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Research Article

End-Regularity of the Join of n Split Graphs

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Abstract: A graph X is said to be End-regular if its endomorphism monoid End (X) is a regular semigroup. In this study, End-regular graphs which are the join of n split graphs are characterized. We give the conditions under which the endomorphism monoid of the join of n splits graphs is regular.

Keywords: Endomorphism, join of n graphs, regular semigroup, split graph

INTRODUCTION

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is try to establish the relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. Just as Petrich and Reilly pointed out in Petrich *et al.* (1999), in the great range of special classes of semigroups, regular semigroups take a central position from the point of view of richness of their structural "regularity". So it is natural to ask for which graph G the endomorphism monoid of G is regular (such an open question raised in Marki (1988). However, it seems difficult to obtain a general answer to this question. So the strategy for answering this question is to find various kinds of conditions of regularity for various kinds of graphs. In Wilkeit (1996) the connected bipartite graphs whose endomorphism monoids are regular were explicitly found. An infinite family of graphs with regular endomorphism monoids were provided in Li (2003) and the joins of two trees with regular endomorphism monoids were also characterized. Hou *et al.* (2008) explored the endomorphism monoid of the complement of a path p_n with n vertices. It was shown that the endomorphism monoid of the complement of a path is an orthodox semigroup. The split graphs and the join of split graphs with regular endomorphism monoids were studied in Li *et al.* (2001), Fan (1997) and Hou *et al.* (2012), respectively. The split graphs with orthodox endomorphism monoids were characterized in Fan (2002). In this study, we continue to explore the endomorphisms monoids of the joins of n split graphs and characterize such graphs whose endomorphism monoids are regular.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by V(X) and the edge set of X is denoted by E(X). The cardinality of the set V(X) is called the order of X. If two vertices x_1 and x_2 are adjacent in the graph X, the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and write $\{x_1, x_2\} \in E(X)$. For a vertex v of X, denote by $N_X(v)$ the set $\{x \in V(X) | \{x, v\} \in E(X)\}$ and called it the neighborhood of v in X, the cardinality of $N_x(v)$ is called the degree or valency of v in X and is denoted by $d_X(v)$. A subgraph H is called an induced subgraph of X if for any $a, b \in H$, $\{a, b\} \in H$ if and only if $\{a, b\} \in E(X)$. We denote by K_n a complete graph with n vertices. A clique of a graph X is the maximal complete sub graph of X. The clique number of X, denoted by $\varpi(X)$, is the maximal order among the cliques of X. Let $X_1, X_2, \ldots, X_n$ be n graphs. The join of $X_1, X_2, \ldots, X_n$, denoted by $X_1 + X_2 + \ldots + X_n$, is a graph with $V(X_1 + X_2 + \ldots + X_n) = V(X) \cup V(X_2) \cup \cdots \cup V(X_n)$ and $E(X_1 + X_2 + \ldots + X_n) = E(X_1) \cup E(X_2) \cup \cdots \cup E(X_n) \cup \{\{a, b\} | a \in V(X_i), b \in V(X_j)\}$, (where $i \neq j$).

Let G be a graph. A subset $K \subseteq V(G)$ is said to be complete if $\{a, b\} \in E(G)$ for any two vertices $a, b \in K$. A subset $S \subseteq V(G)$ is said to be independent if $\{a, b\} \notin E(G)$ for any two vertices $a, b \in S$. A graph X is called split graph if its vertex set V(X) can be partitioned into two disjoint (non-empty) sets S and K, such that S is an independent set and K is a complete set. We can always assume that any split graph X has a

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unique partition $V(X)=K\bigcup S$, where K a maximal is complete set and S is an independent set. Since K is a maximal complete set of X, it is easy to see that for any $y\in S$, $0\le d_X(y)\le n-1$, where $n=\varpi(X)$.

Let X and Y be two graphs. A mapping from V(X) to V(Y) is called a homomorphism if $\{a,b\}\in E(X)$ implies that $\{f(a),f(b)\}\in E(Y)$. A homomorphism from X to itself is called an endomorphism of X. An endomorphism f of X is said to be half-strong if $\{f(a),f(b)\}\in E(X)$ implies that there exist $c\in f^{-1}(a)$ and $d\in f^{-1}(b)$ such that $\{c,d\}\in E(X)$. Denote by End(X) and hEnd(X) the set of endomorphisms and half-strong endomorphisms of X. It is known that End(X) forms a monoid with respect to the composition of mappings and is called the endomorphism monoid (or briefly monoid) of X. Denote by Idpt(X) the set of all idempotents of End(X). It is known that every idempotent endomorphism is half-strong.

A retraction is a homomorphism f from a graph X to a sub graph Y of X such that the restriction f|Y of f to V(Y) is the identity map on V(Y). It is easy to see that the idempotents of End(X) are retractions. Let f be an endomorphism of a graph X. A sub graph of G is called the endomorphic image of G under f, denoted by I_f, if $V(I_f) = f(V(I_f) = f(V(G))$ and $\{f(a),f(b)\}\in E(I_f)$ if and only if there exist $c\in f^{-1}(f(a))$ and $d\in f^{-1}(f(b))$ such that $\{c,d\}\in E(G)$. By ρ_f we denote the equivalence relation on V(X) induced by f, i.e., for $a,b\in V(X)$, $\{a,b\}\in\rho_f$ if and only if f(a) = f(b). Denote by $[a]\rho_f$ the equivalence class containing $a\in V(X)$ with respect to ρ_f.

An element a of a semigroup S is called regular if there exists $x\in S$ such that axa = a. A semigroup S is called regular if all its elements are regular. A graph X is said to be End-regular if its endomorphism monoid End(X) is regular. The reader is referred to Godsil *et al.* (2000) and Howie (1995) for all the notation and terminology not defined here. We list some known results which will be used frequently in the sequel to end this section.

Lemma 1 (Li, 2003): Let X be a graph and $f\in End(X)$. Then:

- $f\in hEnd(X)$ if and only if I_f is an induced sub graph of X
- If f is regular, then $f\in hEnd(X)$

Lemma 2 (Li, 1994): Let X be a graph and $f\in End(X)$. Then f is regular if and only if there exists $g,h\in Idpt(X)$ Such that $\rho_g=\rho_f$ and $I_h=I_f$.

Lemma 3 (Li, 2003): Let X and Y be two graphs. If X + Y is End-regular, then both X and Yare End-regular.

Lemma 4 (Li, 2003): Let X be a graph. Then X is End-regular if and only if X + K_n is End-regular for any n≥1.

The following are some known results about split graphs which are essential for our consideration.

Lemma 5 (Li *et al.*, 2001): Let X be a connected split graph with $V(X)=K\bigcup S$, where S is an independent set and K is a maximal complete set, |K| = n. Then X is End-regular if and only if there exists $r\in\{1,2,\cdots,n-1\}$ such thatd(x) = r for any $x\in S$.

Lemma 6 (Li *et al.*, 2001): A non-connected split graph X is End-regular if and only if X exactly consists of a complete graph and several isolated vertices.

END-REGULAR JOINS OF n SPLIT GRAPHS

The End-regular split graphs have been characterized in Lemma 5 and 6. In this section, we will characterize the End-regular graphs which are the join of n split graphs.

Let X_i be a split graph with $V(X_i)=V(K_i)\bigcup S_i$, where, $S_i = \{x_{i1}, \ldots, x_{ip_i}\}$ is an independent set and $V(K_i) = \{k_{i1}\ k_{i2}\ \cdots, k_{iq_i}\}$ is a maximal complete set. Then the vertex set $V(X_1 + X_2 + \ldots + X_n)$ of $X_1 + X_2 + \ldots + X_n$ can be partitioned into n + 1 parts K , S_1, S_2, …, S_n i.e., $V(X_1 + X_2 + \ldots + X_n) = K\bigcup S_1\bigcup S_2\bigcup\cdots\bigcup S_n$, where $V(K) = V(K_1)\ \cup V(K_2)\ \cup\ \cdots \cup V(K_n)$ is a complete set, S_1, S_2, …, S_n are independent sets. Obviously the subgraph of $X_1 + X_2 + \ldots + X_n$ induced by K is a complete graph and the subgraph of $X_1 + X_2 + \ldots + X_n$ induced by $S_1 \cup S_2 \cup \cdots \cup S_n$ is a complete n partite graph. By Lemma 3, we know if X + Y is End-regular, then both of X and Y are End-regular. Clearly, If $X_1 + X_2 + \ldots + X_n$ is End-regular, then X_i is End-regular for any 1≤i≤n. So we always assume that X_i are End-regular sprit graphs in the sequel unless otherwise stated. Moreover, let d_i be the valency of the vertices of S_i in X_i. Clearly, if X_i is connected, then $0<d_i$ n-1; if X_i is non-connected, then $d_i = 0$.

Lemma 7: If $X_1 + X_2 + \ldots + X_n$ is End-regular, then $q_i - d_i = q_j - d_j$ for any 1≤i≤n and 1≤j≤n.

Proof: Supose that $q_i - d_i \neq q_j - d_j$, Then we have $q_1 + q_2 + \ldots + q_n - d_i \neq q_1 + q_2 + \ldots + q_n - d_j$. Let $q_1 + q_2 + \ldots + q_n - d_i < q_1 + q_2 + \ldots + q_n - d_j$. As $q_i<n$, for any $x\in S_i$, x is not adjacent to exactly $q_i - d_i$ vertices of $V(K_i)$ in X_i, so x is not adjacent to exactly $q_i - d_i$ vertices of $V(K_i)$ in $X_1 + X_2 + \ldots + X_n$, take such a vertex and write k_x.

Let x_1 be a vertex of S_i and y_1 be a vertex of S_j, since we have $|V(K)\cap N(x_1)| = d_i + q_i < d_j + q_j =$

$|V(K) \cap N(y_1)|$, there exists a permutation τ on $V(K)$ such that $\tau(V(K) \cap N(x_1)) \subset V(K) \cap N(y_1)$.
Let f be a mapping from V(X) to itself defined by:

$$f(x) = \begin{cases} y_1, & if \quad x = x_1, \\ \tau(x), & if \quad x \in V(K), \\ \tau(k_x), & if \quad x \in S_1 \cup S_2 \cup \cdots \cup S_n \; and \; x \neq x_1. \end{cases}$$

Then it is easy to see that $f \in End(X_1 + X_2 + ... + X_n)$. Sinc $|V(K) \cap N(x_1)| < |V(K) \cap N(y_1)|$, I_f is not an induced sub graph of $X_1 + X_2 + \ldots + X_n$. Hence $f \notin hEnd(X_1 + X_2 + \cdots + X_n)$. It follows from Lemma 1 that $X_1 + X_2 + \ldots + X_n$ is not End-regular. A contradiction. Therefore $q_i - d_i = q_j - d_j$.

Lemma 8: Let $X_1, X_2, \ldots, X_n$ be n End-regular split graphs, $q_i - d_i = q_j - d_j$ for any $1 \le i \le n$ and $1 \le j \le n$. If $X_1 +$ X2 $+ \ldots + X_n$ is End-regular, then there are no two vertices $x_1, x_2 \in S_t$ such that $N_{X_t}(x_1) \cup N_{X_t}(x_2) = V(K_t)$ for any $1 \le i \le n$.

Proof: Suppose there exist two vertices x_1 and x_2 of S_t such that $N_{X_t}(x_1) \cup N_{X_t}(x_2) = V(K_t)$ for some $1 \le t \le n$. Then we can obtain that $(V(K_t) \setminus N_{X_t}(x_1)) \cap (V(K_t) \setminus N_{X_t}(x_2))$ $= \phi$. Let y_1 be a vertex of S_k ($1 \le k \le n$ and $k \ne t$). Then there exists a permutation τ of V(K) such that $\tau(V(K) \setminus N_{Xt}(x_1)) = (V(K) \setminus N_{Xt}(y_1))$ = and $\tau(V(K) \setminus N_{Xt}(x_2)) = (V(K) \setminus NX_t(x_2))$. For $x \in S_i$, k_x have the same meaning as in the proof of Lemma 7. Let f be a mapping from $V(X_1 + X_2 + \ldots + X_n)$ to itself defined by:

$$f(x) = \begin{cases} y_1, & x = x_1, \\ x_2, & x = x_2, \\ \tau(x), & x \in V(K), \\ \tau(k_x), & otherwise. \end{cases}$$

Then $f \in End(X_1 + X_2 + \cdots + X_n)$. It is easy to see $\{y_1, x_2\} \in E(X_1 + X_2 + \ldots + X_n)$. But $f^{-1}(x_2) = x_2$, $f^{-1}(y_1) = x_1$, and $\{x_1, x_2\} \notin E(X)$. Thus $f \notin hEnd(X)$ and so $X_1 + X_2 + \ldots + X_n$ is not End-regular.

We next prove the conditions in Lemma 7 and 8 are the sufficient conditions such that $X_1 + X_2 + \ldots + X_n$ being End-regular. Note that in case of $q_i - d_i = q_j - d_j$, $X_1 + X_2 + \ldots + X_n$ has a unique clique of order $q_1 + q_2 + \ldots + q_n$ if and only if $d_i \le q_i - 2$ for any $1 \le i \le n$. So we can go process into two cases: $d_i \le q_i - 2$ and $d_i = q_i - 1$ for any $1 \le i \le n$.

Lemma 9: Let $X_1, X_2, \ldots, X_n$ be n split graphs with $d_i \le q_i - 2$, $q_i - d_i = q_j - d_j$ for any $1 \le i \le n$ and $1 \le j \le$. Then for any endomorphism f of $X_1 + X_2 + \ldots + X_n$, I_f is an induced subgraph of $X_1 + X_2 + \ldots + X_n$ if and only if there are no two vertices $x_1, x_2 \in S_t$ such that $N_{Xt}(x_1) \cup N_{Xt}(x_2) = V(K_t)$ for any $1 \le t \le n$.

Proof: Necessity follows from the proof of Lemma 8. Conversely, assume there are no two vertices $x_1, x_2 \in S_t$ such that $N_{X_t}(x_1) \cup N_{X_t}(x_2) = V(K_t)$ for any $1 \le t \le n$. As the proof of Lemma 8, it is easy to show that for any two vertices $s_1, s_2 \in S_t$, there is no endomorphism f such that $f(s_1) \in S_i$ and $f(s_2) \in S_j$ for any $1 \le i, j \le n$ and $i \ne j$.

Let $f \in End(X)$ and let $a, b \in I_f$ with $\{a, b\} \in E\{X_1 + X_2 + ... + X_n)$. We need to prove that there exist $c \in f^{-1}(a)$, $d \in f^{-1}(b)$ such that $\{c, d\} \in E(X_1 + X_2 + \ldots + X_n)$. If both of a and b are in $f(V(K))$, then there exist two vertices $c \in f^{-1}(a)$, $d \in f^{-1}(b)$ such that $\{c, d\} \in E(X)$ since $f(V(K)) = V(K)$. If exactly one of a and b is in f(V(K)), without loss of generality, assume that $a \in f(V(K))$, $b \notin f(V(K))$, then there exists a vertex $c \in f(V(K))$ such that $f(c) = a$. Suppose that $\{c, v\} \notin E(X_1 + X_2 + \ldots X_n)$ for any vertex $v \in f^{-1}(b)$, let $u \in f^{-1}(b)$. Then u is adjacent to exactly $q_1 + q_2 = \ldots + q_n - d_i$ ($1 \le i \le n$) vertices in $V(K) \setminus \{c\}$, say, $x_1, x_2 \cdots, x_{q1 = q2 + \ldots + qn - di}$. So b is adjacent to $f(x_1), f(x_2), \ldots, f(xq_1 + q_2 + \ldots + q_n - d_i)$. Clearly $f(x_1), f(x_2), \ldots, f(x_{q1 + q2 + \ldots + qn})$ a are distinct. We get that b is adjacent to $q_1 + q_2 + \ldots + q_n - d_i + 1$ vertices in V(K), a contradiction. If both a and b are not in f(V(K)) and $\{c, d\} \notin E(X_1 + X_2 + \cdots + X_n)$ for any $c \in f^{-1}(a)$, $d \in f^{-1}(b)$, then $f^{-1}(a)$ and $f^{-1}(b)$ are contained in the same S_i ($i = 1, 2, \ldots, n$). From the discussion in the last paragraph, we have that $a = f(f^{-1}(a))$ and $b = f(f^{-1}(b))$ are in the same S_i ($i = 1, 2$) and so $\{a, b\} \notin E(X_1 + X_2 + \ldots + X_n)$, a contradiction, as required.

Lemma 10: Let $X_1, X_2, \ldots, X_n$ be n split graphs with $d_i \le q_i - 2$ for any $1 \le i \le n$. Then $X_1 + X_2 + \ldots + X_n$ is End-regular if and only if:

- $q_i - d_i = q_j - d_j$ For any $1 \le i \le n$ and $1 \le j \le n$
- There are no two vertices $x_1, x_2 \in S_t$ such that $N_{X_t}(x_1) \cup N_{X_t}(x_2) = V(K_t)$ for any $1 \le t \le n$

Proof: Necessity follows immediately from Lemmas 7 and 8.

Conversely, let $f \in End(X_1 + X_2 + \cdots + X_n)$. To show that f is regular, we need to prove that there exist two idempotents g and h in End(X) such that $\rho_g = \rho_f$ and $I_h = I_f$.

Since $d_1 \le n - 2$ and $d_2 \le m - 2$, f(V(K)) = V(K) and for any $x \in S_1 \cup S_2 \cup \cdots \cup S_n$, there exists a vertex $k_x \in V(K)$ such that x is not adjacent to k_x. Let h be the mapping from V(X) to itself defined by:

$$h(x) = \begin{cases} x, & x \in f(X_1 + X_2 + \cdots + X_n), \\ k_x, & otherwise. \end{cases}$$

Then $h \in End(X)$ and $h(V(K)) = V(K)$. If $x \in f(X_1 + X_2 + \cdots + X_n)$, then $h^2(x) = h(x) = x$; If $x \in V(X_1 + X_2 + \ldots + X_n) \setminus f(X_1 + X_2 + \ldots + X_n)$, then it is easy to check that $h^2(x) = h(k_x) = k_x$ $h(x) = k_x = h(x)$ since $k_x \in V(K) \subseteq f(X)$. Henc $f \in Idpt(X_1 + X_2 + \cdots + X_n)$. Clearly, I_f and I_h have the same set of vertices. Note that an idempotent endomorphism is half-strong. It follows from Lemmas 1 and 9 that both I_h and I_f are induced sub graph of $X_1 + X_2 + \ldots + X_n$. Therefore $I_h = I_f$

Since f(V(K)) = V(K), $[x]_{\rho f}$ contains at most one vertex of V(K) for any $x \in V(X_1 + X_2 + \cdots + X_n)$. Without loss of generality, we can suppose that $V(X_1 + X_2 + \ldots + X_n) / \rho_f = \{[k_1]_{\rho f}, [k_2]_{\rho f}, \ldots, \cdots, [k_r]_{\rho f}, [s_1]_{\rho f}, \ldots, [s_t]_{\rho f}$ where $k_i \in K$, and $s_i \in S_1 \cup S_2 \cup \cdots \cup S_n$. Let g be a mapping from $V(X_1 + X_2 + \ldots + X_n)$ to itself defined by:

$$g(x) = \begin{cases} k_i, & if \quad x \in [k_i]_{\rho_f}, \\ s_i, & if \quad x \in [s_i]_{\rho_f}. \end{cases}$$

Then $g \in End(X_1 + X_2 + \cdots + X_n)$. If any $x \in [k_i]_{\rho_f}$, then $g^2(x) = g(k_i) = k = g(x)_i$; if $x \in [s_i]_{\rho_f}$, then $g^2(x) = g(s_i) = s_i = g(x)$. Hence $g^2 = g$. Clearly, $\rho_g = \rho_f$, as required.

Lemma 11: Let $X_1 + X_2 + \ldots + X_n$ be End-regular split graphs, $d_i = q_i - 1$ for any $1 \le i \le n$. Then $X_1 + X_2 + \ldots + X_n$ is End-regular if and only if $N_{Xi}(x_1) = N_{Xi}(x_2)$ for any $x_1, x_2 \in S_i$ (where $1 \le i \le n$).

Proof: Necessity follows immediately from Lemma 8. Conversely, since $N_{Xi}(x_1) = N_{Xi}(x_2)$ for any $x_1, x_2 \in S_i$, there is a unique vertex k_i in K_i such that $\{x_i, k_i\} \notin E(X_1 + X_2 + \ldots + X_n)$ for any $x_i \in S_i$. Now the subgraph induced by $S_1 \cup S_2 \cup \cdots \cup S_n \cup \{k_1, k_2, \cdots, k_n\}$ is a complete *n* partite graph, denote it by T. Hence $X_1 + X_2 + \ldots + X_n$ is isomorphic to $K_{q1 + q2 + \ldots + qn - n} + T$. Since any complete n partite graph is End-regular, by Lemma 4, $X_1 + X_2 + \ldots + X_n$ is End-regular.

Now we are ready for our main result in this section.

Theorem 12: Let $X_1, X_2, \ldots, X_n$ be n split graphs. Then $X_1 + X_2 + \ldots + X_n$ is End-regular if and only if:

- X_i is End-regular for any $1 \le i \le n$.
- $q_i - d_i = q_j - d_j$ For any $1 \le i \le n$ and $1 \le i \le n$
- There are no two vertices $x_1, x_2 \in S_t$ such that $N_{X_t}(x_1) \cup N_{X_t}(x_2) = V(K_t)$ for any $1 \le i \le n$

Proof: It follows directly from Lemmas 3, 5, 6, 10 and 11.

END-ORTHODOX JOINS OF *n* SPLIT GRAPHS

In this section, we will give the conditions under which the endomorphism monoids of the joins of the split graphs is orthodox.

Lemma 13: Let $G_1, G_2, \ldots, G_n$ be n graphs. If $G_1 + G_2 + \ldots + G_n$ is End-orthodox, then G_i is End- orthodox for any $1 \le i \le n$.

Proof: Since $G_1 + G_2 + \ldots + G_n$ is End-orthodox, $G_1 + G_2 + \ldots + G_n$ is End-regular. By Lemma 3, G_i is End-regular for any $1 \le i \le n$. To show G_i is End-orthodox; we only need to prove that the composition of any two idempotent endomorphisms of G_i is also an idempotent.

Let f_1 and f_2 be two idempotents in End(G_i). Define two mappings g_1 and g_2 from $V(G_1 + G_2 + \ldots + G_n)$ to itself by:

$$g_1(x) = \begin{cases} f_1(x), & if \ x \in V(G_i) \\ x, \ if \ x \in V(G_1 + G_2 + \cdots + G_n) \setminus V(G_i) \end{cases}$$

$$g_2(x) = \begin{cases} f_2(x), & if \ x \in V(G_i) \\ x, \ if \ x \in V(G_1 + G_2 + \cdots + G_n) \setminus V(G_i) \end{cases}$$

Then g_1 and g_2 are two idempotents of $End(G_1 + G_2 + \ldots + G_n)$ and so g_1g_2 is also an idempotent of $End(G_1 + G_2 + \ldots + G_n)$ since $G_1 + G_2 + \ldots + G_n$ is End-orthodox. Clearly, $f_1f_2 = (g_1g_2)|G_i$, the restriction of g_1g_2 to G_i. Hence f_1f_2 is an idempotent of End(G_i) as required.

Lemma 14: Let G be a graph. Then G is End-orthodox if and only if G + K_n is End-orthodox for any positive integer n.

Proof: If G + K_n is End-orthodox, then by Lemma 4, G is End-orthodox.

Conversely, for any positive integer n, by Lemma 4, if X is End- regular, then X + K_n is End-regular. Let f be an idempotent of End(G + K_n). Note that $\varpi(G+K_n)=\varpi(G)+n$, $V(K_n)\subset I_f$ and $f|_{K_n}=1|_{K_n}$, the identity mapping on K_n. Hence $f(V(G))\subseteq V(G)$ and $f|_G \in Idpt(G)$.

If f_1 and $f_2$are two idempotents of End(G + K_n), let $g_1 = f_1|G$ and $g_2 = f_2|G$. Then $g_1, g_2 \in Idpt(G)$ and so $g_1 g_2 \in Idpt(G)$. Now (f_1f_2) $|K_n = 1|K_n$ and $(f_1f_2)|G = g_1g_2$ imply that f_1f_2 is an idempotent of End(G + K_n). Consequently G + K_n is End-orthodox.

Let X_i (i = 1, 2, …, n) be two split graphs. If $X_1 + X_2 + … + X_n$ is End-orthodox, then X is End-regular and X_i is End-orthodox for any 1≤i≤n. The following lemma describes the idempotent endomorphisms of certain End-regular graphs $X_1 + X_2 + … + X_n$.

Lemma 15: Let X_1, X_2, …, X_n be n split graphs with $d_i \le q_i$ - 2 for any 1≤i≤n. If $N_X(x_1) \neq N_X(x_2)$ for any two vertices x_1, $x_2 \in S_1 \cup S_2 \cup \cdots \cup S_n$, then $f \in$ End($X_1 + X_2 + … + X_n$) is a retraction (idempotents) if and only if:

- f(x) = x For any $x \in V(K)$
- For any $y \in S_1 \cup S_2 \cup \cdots \cup S_n$, either $f(y) \in V(K) \setminus N(y)$, or $f(y) = y$

Proof: Note that under the hypothesis of lemma $X_1 + X_2 + … + X_n$ has a unique maximum clique K.

Lemma 16: Let $X_1 + X_2 + … + X_n$ be n split graphs with $d_i \le q_i$ - 2 for any 1≤i≤n. Then $X_1 + X_2 + … + X_n$ is End-orthodox if and only if:

- $X_1 + X_2 + … + X_n$ is End-regular
- $N_X(x_1) \neq N_X(x_2)$ for any two vertices x_1, $x_2 \in S_1 \cup S_2 \cup \cdots \cup S_n$

Proof: Necessity is obvious.

Conversely, since $X_1 + X_2 + … + X_n$ is End-regular, we only need to prove that the composition of two idempotent endomorphisms is also an idempotent. Let f be an arbitrary idempotent of End ($X_1 + X_2 + … + X_n$). Then $f|_{V(K)}$ = V(K) and either f(x) = x or f(x) = k_x for any $x \in S_1 \cup S_2 \cup \cdots \cup S_n$, where k_x is a vertex in V(K) such that $\{x, k_x\} \notin E(X_1 + X_2 + \cdots + X_n)$. Now the assertion follows immediately.

Lemma 17: Let X_1, X_2, …, X_n be n split graphs with $d_i \le q_i$ - 1 for any 1≤i≤n. Then $X_1 + X_2 + … + X_n$ is End-orthodox if and only if $|S_1| = |S_2| = … = |S_n|$.

Proof: Necessity is obvious.

Conversely, $X_1 + X_2 + … + X_n$ is a join of a complete graph and a complete n partite graph. Since any complete n partite graph is End-orthodox, it follows from Lemma 14 that $X_1 + X_2 + … + X_n$ is End-orthodox.

Theorem 18: Let $X_1 + X_2 + … + X_n$ be n split graphs. Then $X_1 + X_2 + … + X_n$ is End-orthodox if and only if:

- X_I is End-regular for any 1≤i≤n
- $q_i - d_i = q_j - d_j$ For any 1≤i≤n and 1≤j≤n
- There are no two vertices $x_1, x_2 \in S_t$ such that $N_{X_t}(x_1) \cup N_{X_t}(x_2) = V(K_t)$ for any 1≤i≤n
- $N_X(s_1) \neq N_X(s_2)$ for any two vertices $s_1, s_2 \in S_1 \cup S_2 \cup \cdots \cup S_n$

Proof: If $X_1 + X_2 + … + X_n$ is orthodox, then $X_1 + X_2 + … + X_n$ is regular and so both of X_i is regular for any 1≤i≤n. Now it follows immediately from Lemma 3, 16 and 17.

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REFERENCES

Fan, S., 1997. The regularities of the endomorphism monoid of split graphs. Acta Mathrmatica Sinica, 40(3): 419-421.

Fan, S., 2002. Retractions of split graphs and end-orthodox split graphs. Discret. Math., 257(1): 161-164.

Godsil, C. and G. Royle, 2000. Algebraic Graph Theory. Springer-Verlag, New York, USA.

Hou, H., Y. Luo and Z. Cheng, 2008. The endomorphism monoid of pn. Eur. J. Combin., 29(5): 1173-1185.

Hou, H., Y. Luo and X. Fan, 2012. End-regular and End-orthodox join of Split Graphs. Ars Combin., 105: 305-318.

Howie, J.M., 1995. Fundamentals of Semigroup Theory. Clarendon Press, Oxford, England.

Li, W., 1994. A regular endomorphism of a graph and its inverse. Mathematik., 41(01): 189-198.

Li, W., 2003. Graphs with regular monoid. Discrete. Math., 265(1-3): 105-118.

Li, W. and J. Chen, 2001. Endomorphism-regularity of split graphs. Eur. J. Combin., 22(20): 207-216.

Marki, L., 1988. Problems raised at the problem session of the colloquium on semi groups in Szeged, August 1987. Semi Group Forum, 37(1): 367-373.

Petrich, P. and R. Reilly, 1999. Completely Regular Semigroups. John Wiley and Sons Inc., New York, USA.

Wilkeit, E., 1996. Graphs with a regular endomorphism monoid. Arch. Math, 66(4): 344-352.