q(1-E[x])} + \frac{C_s E[x] \sum_{i=1}^m \lambda_i}{(1-E[x])} \\
 &+ \sum_{i=1}^m C_i \lambda_i + \frac{hQ \sum_{i=1}^m \lambda_i}{2P(1-E[x])} + \left(\frac{n-1}{n}\right) \left(\frac{hQ \sum_{i=1}^m \lambda_i}{2}\right) \left[\frac{(1-E[x])}{\sum_{i=1}^m \lambda_i} - \frac{1}{P}\right] \\
 &+ \left(\frac{n-1}{n}\right) \left(\frac{\sum_{i=1}^m h_{2i} \lambda_i Q}{2}\right) \left[\frac{1}{P}\right] + \left(\frac{1}{n}\right) \frac{\sum_{i=1}^m h_{2i} \lambda_i Q (1-E[x])}{2 \sum_{i=1}^m \lambda_i}
 \end{aligned} \tag{2}$$

THE PROPOSED ALGEBRAIC APPROACH

One notes that Eq. (2) contains Q and n two decision variables and there are several forms of decision variables Q and n in the right-hand side of Eq. (2): such as Q , Q^{-1} , nQ^{-1} and Qn^{-1} . Therefore, one can rearrange Eq. (2) as:

$$\begin{aligned}
 E[TCU(Q, n)] &= \frac{C \sum_{i=1}^m \lambda_i}{(1-E[x])} + \frac{C_s E[x] \sum_{i=1}^m \lambda_i}{(1-E[x])} + \sum_{i=1}^m C_i \lambda_i + \frac{(K) \sum_{i=1}^m \lambda_i}{(1-E[x])} (Q^{-1}) \\
 &+ \frac{1}{2} \left\{ \left(h \sum_{i=1}^m \lambda_i \right) \left[\frac{E[x]}{P(1-E[x])} + \frac{(1-E[x])}{\sum_{i=1}^m \lambda_i} \right] + \left(\frac{\sum_{i=1}^m h_{2i} \lambda_i}{2} \cdot \frac{1}{P} \right) \right\} (Q) \\
 &+ \frac{\left(\sum_{i=1}^m K_{li}\right) \sum_{i=1}^m \lambda_i}{(1-E[x])} (nQ^{-1}) + \frac{1}{2} \left(\sum_{i=1}^m h_{2i} \lambda_i - h \sum_{i=1}^m \lambda_i \right) \left[\frac{(1-E[x])}{\sum_{i=1}^m \lambda_i} - \frac{1}{P} \right] (n^{-1}Q)
 \end{aligned} \tag{3}$$

Let:

$$\gamma_1 = \frac{C \sum_{i=1}^m \lambda_i}{(1-E[x])} + \frac{C_s E[x] \sum_{i=1}^m \lambda_i}{(1-E[x])} + \sum_{i=1}^m C_i \lambda_i \tag{4}$$

$$\gamma_2 = \frac{(K) \sum_{i=1}^m \lambda_i}{(1-E[x])} \tag{5}$$

$$\gamma_3 = \frac{1}{2} \left\{ \left(h \sum_{i=1}^m \lambda_i \right) \left[\frac{E[x]}{P(1-E[x])} + \frac{(1-E[x])}{\sum_{i=1}^m \lambda_i} \right] + \left(\frac{\sum_{i=1}^m h_{2i} \lambda_i}{2} \cdot \frac{1}{P} \right) \right\} \tag{6}$$

$$\gamma_4 = \frac{\left(\sum_{i=1}^m K_{li}\right) \sum_{i=1}^m \lambda_i}{(1-E[x])} \tag{7}$$

$$\gamma_5 = \frac{1}{2} \left(\sum_{i=1}^m h_{2i} \lambda_i - h \sum_{i=1}^m \lambda_i \right) \left[\frac{(1-E[x]) - \frac{1}{P}}{\sum_{i=1}^m \lambda_i} \right] \quad (8)$$

Eq. (3) becomes:

$$E[TCU(Q, n)] = \gamma_1 + \gamma_2(Q^{-1}) + \gamma_3(Q) + \gamma_4(nQ^{-1}) + \gamma_5(n^{-1}Q) \quad (9)$$

With rearrangement of Eq. (9) one obtains:

$$E[TCU(Q, n)] = \gamma_1 + (Q^{-1})[\sqrt{\gamma_2} - \sqrt{\gamma_3} \cdot Q]^2 + (nQ^{-1})[\sqrt{\gamma_4} - \sqrt{\gamma_5}(n^{-1}Q)]^2 + 2\sqrt{\gamma_2\gamma_3} + 2\sqrt{\gamma_4\gamma_5} \quad (10)$$

One notes that Eq. (10) will be minimized if its second and third terms equal zero. That is:

$$Q = \sqrt{\frac{\gamma_2}{\gamma_3}} \quad (11)$$

and:

$$n = \sqrt{\frac{\gamma_5}{\gamma_4}} \cdot Q \quad (12)$$

Substituting Eq. (5) and (6) in Eq. (11) and substituting Eq. (7), (8) and (11) into Eq. (12), the optimal number of shipments n^* is:

$$n^* = \sqrt{\frac{K \left(\sum_{i=1}^m h_{2i} \lambda_i - h \sum_{i=1}^m \lambda_i \right) (1-E[x]) \left[(1-E[x]) \left(\sum_{i=1}^m \lambda_i \right)^{-1} - \left(\frac{1}{P} \right) \right]}{\sum_{i=1}^m K_{1i} \left\{ \frac{h}{P} \sum_{i=1}^m \lambda_i + \left(\sum_{i=1}^m h_{2i} \lambda_i - h \sum_{i=1}^m \lambda_i \right) \left(\frac{1}{P} \right) (1-E[x]) \right\}}} \quad (13)$$

One notes that Eq. (13) is identical to that obtained by using the conventional differential calculus method (Chiu *et al.*, 2012c).

Now, in order to find the integer value of n^* that minimizes the expected system cost, the two adjacent integers to n must be examined respectively for cost minimization (Chiu *et al.*, 2012d). Let n^+ denote the smallest integer greater than or equal to n (derived from Eq. (13)) and n^- denote the largest integer less than or equal to n . Because n^* is either n^+ or n^- , we can first treat $E[TCU(Q, n)]$ (Eq. (2)) as a cost function with a single decision variable Q and do the following rearrangements:

$$E[TCU(Q, n)] = \gamma_1 + (\gamma_2 + \gamma_4 n)(Q^{-1}) + (\gamma_3 + \gamma_5 n^{-1})(Q) \quad (14)$$

or:

$$E[TCU(Q, n)] = \gamma_1 + (Q^{-1}) \left[\sqrt{\gamma_2 + \gamma_4 n} - (Q) \sqrt{\gamma_3 + \gamma_5 n^{-1}} \right]^2 + 2 \left(\sqrt{\gamma_2 + \gamma_4 n} \right) \left(\sqrt{\gamma_3 + \gamma_5 n^{-1}} \right) \tag{15}$$

In Eq. (15), one notes that E [TCU (Q, n)] will be minimized if the second term of the right-hand size of Eq. (15) equals zero. That is:

$$Q = \frac{\sqrt{\gamma_2 + \gamma_4 n}}{\sqrt{\gamma_3 + \gamma_5 n^{-1}}} \tag{16}$$

Substituting Eq. (5), (6), (7) and (8) in Eq. (16), the optimal production lot size is:

$$Q^* = \frac{2 \left(K + n \sum_{i=1}^m K_{1i} \right) \sum_{i=1}^m \lambda_i}{\left\{ \frac{h \sum_{i=1}^m \lambda_i}{P} + \left(\frac{n-1}{n} \right) \left[h(1-E[x])^2 + \left(\sum_{i=1}^m h_{2i} \lambda_i - h \sum_{i=1}^m \lambda_i \right) \left(\frac{1}{P} \right) (1-E[x]) \right] \right\} + \frac{\sum_{i=1}^m h_{2i} \lambda_i (1-E[x])^2}{n \sum_{i=1}^m \lambda_i}} \tag{17}$$

One notes Eq. (17) is identical to that obtained by using the conventional differential calculus method (Chiu *et al.*, 2012c).

Finally, in order to find the optimal (Q*, n*) policy one substitutes all related system parameters along with n+ and n- in Eq. (17) and then applying (Q, n+) and (Q, n-) in Eq. (3) respectively and selecting the one that gives the minimum expected system cost as the optimal (Q*, n*) policy.

Numerical example: This section uses a numerical example to verify the proposed algebraic approach and its resulting Eq. (13), (17) and (3), by using the same numerical example as given in Chiu *et al.* (2012c). In a single-producer multi-retailer integrated inventory model with scrap, suppose a product can be made at a rate P = 60,000 units per year and its annual demands λ_i from 5 different retailers are 400, 500, 600, 700 and 800, respectively (total demand λ = 3000 per year). The random defective rate x follows a uniform distribution over the range of [0, 0.3]. All defective items are considered to be scrapped items.

Other values of parameters are C = \$ 100, K = \$ 35000, h = \$ 20, h₁ = \$ 60, C_s = \$ 20, K_{1i} for retailer i are 100, 200, 300, 400 and 500 \$, respectively, h_{2i} for retailer i are 75, 70, 65, 60 and 55 \$ per item,

respectively, C_i for retailer i are 0.5, 0.4, 0.3, 0.2 and 0.1 \$, respectively.

We first determine the optimal number delivery by computing Eq. (13), one has n = 5.39. Then, examine the two adjacent integers to n and applying Eq. (17), one obtains (Q, n+) = (3231, 6) and (Q, n-) = (3122, 5). Finally, substituting these (Q, n+) and (Q, n-) in Eq. (3), respectively. Choosing the one that gives the minimum system cost as our optimal policy, one obtains (Q, n+) = (3122, 5) and total expected cost E [TCU (Q*, n*)] = \$ 460, 408.

One notes that the results from the aforementioned numerical example are confirmed to be identical to those obtained in Chiu *et al.* (2012c).

CONCLUSION

This study proposed a mathematical modeling along with algebraic derivation for resolving for the optimal production-shipment policy for a single-producer multi-retailer integrated inventory model with scrap in production. Unlike the conventional method by the use of differential calculus to find the optimal policy (Chiu *et al.*, 2012c), the proposed algebraic approach demonstrate a straightforward derivation that may enable students and/or practitioners with little

knowledge of calculus to understand and manage such a real-life supply chains system more effectively.

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