Research Article
Solution of Seventh Order Boundary Value Problems by Variation of Parameters Method

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Abstract: The induction motor behavior is represented by a fifth order differential equation model. Addition of a torque correction factor to this model accurately reproduces the transient torques and instantaneous real and reactive power flows of the full seventh order differential equation model. The aim of this study is to solve the seventh order boundary value problems and the variation of parameters method is used for this purpose. The approximate solutions of the problems are obtained in terms of rapidly convergent series. Two numerical examples have been given to illustrate the efficiency and implementation of the method.

Keywords: Approximate solution, boundary value problems, linear and nonlinear problems, variation of parameters method

INTRODUCTION

The theory of seventh order boundary value problems is not much available in the numerical analysis literature. The seventh order boundary value problems generally arise in modelling induction motors with two rotor circuits. The behavior of induction motor is represented by a fifth order differential equation model. This model contains two stator state variables, two rotor state variables and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth order and rotor impedance is algebraically altered as function of rotor speed. This is done under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. So, the behaviors of such models show up in the seventh order differential equation model (Richards and Sarma, 1994).

Siddiqi and Akram (2006a, b) presented the solutions of fifth and sixth order boundary value problems using non-polynomial spline. Noor *et al.* (2008) used the variation of parameters method for solving the fifth order boundary value problems. In

Akram and Rehman (2011), solved the fifth order boundary value problems using the Reproducing Kernel space method. Akram and Siddiqi (2012) presented the solution of seventh order boundary value problem using octic spline.

In this study, the solution of seventh order boundary value problem is presented using variation of parameters method, following variation of parameters method in Mohyud-Din *et al.* (2009) for the solution of sixth order boundary value problems. This method does not require the identification of Lagrange multipliers and applied in a direct way.

VARIATION OF PARAMETERS METHOD

Consider the seventh order boundary value problem of the form:

$$u^{(7)}(x) = f(x, u(x)), \quad a \le x \le b \qquad (1)$$

with boundary conditions:

$$
\left.
\begin{array}{l}
u(a) = A_0, u^{(1)}(a) = A_1, u^{(2)}(a) = A_2, u^{(3)}(a) = A_3, \\[4pt]
u(b) = A_4, u^{(1)}(b) = A_5, u^{(2)}(b) = A_6 \\[4pt]
OR \\[4pt]
u(a) = A_0, u^{(2)}(a) = A_1, u^{(4)}(a) = A_2, u^{(6)}(a) = A_3, \\[4pt]
u(b) = A_4, u^{(2)}(b) = A_5, u^{(4)}(b) = A_6
\end{array}
\right\} \qquad (2)
$$

The variation of parameters method provides the solution of Eq. (1) as:

$$u(x) = \sum_{i=0}^{n} A_i u_i(k) + \sum_{i=0}^{n} A_i u_i(k) \int_{0}^{x} f(t,u)g(t)dt \quad (3)$$

The first term on right hand side of the Eq. (3) is said to be the complementary solution of equation Eq. (1) and second term is said to be the particular solution. In variation of parameters method the constants A_i s are replaced by the parameters and using this modified expression in Eq. (1) a system of equations is obtained after some making some assumptions. The solution of this system gives the values of the parameters in terms of the integrals and hence the particular solution is obtained as in Eq. (3). Consequently, applying the boundary conditions (2) on Eq. (3), the following reccurence relation is obtained:

$$u_{n+1}(x) = v(x) + \int_{a}^{b} f(t,u_n)g(x,t) + \int_{0}^{x} f(t,u_n)g(x,t)dt \quad (4)$$

Using initial approximation as $u_0(x) = v(x)$.

To implement the method, two numerical examples are considered in the following section.

NUMERICAL EXAMPLES

Example 1: Consider the linear seventh order boundary value problem:

$$\left. \begin{array}{l} u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), 0 \le x \le 1 \\ u(0) = 0, u^{(1)}(0) = 1, u^{(2)}(0) = 0, u^{(3)}(0) = -3, \\ u(1) = 0, u^{(1)}(1) = -e, u^{(2)}(1) = -4e \end{array} \right\} \quad (5)$$

The exact solution of the Example 1 is $u(x) = x(1-x)e^x$ (Akram and Siddiqi, 2012).

Using the variation of parameters method (4), the given seventh order boundary value problem (5) can be written as:

$$u_{n+1}(x) = A_0 + A_1 x + A_2 \frac{x^2}{2!} + A_3 \frac{x^3}{3!}$$
$$+ A_4 \frac{x^4}{4!} + A_5 \frac{x^5}{5!} + A_6 \frac{x^6}{6!}$$
$$+ \int_{0}^{x} \left(\frac{t^6}{720} - \frac{xt^5}{120} + \frac{x^2 t^4}{48} - \frac{x^3 t^3}{36} + \frac{x^4 t^2}{48} - \frac{x^5 t}{120} + \frac{x^6}{720} \right) \quad (6)$$
$$(-u_n(t) - e^t(35 + 12t + 2t^2))dt.$$

Applying the boundary conditions (5), on Eq. (6), yields:

$$A_0 = 0, \ A_1 = 1, \ A_2 = 0, \ A_3 = -3,$$

$$A_4 = 12(-17 - 6e -$$
$$\int_{0}^{1} \frac{-1}{24}(-1+t)^4(e^t(35 + 12t + 2t^2) + u_n(t))dt$$

$$+ 10\int_{0}^{1} \frac{1}{120}(-1+t)^5(e^t(35 + 12t + 2t^2) + u_n(t))dt$$

$$- 30\int_{0}^{1} \frac{-1}{720}(-1+t)^6(e^t(35 + 12t + 2t^2) + u_n(t))dt),$$

$$A_5 = -60(-27 + 10e - 2$$
$$\int_{0}^{1} \frac{-1}{24}(-1+t)^4(e^t(35 + 12t + 2t^2) + u_n(t))dt$$

$$+ 18\int_{0}^{1} \frac{1}{120}(-1+t)^5(e^t(35 + 12t + 2t^2) + u_n(t))dt$$

$$- 48\int_{0}^{1} \frac{-1}{720}(-1+t)^6(e^t(35 + 12t + 2t^2) + u_n(t))dt),$$

$$A_6 = 360(-11 + 4e -$$
$$\int_{0}^{1} \frac{-1}{24}(-1+t)^4(e^t(35 + 12t + 2t^2) + u_n(t))dt$$

$$+ 8\int_{0}^{1} \frac{1}{120}(-1+t)^5(e^t(35 + 12t + 2t^2) + u_n(t))dt$$

$$- 20\int_{0}^{1} \frac{-1}{720}(-1+t)^6(e^t(35 + 12t + 2t^2) + u_n(t))dt).$$

Finally, the series solution is given as:

$$u(x) = u_3(x) = 1 - 0.5x^3 - 0.33333x^4 - 0.125x^5$$
$$- 0.0333334x^6 - 0.00694444x^7 - 0.00119048x^8$$

$$- 0.000173611x^9 - 0.0000220459x^{10}$$
$$- 2.48016 \times 10^{-6} x^{11} - 2.50521 \times 10^{-7} x^{12}$$

$$- 2.29644 \times 10^{-8} x^{13} - 2.32856 \times 10^{-9} x^{14}$$
$$- 1.85826 \times 10^{-10} x^{15} - 1.37171 \times 10^{-11} x^{16} + O(x^{17})$$

The comparison of the exact solution with the series solution of the Example 1 is given in Table 1. Maximum absolute errors for Example 1 are compared with the octic spline method (Akram and Siddiqi, 2012)

Table 1: Comparison of numerical results for Example 1

x	Exact solution	Approximate series solution	Absolute error present method
0.0	0.0000	0.0000	0.0000
0.1	0.9946	0.9946	8.55607E-13
0.2	0.1954	0.1954	9.94041E-12
0.3	0.2835	0.2835	3.52244E-11
0.4	0.3580	0.3580	7.3224E-10
0.5	0.4122	0.4122	1.08769E-10
0.6	0.4373	0.4373	1.29035E-10
0.7	0.4229	0.4229	1.51466E-10
0.8	0.3561	0.3561	2.717974E-10
0.9	0.2214	0.2214	7.48179E-10
1.0	0.0000	-2.1729E-09	2.1729E-09

Table 2: Comparison of maximum absolute errors for Example 1

Present method (for u_3)	Octic spline method (Akram and Siddiqi, 2012)	
2.1729E-09	$n = 10$,	5.5071E-06
	$n = 20$,	2.2960E-07
	$n = 30$,	2.5180E-08
	$n = 40$,	3.1744E-09

in Table 2, which shows that the present method is quite efficient.

Example 2: The following seventh order nonlinear boundary value problem is considered:

$$\left.\begin{array}{l} u^{(7)}(x) = e^{-x}u^2(x), 0 \leq x \leq 1 \\ u(0) = u^{(2)}(0) = u^{(4)}(0) = u^{(6)}(0) = 1, \\ u(1) = u^{(2)}(1) = u^{(4)}(1) = e, \end{array}\right\} \quad (7)$$

The exact solution of the problem (6) is $u(x) = e^x$.

Using the variation of parameters method (4), the given seventh order boundary value problem (7) can be written as:

$$u_{n+1}(x) = A_0 + A_1 x + A_2 \frac{x^2}{2!} +$$

$$A_3 \frac{x^3}{3!} + A_4 \frac{x^4}{4!} + A_5 \frac{x^5}{5!} + A_6 \frac{x^6}{6!}$$

$$+ \int_0^x \left(\frac{t^6}{720} - \frac{xt^5}{120} + \frac{x^2 t^4}{48} - \frac{x^3 t^3}{36} + \frac{x^4 t^2}{48} - \frac{x^5 t}{120} + \frac{x^6}{720} \right) \quad (8)$$
$$(u_n(t))^2 e^{-t} dt$$

Applying the boundary conditions (7), on Eq. (8), gives:

$$A_0 = 1,$$

$$A_1 = \frac{1}{720}(-947 + 614e - 14$$

$$\int_0^1 (\frac{1}{2} - t + \frac{t^2}{2}) e^{-t} (u_n(t))^2 dt$$

$$+120 \int_0^1 (\frac{1}{24} - \frac{t}{6} + \frac{t^2}{4} - \frac{t^3}{6} + \frac{t^4}{24}) e^{-t} (u_n(t))^2 dt$$

$$+720 \int_0^1 \left(\frac{1}{720} - \frac{t}{120} + \frac{t^2}{48} - \frac{t^3}{36} + \frac{t^4}{48} - \frac{t^5}{120} + \frac{t^6}{720} \right) e^{-t} (u_n(t))^2 dt,$$

$$A_2 = 1,$$

$$A_3 = \frac{1}{24}(43 - 20e + 4 \int_0^1 (\frac{1}{2} - t + \frac{t^2}{2}) e^{-t} (u_n(t))^2 dt$$

$$-24 \int_0^1 (\frac{1}{24} - \frac{t}{6} + \frac{t^2}{4} - \frac{t^3}{6} + \frac{t^4}{24}) e^{-t} (u_n(t))^2 dt,$$

$$A_4 = 1,$$

$$A_5 = \frac{3}{2} + e - \int_0^1 (\frac{1}{2} - t + \frac{t^2}{2}) e^{-t} (u_n(t))^2 dt,$$

$$A_6 = 1$$

Finally, the series solution can be written as:

$$u(x) = u_3(x) = 1 + 0.999998x - 0.5x^2 + 0.16667x^2$$
$$+ 0.0416667x^3 - 0.125x^4 + 0.0083319x^5$$

$$+ 0.00138889x^6 + 0.000198413x^7$$
$$+ 0.0000248015x^8 + 2.9904 \times 10^{-6} x^9 + 3.22348$$
$$\times 10^{-8} x^{10} + 1.1742 \times 10^{-7} x^{11} - 1.43111 \times 10^{-8} x^{12}$$
$$- 7.38378 \times 10^{-10} x^{13} + 1.52458 \times 10^{-9} x^{14}$$

$$- 4.02895 \times 10^{-10} x^{15} - 1.29386 \times 10^{-11} x^{16} + O(x^{17})$$

In Table 3, the exact solution and the series solution of the Example 2 are compared, which shows that the method is quite accurate.

Table 3: Comparison of numerical results for Example 2

x	Exact solution	Approximate series solution	Absolute error present method
0.0	1.0000	1.0000	0.0000
0.1	1.1051	1.1051	2.26257E-07
0.2	1.2214	1.2214	4.38942E-07
0.3	1.3498	1.3498	6.1274E-07
0.4	1.4918	1.4918	7.71759E-07
0.5	1.6487	1.6487	7.71759E-07
0.6	1.8221	1.8221	7.37682E-07
0.7	2.0137	2.0137	6.25932E-07
0.8	2.2255	2.2255	4.68244E-07
0.9	2.4596	2.4596	2.95852E-07
1.0	2.7182	2.7182	1.25922E-07

CONCLUSION

In this study, the variation of parameters method has been applied to obtain the numerical solutions of linear and nonlinear seventh order boundary value problems. The method applied directly without using any linearization, discretization or perturbation assumptions. This method gives rapidly converging series solutions in both linear and nonlinear cases. The numerical results show that the present method is more accurate.

REFERENCES

Akram, G. and H. Rehman, 2011. Solution of fifth order boundary value problems in reproducing kernel space. Middle-East J. Sci. Res., 10(2): 191-195.

Akram, G. and S.S. Siddiqi, 2012. Solution of seventh order boundary value problem using octic spline. Arch. Des Sci., 65(1), Accepted for Publication (In Press).

Mohyud-Din, S.T., M.A. Noor and A. Waheed, 2009. Variation of parameters method for solving sixth-order boundary value problems. Comm. Korean Math. Soc., 24(4): 605-615.

Noor, M.A., S.T. Mohyud-Din and A. Waheed, 2008. Variation of parameters method for solving fifth-order boundary value problems. Appl. Math. Inf. Sci. Int. J., 2(2): 135-141.

Richards, G. and P.R.R. Sarma, 1994. Reduced order models for induction motors with two rotor circuits. IEEE Trans. Energ. Conv., 9(4): 673-678.

Siddiqi, S.S. and G. Akram, 2006a. Solutions of fifth order boundary-value problems using nonpolynomial spline technique. Appl. Math. Comput., 175(2): 1574-1581.

Siddiqi, S.S and G. Akram, 2006b. Solutions of sixth order boundary-value problems using nonpolynomial spline technique. Appl. Math. Comput., 181: 708-720.