Research Article
Error Analysis for Shannon Sampling Series Approximation with Measured Sampled Values

Peixin Ye

School of Mathematics and LPMC, Nankai University, Tianjin, 300071, P.R. China   858-864

Abstract: Errors appear when the Shannon sampling series is applied to reconstruct a signal in practice. In this paper, we study a general model that uses linear functional to cover several errors in one formula, and consider sampling series with measured sampled values for not band-limited signals but satisfying some decay condition. We obtain the uniform truncated error bound of Shannon series approximation for multivariate Besov class. The results show this kind of Shannon series can approximate a smooth signal well.

Keywords: Besov class, local averages, measured sampled values, modulus of continuity, Shannon sampling theorem, truncation error

INTRODUCTION

Since Shannon introduced the sampling series in the context of communication in the land-mark study (Shannon, 1948), the Shannon sampling series has become more and more important. By now, sampling theory has grown to an independent research area. The theorem states that a band-limited signal can be exactly recovered from its sample values. However errors appear when the Shannon is applied to approximate a signal in practice. Typical errors are jitter errors, amplitude errors, truncated error and aliasing error.

Motived by Butzer and Lei (1998), we consider the sampled values which are the results of a linear functional and its integer translates acting on a undergoing signal (Burchard and Lei, 1995). Following (Casey and Walnut, 1994) we call such sampled values measured sampled values because they are closer to the true measurements taken from a signal. This model covers jitter errors, amplitude errors and the errors arising from sampled values obtained by averaging a signal (Boche, 2010).

In this study we investigate the approximation sampling series with measured sampled values for not band-limited functions from Besov class and obtain the uniform bound of the truncation errors. Note that the strong "band-limited" assumption is the first time replaced by a weaker one, a smooth condition.

Let's begin with some concepts. For $1 \leq p \leq \infty$, let $L^p(\mathbf{R}^d)$ be the space of all p-th power Lebesgue integrable functions on $\mathbf{R}^d$ equipped with the usual norm:

$$\|f\|_p := \begin{cases} \left(\int_{\mathbf{R}^d} |f(t)|^p \, dt \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in \mathbf{R}^d} |f(t)|, & p = \infty \end{cases}$$

In the following, $\underline{t} \in R^d$ denotes vector $\underline{t} = (t_1, \cdots, t_d)$ with $t_j \in \mathbf{R}$ and $a\underline{t} = (at_1, \cdots, at_d)$ is the product of $\underline{t}$ with the scalar $a \in \mathbf{R}$, $\underline{t}\underline{k} = (t_1 k_1, \cdots, t_d k_d)$ is the product of $\underline{t}$, $\underline{k} \in R^d$. By $[-\underline{t}, \underline{t}]$ we understand the d-dimensional rectangle given by all vectors $\underline{k} \in \mathbf{R}^d$ with $-t_j \leq k_j \leq t_j$ j = 1, …d.

Set $\|k\| = \max \{|k_i| : i = 1, \cdots, d\}$, $\underline{k} \in \mathbf{R}^d$

Let $B_{\pi\underline{\Omega}}^p(\mathbf{R}^d)$ be the set of all bounded functions from $L^p(\mathbf{R}^d)$ which can be extended to entire functions of exponential type $\pi\underline{\Omega}$, In view of Schwartz's theorem, every function $f \in B_{\pi\underline{\Omega}}^p$ is band-limited to $[-\pi\underline{\Omega}, \pi\underline{\Omega}]$, i.e.,

$$B_{\pi\underline{\Omega}}^p := \left\{ f \in L^p(\mathbf{R}^d) : \operatorname{supp} \hat{f} \subset [-\pi\underline{\Omega}, \pi\underline{\Omega}] \right\}$$

where, $\hat{f}$ is the Fourier transform of f in the sense of distribution. In the special case p = 2 it reduces to the Paley-Wiener theorem (Nikolskii, 1975).

The famous Whittaker-Shannon-Nyquist sampling theorem, or simply Shannon theorem, states every

signalfunction $f \in B_{\pi\Omega}^2$ can be completely reconstructed from its sampled values taken at instances $k/2\Omega$. In this case the representation of f is given by:

$$f(t) = (S_\Omega f)(t) := \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{2\Omega}\right) \sin c(2\Omega t - k)$$

where, $\sin c(t) = \sin \pi t / (\pi t)$ and $\sin c(0) = 1$. The Shannon theorem plays an important role in signal analysis as it provides the foundation for digital signal processing. It has been generalized to the case of non-band-limited signals, irregular sampling, multidimensional, and stochastic signal and so on (Zayed, 1993).

Shannon's expansion requires us to know the exact values of a signal f at infinitely many points and to sum an infinite series. But in practical situations, only finitely many samples are available, so it is natural to use:

$$(S_{\Omega,N} f)(t) := \sum_{k=-N}^{+N} f\left(\frac{k}{2\Omega}\right) \sin c(2\Omega t - k)$$

to approximate $f(t)$ and study the bounds on the truncation error, which is defined by:

$$(E_{\Omega,N} f)(t) := \left| f(t) - (S_{\Omega,N} f)(t) \right|$$

There were some papers concerned the estimate of truncation error (Zayed, 1993).

We proceed to state our own generalization of Shannon Theorem. We consider multi-dimensional signal. To obtain our main results, we need the error modulus:

$$\omega_\Omega(f, \lambda) := \sup \left| \lambda_k f(\cdot + \underline{k}/\underline{\Omega}) - f(\underline{k}/\underline{\Omega}) \right|, \underline{\Omega} > 0$$

where, $\lambda = \{\lambda_k\}$ is any sequence of continuous linear functional $C_0(\mathbf{R}^d) \to \mathbf{R}$ with $C_0(\mathbf{R}^d)$ being the Banach space consisting of all continuous functions defined on $\mathbf{R}^d$ and tending to zero at infinity. The error modulus $\omega_\Omega(f, \lambda)$ provides a quantity for the quality of a signal's measured sampling values. When λ is concrete, we may get some reasonable estimates for $\omega_\Omega(f, \lambda)$.

Our aim is to study the sampling series with the measured sampled values, namely:

$$(S_{\underline{\Omega}}^\lambda f)(\underline{t}) := \sum_{\underline{k} \in \mathbf{Z}^d} \lambda_k f(\cdot + \underline{k}/\underline{\Omega}) \sin c(\underline{\Omega t} - \underline{k})$$

and consider the finite sum:

$$(S_{\underline{\Omega},N}^\lambda f)(\underline{t}) := \sum_{\underline{k} \in \mathbf{Z}^d, \|\underline{k}\| \le N} \lambda_k f(\cdot + \underline{k}/\underline{\Omega}) \sin c(\underline{\Omega t} - \underline{k}) \qquad (1)$$

where, $\sin c(\underline{t}) = \prod_{j=1}^d \sin c(t_j)$ for all $\underline{t} \in \mathbf{R}^d$. In this way we can study the bounds of truncation error:

$$(E_{\underline{\Omega},N}^\lambda f)(\underline{t}) := \left| f(\underline{t}) - (S_{\underline{\Omega},N}^\lambda f)(\underline{t}) \right| \qquad (2)$$

We organize the study as follows. In section two we derive the truncation error for band-limited functions from the class $B_{\pi\Omega}^p(\mathbf{R}^d)$. In section three we obtain truncation error for non-band-limited functions from Besov class $\mathsf{A}(B_{p,\theta}^r(\mathbf{R}^d))$. In section four we provide some applications.

Trunction error for band-limited functions from class $B_{\pi\Omega}^p(\pmb{R}^d)$: In this section we derive the truncation error for band-limited functions from the class $B_{\pi\Omega}^p(\mathbf{R}^d)$. For a real number x, let $[x]$ denote the largest integer less than or equal to x. In what follows, we often use the same symbol C for possibly different positive constants. These constants are independent of N, Ω and σ. Let $B_\Omega^p(\pmb{R}^d) := B_{\underline{\Omega}}^p(\mathbf{R}^d)$, if $\underline{\Omega} = (\Omega, \ldots, \Omega)$.

The main result of this section is the following theorem.

Theorem 1: *Let* $1 \le p < \infty$, $f \in B_{\pi\Omega}^p(\mathbf{R}^d)$ *and satisfy the decay condition inequality*

$$\left| f(t) \right| \le A / \|\underline{t}\|^\delta, \underline{t} \ne 0 \qquad (3)$$

where, A > 0 is a constant. Then for $e^{(d+1)/\delta} \le N \le N_0 = [\omega_\Omega(f, \lambda)^{-1/\delta} \Omega]$

we have:

$$(E_{\Omega,N}^\lambda f)(\underline{t}) \le C \cdot \left(\frac{\Omega}{N}\right)^\delta \cdot \ln^d N,$$

And for $N > N_0$

$$(E_{\Omega,N}^\lambda f)(\underline{t}) \le C \cdot \omega_\Omega(f, \lambda) \cdot \left(\ln \frac{1}{\omega_\Omega(f, \lambda)}\right)^d$$

To prove Theorem 1 we need some auxiliary assertions.

Lemma 1: Let $f \in B_{\pi\Omega}^p(\mathbf{R}^d)$, $1 < p < \infty$: Then:

$$f(\underline{t}) = (S_\Omega f)(\underline{t}) = \sum_{\underline{k} \in \mathbf{Z}^d} f\left(\frac{\underline{k}}{\Omega}\right) \sin c(\Omega \underline{t} - \underline{k})$$

where, the series on the right-hand side converges uniformly on $\mathbf{R}^d$.

Lemma 2: For $\underline{t} \in \mathbf{R}^d$, q<1 and $\Omega > 0$, we have:

$$\left(\sum_{\underline{k} \in \mathbf{Z}^d} \left| \sin c(\Omega \underline{t} - \underline{k}) \right|^q \right)^{1/q} \leq \left(\frac{q}{q-1} \right)^d$$

Lemma 3: Let $0 < p, p_0 < \infty$, $f \in B_{\pi\Omega}^p(\mathbf{R}^d)$. Then we have:

$$\left| f(\underline{t}) - \sum_{\underline{k} \in \mathbf{Z}^d} \lambda_k(f) \sin c(\Omega \underline{t} - \underline{k}) \right|$$

$$\leq \left(\sum_{\|\underline{k}\| \leq N} \left| f\left(\frac{\underline{k}}{\Omega} \right) - \lambda_k(f) \right|^{p_0} \right)^{1/p_0} p_0^{\,d}$$

$$+ \left(\sum_{\|\underline{k}\| > N} \left| f\left(\frac{\underline{k}}{\Omega} \right) \right|^{p_0} \right)^{1/p_0} p_0^{\,d}$$

where, $1/p_0 + 1/q_0 = 1$.

Proof of Theorem 1: Let's consider the case $N \leq N_0$ first. For $1 \leq p \leq \infty$ set:

$$F_{\pi\Omega}^p := \left\{ f : f \in B_{\pi\Omega}^p(\mathbf{R}^d), |f(\underline{t})| \leq A/\|\underline{t}\|^\delta, \underline{t} \neq 0, \delta > 0 \right\}$$

For a given class $F_{\pi\Omega}^p$, we show that there exists a $p^* \in (1, \infty)$ such that $F_{\pi\Omega}^p \subset B_{\pi\Omega}^{p^*}$. Actually when $1 < p < \infty$, we simply take

$p^* = p$. If $f \in F_{\pi\Omega}^1 \cup F_{\pi\Omega}^\infty$, then f is measurable. Let p*>max{1, 1/δ}, applying the decay condition of f it is easy to prove that $f \in L^{p^*}(\mathbf{R}^d)$. Thus $f \in B_{\pi\Omega}^{p^*}$. Observe that $f \in F_{\pi\Omega}^p$ implies $f \in C_o(\mathbf{R}^d)$. According to Lemma 1 and Lemma 3, we have:

$$\left| f(\underline{t}) - \sum_{\|\underline{k}\| \leq N} \lambda_k(f)(\cdot + \underline{k}/\Omega) \sin c(\Omega \underline{t} - \underline{k}) \right|$$

$$\leq \left(\sum_{\|\underline{k}\| \leq N} \left| f\left(\frac{\underline{k}}{\Omega} \right) - \lambda_k(f) \right|^{p_0} \right)^{1/p_0} p_0^{\,d}$$

$$+ \left(\sum_{\|\underline{k}\| > N} \left| f\left(\frac{\underline{k}}{\Omega} \right) \right|^{p_0} \right)^{1/p_0} p_0^{\,d}$$

where, $1/p_0 + 1/q_0 = 1$. We first derive the following estimate of the first term of right hand-side by the definition of $\omega_\Omega(f, \lambda)$:

$$\left(\sum_{\|\underline{k}\| \leq N} \left| f\left(\frac{\underline{k}}{\Omega} \right) - \lambda_k(f) \right|^{p_0} \right)^{1/p_0} p_0^{\,d} \leq$$

$$C(2N+1)^{d/p_0} \omega_\Omega(f, \lambda) p_0^{\,d} \qquad (4)$$

Let f satisfy the decay condition (3), then for $p_0 = \ln N \geq \max\{(d+1)/\delta, 1\}$ we have:

$$\left(\sum_{\|\underline{k}\| > N} \left| f\left(\frac{\underline{k}}{\Omega} \right) \right|^{p_0} \right)^{1/p_0} \leq A \left(\sum_{\|\underline{k}\| > N} \left(\frac{\underline{k}}{\Omega} \right)^{-\delta p_0} \right)^{1/p_0}$$

$$\leq (2d3^{d-1})^{1/p_0} \Omega^\delta \frac{N^{d/p_0 - \delta}}{(p_0\delta - d)^{1/p_0}} \qquad (5)$$

where, in the last inequality we use the inequality from. The assumption $p_0 = \ln N \geq (d+1)/\delta$ implies $p_0\delta - d \geq 1$ and $N^{d/p_0} \leq e^d$. Then combining (4) and (5) we get:

$$(E_{\Omega,N}^\lambda f)(\underline{t}) \leq C \left(\omega_\Omega(f, \lambda) + \left(\frac{\Omega}{N} \right)^\delta \right) (\ln N)^d \leq C \cdot \left(\frac{\Omega}{N} \right)^\delta \cdot \ln^d N$$

Now we take $N > N_0$, similarly we have:

$$(E_{\Omega,N}^\lambda f)(\underline{t}) \leq C \left(\frac{\Omega}{N_0} \right)^\delta (\ln N_0)^d \leq C \omega_\Omega(f, \lambda) \left(\ln \frac{1}{\omega_\Omega(f, \lambda)} \right)^d$$

Thus the proof of Theorem 1 is complete.

TRUNCTION ERROR FOR NOT BAND LIMITED FUNCTIONS FROM BESOV CLASSES

Since the functions encountered in applications of the sampling theory are not always exactly band-limited. Finding a bound for the error when a non-band-limited function is approximated by the sampling expansion is a very practical issue. In this paper we study this problem in a considerable generality, that is, we assume that the signal functions belong to Besov class which is very commonly used in many applications.

Let's recall the definition of Besov space. Suppose that $l \in N$, $r \in \mathbf{R}^+$. For any $f \in L^p(\mathbf{R}^d)$:

$$\Delta_{\underline{u}}^l f(\underline{t}) = \sum_{j=0}^l (-1)^{(l+j)} \binom{l}{j} f(\underline{t} + j\underline{u})$$

is the l-th difference of the function f with step $\underline{u}$ and $l = [r]+1$. We say $f \in B^r_{p,\theta}(\mathbf{R}^d)$, $1 \le p, \theta \le \infty$ if $f \in L^p(\mathbf{R}^d)$ and the following semi-norm is finite:

$$|f|_{B^r_{p,\theta}(\mathbf{R}^d)} := \begin{cases} \left\{ \int_{\mathbf{R}^d} \left(\frac{\left\| \Delta^l_{\underline{u}} f(\cdot) \right\|_p}{\|\underline{u}\|_r} \right)^\theta \frac{d\underline{u}}{\|\underline{u}\|^d} \right\}^{1/\theta}, & 1 \le \theta < \infty, \\ \sup_{\|\underline{u}\| \ne 0} \frac{\left\| \Delta^l_{\underline{u}} f(\cdot) \right\|_p}{\|\underline{u}\|^r}, & \theta = \infty. \end{cases}$$

The linear space $B^r_{p,\theta}(\mathbf{R}^d)$ is the Banach space with the norm:

$$\|f\|_{B^r_{p,\theta}(\mathbf{R}^d)} := \|f\|_p + |f|_{B^r_{p,\theta}(\mathbf{R}^d)}$$

We assume $r > d/p$, by the embedding theorem cf., functions from $B^r_{p,\theta}(\mathbf{R}^d)$ are continuous, and therefore function values are well-defined. Let $\mathsf{A}(B^r_{p,\theta}(\mathbf{R}^d))$ be the unit ball of the space $B^r_{p,\theta}(\mathbf{R}^d)$. In this section we will develop uniform estimates for $(E^\lambda_{\Omega,N} f)(\underline{t})$ when $f \in \mathsf{A}(B^r_{p,\theta}(\mathbf{R}^d))$. Our result is the following theorem.

Theorem 2: *let* $\mathsf{f} \in \mathsf{A}(B^r_{p,\theta}(\mathbf{R}^d))$, $1 \le p, \theta \le \infty$, $r > d/p$, *satisfying the inequality* (3) *and* $\Omega \ge e^{(d+1)/(\delta+(r-d/p))}$. *Then:*

$$(E^\lambda_{\Omega,N} f)(\underline{t}) \le C \cdot \Omega^{-r+d/p} (\ln \Omega)^d$$

provided $N := \Omega^{1+(r-d/p)/\delta}$ and $\omega_\Omega \le \Omega^{-r+d/p}$

To prove Theorem 2 we will choose an intermediate function which is a good approximation for both f and $S_\Omega f$. Now we begin with a description of how to choose this function. For any positive real number v we define the function:

$$g_{\pi v}(x) = A_s (\sin cvx)^{2s} \quad (x \in \mathbf{R}, 2s > 1)$$

where, the constant A_s is taken such that $\int_{\mathbf{R}} g_{\pi v}(x) dx = 1$

Let $\underline{\Omega} = (\Omega_1, \cdots, \Omega_d) > 0$. For $f \in B^r_{p,\theta}(\mathbf{R}^d)$, set

$$(T^s_{\pi\Omega_i} f)(\underline{t}) := \int_{\mathbf{R}} g_{\pi\Omega_i}(x_i)((-1)^{k_i+1} \Delta^{k_i}_{t_i} f(\underline{t}) + f(\underline{t})) dx_i$$

We introduce the operator $T^s_{\pi\Omega_1, \cdots, \pi\Omega_n}$: $B^r_{p,\theta}(\mathbf{R}^d) \to L^p(\mathbf{R}^d)$ by:

$$(T^s_{\pi\Omega_1, \cdots, \pi\Omega_n} f)(\underline{t}) := \int_{\mathbf{R}^t} G_{\pi\Omega_1}(u_1) \cdots G_{\pi\Omega_n}(u_n) f(t_1+u_1, \cdots, t_n+u_n, t_{n+1}, \cdots, t_d) d\underline{u}$$

where, $G_{\pi\Omega_i}(t_i) = \sum_{j=1}^{k_i} (d_j/j) g_{\pi\Omega_i}(t_i/j)$, $1 \le n \le d$. It is known from Nikolskii (1975) that $T^s_{\pi\underline{\Omega}} f \in B_{2s\pi\underline{\Omega}}(\mathbf{R}^d)$. So if $2s \ge 1$, then $T^s_{\pi\underline{\Omega}} f \in B_{\pi\underline{\Omega}}(\mathbf{R}^d)$. The family of linear operator $T^s_{\pi\underline{\Omega}}$ plays an important role in the representation and approximation of functions from Besov space.

Lemma 4: Let $f \in \mathsf{A}(B^r_{p,\theta}(\mathbf{R}^d))$, $1 \le p, \theta \le \infty$, $r > d/p$, then for any $\underline{t} \in \mathbf{R}^d$:

$$\left| f(\underline{t}) - (T^s_{\pi\underline{\Omega}} f)(\underline{t}) \right| \le C \cdot \Omega^{-r+d/p}$$

Lemma 5: Suppose $f \in \mathsf{A}(B^r_{p,\theta}(\mathbf{R}^d))$ satisfy the inequality (3) and $\Omega \ge 1, 2s > d + \delta$ then there exists a constant $A' > 0$ such that:

$$\left| (T^s_{\pi\underline{\Omega}} f)(\underline{t}) \right| \le A' \|\underline{t}\|^{-\delta}$$

Next we should discuss the convergence of $(S_\Omega f)(\underline{t})$ when f is not band-limited. We begin with the introduction of $l^{p,d}$ space. For $1 < p < \infty$, let $l^{p,d} := l_p(\mathbf{Z}^d)$ be the Banach space of all double infinite bounded sequences $\underline{y} = \{y_{\underline{k}}\}$ with the usual norm:

$$\|y\|_p := \left(\sum_{\underline{k} \in \mathbf{Z}^d} |y_{\underline{k}}|^p \right)^{1/p}$$

Lemma 6: *Let* $\underline{y} = \{y_{\underline{k}}\} \in l^{p,d}$, $1 < p < \infty$, *then the series* $\sum_{\underline{k} \in \mathbf{Z}^d} y_{\underline{k}} \sin c(\Omega\underline{t} - \underline{k})$ *converges uniformly on* $\mathbf{R}^d$ *to a function in* $B^p_{\pi\Omega}(\mathbf{R}^d)$.

Theorem 3: Let $f \in \mathsf{A}(B^r_{p,\theta}(\mathbf{R}^d))$, $1 \le p, \theta \le \infty$, $r > d/p$, satisfying the inequality (3) and $\Omega \ge e^{(d+1)/(\delta+(r-d/p))}$, then for any $\underline{t} \in \mathbf{R}^d$:

$$\left| f(\underline{t}) - (S_\Omega f)(\underline{t}) \right| \le C \cdot \Omega^{-r+d/p} \ln^d \Omega$$

Proof: It's known from (5) that the decay condition of f implies $\left\{ f\left(\frac{\underline{k}}{\Omega} \right) \right\} \in l^{p_0,d}$, where, $p_0 \ge \max\{(d+1)/\delta, 1\}$. Thus by Lemma 6 $(S_\Omega f)(\underline{t})$ converges uniformly on $\mathbf{R}^d$. The proof of Theorem 3 relies on the inequality:

$$\left| f(\underline{t}) - S_\Omega f(\underline{t}) \right| \le \left| f(\underline{t}) - (T_{\pi\Omega}^s f)(\underline{t}) \right| + \left| (T_{\pi\Omega}^s f)(\underline{t}) - (S_\Omega f)(\underline{t}) \right| \quad (6)$$

As was mentioned above that under the assumption $2s > 1 + \delta$, $T_{\pi\Omega}^s f \in B_{\pi\Omega}^p$. This fact, together with (3) imply $T_{\pi\Omega}^s f \in F_{\pi\Omega}^p$. Therefore we have $T_{\pi\Omega}^s f = S_\Omega \left(T_{\pi\Omega}^s f \right)$ and $\left| (T_{\pi\Omega}^s)(\underline{t}) - (S_\Omega f)(\underline{t}) \right| = \sum_{\underline{k}\in Z^d} ((T_{\pi\Omega}^s)\left(\frac{\underline{k}}{\Omega}\right) - f(\frac{\underline{k}}{\Omega}))$ $sinc\ (\Omega\underline{t} - \underline{k})$

By Lemma 3 we have:

$$\left| (T_{\pi\Omega}^s f)(\underline{t}) - (S_\Omega f)(\underline{t}) \right| = (I_1 + I_2 + I_3) p_0^{\ d}$$

where,

$$I_1 = \left(\sum_{\|\underline{k}\| \le N} | T_{\pi\Omega}^s f(\underline{k}/\Omega) - f(\underline{k}/\Omega) |^{p_0} \right)^{1/p_0}$$

$$I_2 = \left(\sum_{\|\underline{k}\| > N} \left| (T_{\pi\Omega}^s f)(\underline{k}/\Omega) \right|^{p_0} \right)^{1/p_0}$$

$$I_3 = \left(\sum_{\|\underline{k}\| > N} \left| f(\underline{k}/\Omega) \right|^{p_0} \right)^{1/p_0}$$

Lemma 4 yields that $I_1 \le C \cdot \Omega^{-r+d/p}$ and the estimate of $I_2 + I_3$ as we did in the proof of Theorem 1. In summary, we have obtained:

$$\left| (T_{\pi\Omega}^s f)(\underline{t}) - (S_\Omega f)(\underline{t}) \right| \le C \cdot \left(\Omega^{-r+d/p} + \left(\frac{\Omega}{N} \right)^\delta \right) \ln^d N$$

and set $N = \Omega^{1+(r-d/p)/\delta}$, we yield:

$$\left| (T_{\pi\Omega}^s f)(\underline{t}) - (S_\Omega f)(\underline{t}) \right| \le C \cdot \Omega^{-r+d/p} \ln^d \Omega \quad (7)$$

Thus combining Lemma 4, 6 and (7) we yield the desired result.

Proof of Theorem 2: We will use the triangle inequality:

$$(E_{\Omega,N}^\lambda f)(\underline{t}) \le \left| f(\underline{t}) - (S_\Omega f)(\underline{t}) \right| + \left| (S_\Omega f)(\underline{t}) - (S_{\Omega,N}^\lambda f)(\underline{t}) \right| \quad (8)$$

And,

$$\left| (S_\Omega f)(\underline{t}) - \sum_{\|\underline{k}\| \le N} \lambda_k(f) sinc(\Omega\underline{t} - \underline{k}) \right| \le (I_1 + I_2) \cdot p_0^{\ d}$$

where,

$$I_1 = \left(\sum_{\|\underline{k}\| \le N} | f(\underline{k}/\Omega) - \lambda_k(f)(\cdot + \underline{k}/\Omega) |^{p_0} \right)^{1/p_0}$$

$$I_2 = \left(\sum_{\|\underline{k}\| > N} \left| f(\underline{k}/\Omega) \right|^{p_0} \right)^{1/p_0}$$

p_0 and q_0 are exponent conjugates, i.e., $1/p_0 + 1/q_0 = 1$. Note that the embedding condition and the decay condition of f implies $C_0(\mathbf{R}^d)$. Therefore using the same arguments as in the proof of Theorem 1 we obtain the same error estimates for:

$$\left| ((S_\Omega f)(\underline{t}) - (S_{\Omega,N}^\lambda f)(\underline{t}) \right| \le C \cdot \left(\frac{\Omega}{N} \right)^\delta \cdot \ln^d N, e^{(d+1)/\delta} \le N \le N_0 \quad (9)$$

and

$$\left| ((S_\Omega f)(\underline{t}) - (S_{\Omega,N}^\lambda f)(\underline{t}) \right| \le C \cdot \omega_\Omega(f, \lambda) \cdot \left(\ln \frac{1}{\omega_\Omega(f, \lambda)} \right)^d, N > N_0$$

For $N = \Omega^{1 + \frac{r-d/p}{\delta}}$ and $\omega_\Omega(f, \lambda) \le \Omega^{-r+d/p}$ we know $N \le N_0$ and $\Omega \ge e^{(d+1)/(\delta+(r-d/p))}$ implies $e^{(d+1)/\delta} \le N$. Therefore the conclusion of theorem 2 follows directly from Nikolskii (1975) and Theorem 3.

Some applications: In this section we apply Theorem 1 to some practical examples, which will be summarized in the following corollarys. The first one is that the measured sampled values are given by local average. Observe that the equation (1) requires values of a signal f that are measured on a discrete set. However, due to the physical limitation, say the inertia, a measuring apparatus may not be able to obtain the exact values of f at epoch tk = $\underline{k}$/Ω. In practice only local averages of the signal near the sampling point can be measured. In 1992, Gröchenig K. used a sequence of weight functions to average the original signal locally near the sampling point, and later, Aldroubi A., Butzer. P.L and Lei J., Sun W and Zhou X. obtained a lot of interesting results on the local average sampling. Specifically, let λ = {λ_k}be a sequence of continuous linear functionals. The sampled values are given by the local averages of a function may be formulated by the following integral representation:

$$\lambda_k(f) := \langle f, u_k \rangle = \int f(v) u_k(v) dv \quad (10)$$

where, u_k for each $k \in \mathbf{Z}$ is a weight function characterizing the inertia of measuring apparatus.

Particularly, in the ideal case, the function u_k is given by Dirac δ-function, $u_k = \delta(\cdot + t_k)$, then:

$$\lambda_k(f) := \langle f, u_k \rangle = f(t_k)$$

is the exact value of t_k Gröchenig first studied the reconstruction of signal from local averages in 1992. After that some authors studied the approximation error when local averages are used as sampled values. Now we assume that the functionals λ_k are given by (10) in terms of the weight functions u_k and u_k satisfy the following properties:

i). $\operatorname{supp} u_k \subset \left[\dfrac{k}{\Omega} - \sigma'_k, \dfrac{k}{\Omega} - \sigma^*_k\right]$, where $\dfrac{\sigma}{4} \leq \|\sigma'_k\|$

$\|\sigma^*_k\| \leq \dfrac{\sigma}{2}$, σ is a positive constant:

ii). $u_k(\underline{t}) \geq 0$, $\int u_k(\underline{t}) d\underline{t} = 1$ (11)

iii). $\alpha = \inf\limits_{k \in \mathbf{Z}}\{\alpha_k\}$, where, $\alpha_k := \int_{\underline{k}/\Omega - \sigma/4}^{k/\Omega + \sigma/4} u_k(\underline{t}) d\underline{t}$

Note that now:

$$\lambda_k(f) = \int_{\underline{k}/\Omega - \sigma'_k}^{k/\Omega + \sigma^*_k} u_k(\underline{t}) f(\underline{t}) d\underline{t}$$

To estimate $(E_{\Omega,N}^\lambda f)(\underline{t})$, we need the concept of the modulus of continuity. For $f \in C_0(\mathbf{R}^d)$, we define the modulus of continuity by:

$$\omega(f, \sigma) = \sup_{\|h\| \leq \sigma} \|f(\cdot + h) - f(\cdot)\|_\infty$$

where, σ is any positive number. Note that the functions in $C_0(\mathbf{R}^d)$ are uniformly continuous on $\mathbf{R}^d$. Therefore we have $\lim_{\sigma \to 0} \omega(f, \sigma) = 0$.

Corollary 1: Let $\Omega \geq e^{(d+1)/(\delta + (r - d/p))}$, $\sigma < 1/\Omega$ and $f \in \mathsf{A}(B_{p,\theta}^r(\mathbf{R}^d))$ satisfying (3). Suppose $\lambda_k(f)$ is obtained by the rule (11). If $\omega(f, \sigma/4) \leq \min\{e^{-d}, \Omega^{-r + d/p}/(2 - \alpha)\}$. Then:

$$\left| f(\cdot) - \sum_{\|\underline{k}\| \leq N} \lambda_k(f) \operatorname{sinc}(\Omega \cdot - \underline{k}) \right| \leq C \cdot \omega(f, \sigma/4) \left(\ln \frac{1}{\omega(f, \sigma/4)} \right)^d$$

where, C is a constant.

Proof: Note that:

$$\lambda_k(f) = \int_{\underline{k}/\Omega - \sigma'_k}^{k/\Omega + \sigma^*_k} u_k(\underline{t}) f(\underline{t}) d\underline{t}$$

and $f(\underline{k}/\Omega) = \int_{\underline{k}/\Omega - \sigma'_k}^{k/\Omega + \sigma^*_k} u_k(\underline{t}) f(\underline{k}/\Omega) d\underline{t}$
Then we have:

$$\left| f(\underline{k}/\Omega) - \int_{\underline{k}/\Omega - \sigma'_k}^{k/\Omega + \sigma^*_k} u_k(\underline{t}) f(\underline{t}) d\underline{t} \right|$$

$$= \left| \int_{\underline{k}/\Omega - \sigma'_k}^{k/\Omega - \sigma/4} + \int_{\underline{k}/\Omega - \sigma/4}^{k/\Omega + \sigma/4} + \int_{\underline{k}/\Omega + \sigma/4}^{k/\Omega + \sigma^*_k} u_k(\underline{t}) (f(\underline{k}/\Omega) - f(\underline{t} + \underline{k}/\Omega)) d\underline{t} \right|$$

$$\leq \left| \left(\int_{-\sigma'_k}^{\sigma/4} + \int_{\sigma/4}^{\sigma^*_k} u_k(\underline{t} + \underline{k}/\Omega) \omega(f, \sigma/2) d\underline{t} \right) \right.$$

$$\left. + \int_{-\sigma/4}^{\sigma/4} u_k(\underline{t} + \underline{k}/\Omega) \omega(f, \sigma/4) d\underline{t} \right|$$

Because, $\omega(f, \sigma/2) \leq 2\omega(f, \sigma/4)$, so

$$\omega_\Omega(f, \lambda) \leq (2 - \alpha) \omega(f, \sigma/4)$$

We also note that the function $x \to x \ln(1/x)^d$ is monotonely increasing for $x \in (0, e^{-d})$ and we complete the proof.

Next we consider the combination of all four errors. The four usual types of errors existing in sampling series are the amplitude error, the time-jitter error, the truncation errors and the aliasing errors. First we give some explanation for the amplitude error and the time-jitter error.

We assume the amplitude error results from quantization, which means the functional value $f(\underline{t})$ of a function f at moment $\underline{t}$ is replaced by the nearest discrete value or machine number $\overline{f}(\underline{t})$ The quantization size is often known before hand or can be chosen arbitrarily. We may assume that the local error at any moment $\underline{t}$ is bounded by a constant $\varepsilon > 0$, i. e., $|\overline{f}(\underline{t}) - f(\underline{t})| \leq \varepsilon$.

The time-jitter error arises if the sampled instances are not met correctly but might differ from the exact ones by σ_k, We assume $\sigma_k \leq \sigma$ for all k and a constant σ.

Let us now consider the combination of all four errors, namely the combined error:

$$E_N f(\underline{t}) := f(\underline{t}) - \sum_{\|\underline{k}\| \leq N} \overline{f}(\underline{k}/\Omega + \sigma_k) \operatorname{sinc}(\Omega \underline{t} - \underline{k})$$

Corollary 2: Let $f \in \mathsf{A}(B_{p,\theta}^r(\mathbf{R}^d))$ *satisfying (3). For* $\Omega \geq e^{(d+1)/(\delta + (r - d/p))}$ we have:

$$|E_N f| \leq C \cdot \Omega^{-r+d/p} \ln^d \Omega$$

provided $N = \Omega^{1+\frac{r-d/p}{\delta}}$, $|\bar{f}(\underline{t}) - f(\underline{t})| \leq c_1 \Omega^{-r+d/p}$ and $\omega(f, \sigma) \leq c_2 \Omega^{-r+d/p}$ for some constants $c_1 > 0$ and $c_2 > 0$ and for all $\underline{t} \in \mathbf{R}^d$ where $c_1 + c_2 \leq 1$ is a constant.

Proof: We define:

$$\lambda_k = \frac{\bar{f}(\underline{k}/\Omega + \sigma_k)}{f(\underline{k}/\Omega + \sigma_k)} \delta(\cdot - \sigma_k)$$

where, δ is the Dirac distribution and $0 < \| \sigma 2 \| \leq \sigma$. Then $\lambda = \{\lambda_k\}$ is a sequence of linear functional on $C_0(\mathbf{R}^d)$. It is clear that $\lambda_k f(\cdot + \underline{k}/\Omega) = \bar{f}(\underline{k}/\Omega + \sigma_k)$ and:

$$\begin{aligned}
&\left|\lambda_k f(\cdot + \underline{k}/\Omega) - f(\underline{k}/\Omega)\right| \\
&\leq \left|\bar{f}(\underline{k}/\Omega + \sigma_k) - f(\underline{k}/\Omega + \sigma_k)\right| + \left|f(\underline{k}/\Omega + \sigma_k) - f(\underline{k}/\Omega)\right| \\
&\leq (c_1 + c_2)\Omega^{-r+d/p}
\end{aligned}$$

Thus $\omega_\Omega(f, \lambda) \leq \Omega^{-r+d/p}$. The theorem follows from Theorem 2.1.

CONCLUSION

In this study we show that a function has a smoothness of order $r > 0$ then the approximation order we can get is $\Omega^{-r+d/p} \ln^d \Omega$. We also apply the results to some practical examples. Finally we would like to mention that we expect the sampling series to work better, i.e., without the factor $\ln^d \Omega$ and it would be of great interest to generalize Theorem 3 to the case of the scalar $\underline{\Omega} = (\Omega_1 ... \Omega_d), \Omega_i \neq \Omega_j$.

REFERENCES

Boche, H., 2010. Approximation of wide-sense stationary stochastic processes by Shannon sampling series. IEEE T. Inform. Theory, 56(12): 6459-6469.

Burchard, H. and J. Lei, 1995. Quasi-interpolation approximation using functional basis. J. Approx. Theory, 82: 240-256.

Butzer, P.L. and J.J. Lei, 1998. Errors in truncated sampling series with measured sampled values for non-necessarily bandlimited functions [J]. Funct. Approx., 26: 25-39.

Casey, S. and D.Walnut, 1994. Systems of convolution equations deconvolution, Shannon sampling and wavelets and Gabor transforms [J]. SIAM Rev., 36: 537-577.

Nikolskii, S.M., 1975. Approixmation of Functions of Several Variables and Imbedding Theorems. Springer-Verlag, Berlin/Heidelberg, New York.

Shannon, C.E., 1948. A mathematical theory of communication [J]. Bell Sys. Tech. J., 27: 397-423, 623-656.

Zayed, A.I., 1993. Advances in Shannon's Sampling Theory. CRC Press, Boca Raton, FL.