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Research Article

Development and Validation of an Analytical Method to the Solution of Modelling the Pollution of a System of Lakes

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Abstract: In this study analytically systems of nonlinear ordinary differential equations such as modeling the pollution of a system of lakes are studied by a powerful analytical method that is called Reconstruction of Variational Iteration Method (RVIM). The RVIM technique provides us a simple way to ensure the convergence of solution series, so that we can always get accurate enough approximations. Also, it is independent of any small parameters at all. The method is capable of reducing the size of calculation and easily overcomes the difficulty of the perturbation technique or Adomian polynomials. The results are compared with the results obtained by MATLAB. Some plots are presented to show the reliability and simplicity of the methods.

Keywords: Modeling the pollution of a system of lakes, Reconstruction of Variational Iteration Method (RVIM)

INTRODUCTION

The Great Lakes are very big and very beautiful, but most of all they are very fragile and delicate.

We can prevent pollution in river water in many ways and some of them are as follows:

- Not discharging untreated sewage and other industrial waste water into rivers
- Not dumping solid wastes, garbage and medical wastes in river banks
- Not using river banks as open toilets
- Not disposing dead human bodies/animal carcasses in rivers
- Not quarrying sand in river beds which is a key factor in river bank stability and erosion of banks
- Not washing vehicles directly in the flowing river water and in many other ways according to the life style of local people

Since the lakes are surrounded by land, once bad toxic substances go into the lake, they will stay there for a long time. This is why it is so important that we work together to make sure the lakes do not become polluted.

Modelling the pollution of a system of lakes is examined at the study (Biazar *et al*., 2006). Figure 1 shows the system of three lakes that are modeled in this study (Hoggard, 2008). Each lake is considered to be a large compartment and the interconnecting channel as pipes between the compartments. The direction of flow in the channels or pipes is indicated by the arrows in (Hoggard, 2008). A pollutant is introduced into the first lake where p (t) denotes the rate at which the pollutant enters the lake per unit time. The function p (t) may be constant or may vary with time. We are interested in knowing the levels of pollution in each lake at any time. The components of the basic three-component model are the amount of the pollutant in lake 1 at any time $t \geq 0$, the amount of the pollutant in lake 2 at any time $t \geq 0$,and the amount of the pollutant in lake 3 at any time $t \geq 0$, are denoted respectively by $x(t)$, $y(t)$ and $z(t)$. These quantities satisfy:

$$\begin{aligned}\frac{dx}{dt} &= \frac{f_{13}}{v_3} z(t) + p(t) - \frac{f_{31}}{v_1} x(t) - \frac{f_{21}}{v_1} x(t)\\ \frac{dy}{dt} &= \frac{f_{21}}{v_1} x(t) - \frac{f_{32}}{v_2} y(t)\\ \frac{dz}{dt} &= \frac{f_{31}}{v_1} x(t) + \frac{f_{32}}{v_2} y(t) - \frac{f_{13}}{v_3} z(t)\end{aligned} \tag{1}$$

with the initial conditions:

$$x(0) = r_1, \quad y(0) = r_2 \quad z(0) = r_3 \tag{2}$$

Throughout this study, we assume the following conditions:

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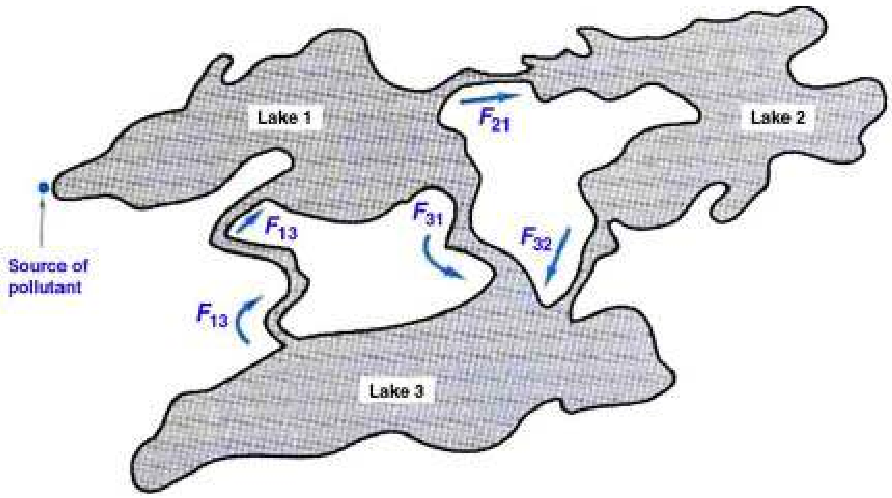

Fig. 1: System of three lakes with interconnecting channels. A pollutant enters the first lake at the indicated source (Giordano and Weir, 1991)

$$lake1: f_{13} = f_{21} + f_{31}$$
$$lake2: f_{21} = f_{32}$$
$$lake3: f_{31} + f_{32} = f_{13}$$

In recent years, several such techniques have drawn special attention, such as the homogeneous balance method (Wang *et al*., 1996), Adomian's decomposition method ADM (Adomian, 1998), the Variational Iteration Method (Nikkar and Mighani, 2012; Saadati *et al*., 2009; He, 1999, 2000a), Homotopy Perturbation Method (Yildirim *et al*., 2010; Taghipour 2010) Energy Balance Method (Nikkar *et al*., 2011), as well as Homotopy Analysis Method (HAM). After that, many types of nonlinear problems were solved by the HAM by others (He, 2000b; Matinfar *et al*., 2012; Khan *et al*., 2012). One of the newest analytical methods to solve nonlinear equations is Reconstruction of Variational Iteration Method (RVIM) which is an accurate and a rapid convergence method in finding the approximate solution for nonlinear equations. By applying Laplace Transform, RVIM overcomes the difficulty of the perturbation techniques and other variational methods in case of using small parameters and Lagrange multipliers, respectively. Reducing the size of calculations and omitting the difficulty arising in calculation of nonlinear intricately terms are other advantages of this method. In this study RVIM has been applied to solve of modeling the pollution of a system of lakes (1). The numerical solutions are compared with the available exact and by MATLAB ode15s.

BASIC IDEA OF RVIM

To clarify the basic ideas of our proposed method in (Hesameddini and Latifizadeh, 2009; Nikkar *et al*., 2012), we consider the following differential equation same as VIM based on Lagrange multiplier (Wazwaz, 2007):

$$Lu(x_1,\ldots,x_k) + Nu(x_1,\ldots,x_k) = f(x_1,\ldots,x_k) \tag{3}$$

By suppose that:

$$Lu(x_1,\cdots,x_k) = \sum_{i=0}^{k} L_{xi}u(x_i) \tag{4}$$

where, L is a linear operator, N a nonlinear operator and $f(x_1,\cdots,x_k)$ an inhomogeneous term.

we can rewrite Eq. (3) down a correction functional as follows:

$$L_{xj}u(x_j) = \underbrace{f(x_1,\cdots,x_k) - Nu(x_1,\cdots,x_k) - \sum_{\substack{i=0\\ i\neq j}}^{k} L_{xi}u(x_i)}_{h((x_1,\cdots,x_k),u(x_1,\cdots,x_k))} \tag{5}$$

therefore:

$$L_{xj}u(x_j) = h((x_1,\cdots,x_k),u(x_1,\cdots,x_k)) \tag{6}$$

with artificial initial conditions being zero regarding the independent variable x_j.

By taking Laplace transform of both sides of the Eq. (6) in the usual way and using the artificial initial conditions, we obtain the result as follows:

$$\begin{aligned} &P(s).U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k) \\ &= H((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u) \end{aligned} \tag{7}$$

where, P (s) is a polynomial with the degree of the highest derivative in Eq. (7), (the same as the highest order of the linear operator L_{x_j}). The following relations are possible:

$$\ell[h] = H \quad (8\text{-a})$$

$$B(s) = \frac{1}{P(s)} \quad (8\text{-b})$$

$$\ell[b(x_i)] = B(s) \quad (8\text{-c})$$

Which that in Eq. (8-a) the function:

$$H((x_1, \cdots, x_{i-1}, s, x_{i+1}, x_k), u)$$

and

$$h((x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k), u)$$

have been abbreviated as H, h, respectively.
Hence, rewrite the Eq. (7) as:

$$\begin{aligned} &U(x_1, \cdots, x_{i-1}, s, x_{i+1}, x_k) \\ &= H((x_1, \cdots, x_{i-1}, s, x_{i+1}, x_k), u).B(s) \end{aligned} \quad (9)$$

Now, by applying the inverse Laplace Transform on both sides of Eq. (9) and by using the (8-a) to (8-c), we have:

$$\begin{aligned} &u(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) \\ &= \int_0^{x_i} h((x_1, \cdots, x_{i-1}, \tau, x_{i+1}, x_k), u).b(x_i - \tau) d\tau \end{aligned} \quad (10)$$

Now, we must impose the actual initial conditions to obtain the solution of the Eq. (3). Thus, we have the following iteration formulation:

$$\begin{aligned} u_{n+1}(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) &= u_0(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) \\ &+ \int_0^{x_i} h((x_1, \cdots, x_{i-1}, \tau, x_{i+1}, x_k), u).b(x_i - \tau) d\tau \end{aligned} \quad (11)$$

where, u_0 is initial solution with or without unknown parameters. Assuming u_0 is the solution of Lu, with initial/boundary conditions of the main problem, In case of no unknown parameters, u_0 should satisfy initial/boundary conditions. When some unknown parameters are involved in u_0, the unknown parameters can be identified by initial/boundary conditions after few iterations, this technology is very effective in dealing with boundary problems. It is worth mentioning that, in fact, the Lagrange multiplier in the He's variational iteration method is $\lambda(\tau) = b(x_i - \tau)$ as shown in (Hesameddini and Latifizadeh, 2009).

The initial values are usually used for selecting the zeroth approximation u_0. With u_0 determined, then several approximations u_n $n > 0$, follow immediately. Consequently, the exact solution may be obtained by using:

$$\begin{aligned} &u(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) \\ &= \lim_{n\to\infty} u_n(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) \end{aligned} \quad (12)$$

Applying RVIM to modelling the pollution of a system of lakes: We close our analysis by studying the Pollution of a System of Lakes:

$$\begin{aligned} \frac{dx}{dt} &= \frac{f_{13}}{v_3} z(t) + p(t) - \frac{f_{31}}{v_1} x(t) - \frac{f_{21}}{v_1} x(t) \\ \frac{dy}{dt} &= \frac{f_{21}}{v_1} x(t) - \frac{f_{32}}{v_2} y(t) \\ \frac{dz}{dt} &= \frac{f_{31}}{v_1} x(t) + \frac{f_{32}}{v_2} y(t) - \frac{f_{13}}{v_3} z(t) \end{aligned} \quad (13)$$

with the initial conditions:

$$x(0) = r_1, \quad y(0) = r_2 \quad z(0) = r_3 \quad (14)$$

At first rewrite Eq. (13) based on selective linear operator as:

$$\begin{aligned} \ell\{x(t)\} &= \frac{dx}{dt} = \overbrace{(\frac{f_{13}}{v_3} z(t) + p(t) - \frac{f_{31}}{v_1} x(t) - \frac{f_{21}}{v_1} x(t)}^{h(x,t,u)} \\ \ell\{y(t)\} &= \frac{dy}{dt} = \overbrace{(\frac{f_{21}}{v_1} x(t) - \frac{f_{32}}{v_2} y(t))}^{h(x,t,v)} \\ \ell\{z(t)\} &= \frac{dz}{dt} = \overbrace{(\frac{f_{31}}{v_1} x(t) + \frac{f_{32}}{v_2} y(t) - \frac{f_{13}}{v_3} z(t))}^{h(x,t,w)} \end{aligned} \quad (15)$$

Now Laplace transform is implemented with respect to independent variable x on both sides of Eq. (15) and by using the new artificial initial condition (which all of them are zero) we have:

$$\begin{cases} s \cup (t) = \ell\{h(x,t)\} \\ s \cup (t) = \ell\{h(x,t)\} \\ s \cup (t) = \ell\{h(x,t\} \end{cases} \quad (16)$$

$$\begin{cases} \cup(t) = \dfrac{\ell\{h(x,t)\}}{s} \\ \cup(t) = \dfrac{\ell\{h(x,t)\}}{s} \\ \cup(t) = \dfrac{\ell\{h(x,t)\}}{s} \end{cases} \tag{17}$$

And whereas Laplace inverse transform of *1/s* is as follows:

$$\ell^{-1}[\frac{1}{s}] = 1 \tag{18}$$

Therefore by using the Laplace inverse transform and convolution theorem it is concluded that:

$$\begin{aligned} x(t) &= \int_0^t h(x,\varepsilon)d\varepsilon \\ y(t) &= \int_0^t h(y,\varepsilon)d\varepsilon \\ z(t) &= \int_0^t h(z,\varepsilon)d\varepsilon \end{aligned} \tag{19}$$

Hence, we arrive the following iterative formula for the approximate solution of subject to the initial condition (14). So, in exchange with applying recursive algorithm, following relations are achieved:

$$\begin{aligned} x_{n+1}(t) &= x_0(t) + \int_0^t \left\{ + \frac{f_{13}}{v_3}\tilde{z}_n(\varepsilon) + p(\varepsilon) - \left(\frac{f_{31}}{v_1} + \frac{f_{21}}{v_1} \right)\tilde{x}_n(\varepsilon) \right\} d\varepsilon \\ y_{n+1}(t) &= y_0(t) + \int_0^t \left\{ + \frac{f_{21}}{v_1}\tilde{x}_n(\varepsilon) - \frac{f_{32}}{v_2}\tilde{y}_n(\varepsilon) \right\} d\varepsilon \\ z_{n+1}(t) &= z_0(t) + \int_0^t \left\{ + \frac{f_{32}}{v_1}\tilde{x}_n(\varepsilon) + \frac{f_{32}}{v_2}\tilde{y}_n(\varepsilon) - \frac{f_{13}}{v_3}\tilde{z}_n(\varepsilon) \right\} d\varepsilon \end{aligned} \tag{20}$$

We start with an initial approximation $x_0(t) = r_1, y_0(t) = r_2 \; and \; x_0(t) = r_3$ by the iteration formula (20), we can obtain the first few components as follows:

$$\begin{aligned} x_1(t) &= r_1 + \left[\frac{f_{13}r_3}{v_3} + p - \left(\frac{f_{31} + f_{21}}{v_1} \right) r_1 \right] t, \\ y_1(t) &= r_2 + \left[\frac{f_{21}r_1}{v_1} - \frac{f_{32}r_2}{v_2} \right] t, \\ z_1(t) &= r_1 + \left[\frac{f_{31}r_1}{v_1} + \frac{f_{32}r_2}{v_2} - \frac{f_{31}r_3}{v_3} \right] t, \end{aligned} \tag{21}$$

$$\begin{aligned} x_2(t) &= r_1 + \left[\frac{f_{13}r_3}{v_3} + p - \left(\frac{f_{31}+f_{21}}{v_1} \right) r_1 \right] t + \frac{1}{2} \left[\begin{array}{l} \frac{f_{13}}{v_3}\left(\frac{f_{31}r_1}{v_1} + \frac{f_{32}r_2}{v_2} - \frac{f_{31}r_3}{v_3} \right) \\ -\left(\frac{f_{31}+f_{21}}{v_1} \right) \\ \left(\frac{f_{13}r_3}{v_3} + p - \left(\frac{f_{31}+f_{21}}{v_1} \right) r_1 \right) \end{array} \right] t^2 \\ y_2(t) &= r_2 + \left[\frac{f_{21}r_1}{v_1} - \frac{f_{32}r_2}{v_2} \right] t + \frac{1}{2} \left[\begin{array}{l} \frac{f_{21}}{v_1}\left(\frac{f_{13}r_3}{v_3} + p - \left(\frac{f_{31}+f_{21}}{v_1} \right) r_1 \right) \\ -\frac{f_{32}}{v_2}\left(\frac{f_{21}r_1}{v_1} - \frac{f_{32}r_2}{v_2} \right) \end{array} \right] t^2 \\ z_2(t) &= r_1 + \left[\frac{f_{31}r_1}{v_1} + \frac{f_{32}r_2}{v_2} - \frac{f_{31}r_3}{v_3} \right] t + \frac{1}{2} \left[\begin{array}{l} \frac{f_{31}}{v_1}\left(\frac{f_{13}r_3}{v_3} + p - \left(\frac{f_{31}+f_{21}}{v_1} \right) r_1 \right) \\ + \frac{f_{32}}{v_2}\left(\frac{f_{21}r_1}{v_1} - \frac{f_{32}r_2}{v_2} \right) - \frac{f_{13}}{v_3} \\ \left(\frac{f_{31}r_1}{v_1} + \frac{f_{32}r_2}{v_2} - \frac{f_{31}r_3}{v_3} \right) \end{array} \right] t^2 \end{aligned} \tag{22}$$

where, as the RVIM method admits the use of:

$$\begin{aligned} \mathrm{x(t)} &= \lim_{n \to \infty} \mathrm{x(t)}_n \\ \mathrm{y(t)} &= \lim_{n \to \infty} \mathrm{y(t)}_n \\ \mathrm{z(t)} &= \lim_{n \to \infty} \mathrm{z(t)}_n \end{aligned}$$

A five terms approximation to the solutions are considered:

$$x(t) \approx x_4, \qquad y(t) \approx y_4, \quad z(t) \approx z_4,$$

This was done with the standard parameter values given above and initial values $r_1 = 0, r_2 = 0$ and $r_3 = 0$ for the three-component model.

A few approximations for $x(t)$, $y(t)$ and $z(t)$ are calculated and presented below:

$$\begin{aligned} x_1(t) &= 100t \\ y_1(t) &= 0 \\ z_1(t) &= 0 \end{aligned} \tag{23}$$

$$
\begin{aligned}
&x_2(t)=100t-0.6552t^2\\
&y_2(t)=0.3103t^2\\
&z_2(t)=0.3448t^2\\
&\vdots
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&x_8(t)=100\,t-0.6552\,t^2+0.399\,e-1t^3-0.31e-2t^4\\
&+0.19961\,e-3t^5-0.60005\,e-6t^6\\
&-0.14047\,e-9t^7+78137\,e-12t^8,\\
&y_8(t)=0.3103\,t^2+0.35e-2t^3+0.80652\,e-4t^4\\
&-0.41347\,e-5t^5+0.22109\,e-6t^6\\
&-0.16894\,e-9t^7-37163\,e-12t^8,\\
&z_8(t)=0.3448\,t^2-0.0363\,t^3+0.003\,t^4-0.19548\,e\\
&-3t^5+0.37896\,e-6t^6+0.30940\,e-9t^7-40974\,e-12t^8
\end{aligned}
\tag{25}
$$

Figure 2 indicates that the differences among the ode15s with Reconstruction of Variational Iteration

Method (RVIM) and the obtained results converge to the ode15s solution and the errors are reduced. Figure 3 indicates that the differences among the ode15s with Reconstruction of Variational Iteration Method (RVIM). Ode15s solution is obtained from the solution of RVIM is a good convergence and the error close to zero.

Figure 4 indicates that the differences among the ode15s with Reconstruction of Variational Iteration Method (RVIM). The solution obtained from RVIM away from the solution is obtained ode15s.

As the plots state the amount of the pollutant in Lake 1, Lake 2 and Lake 3 increase.

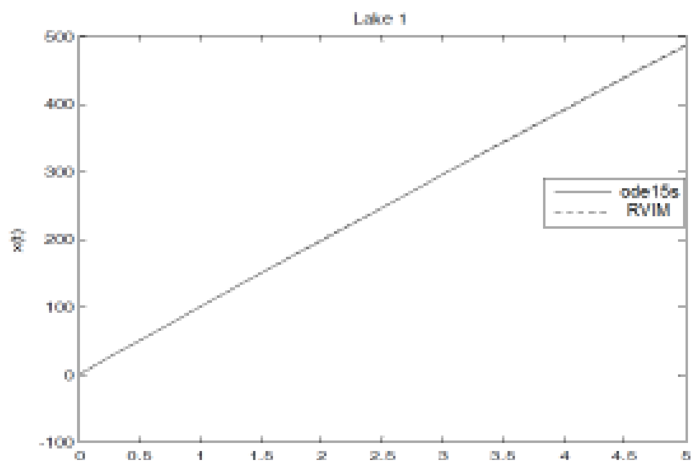

Fig. 2: The comparison of the results of x(t) via the two methods for a system of lakes (1)

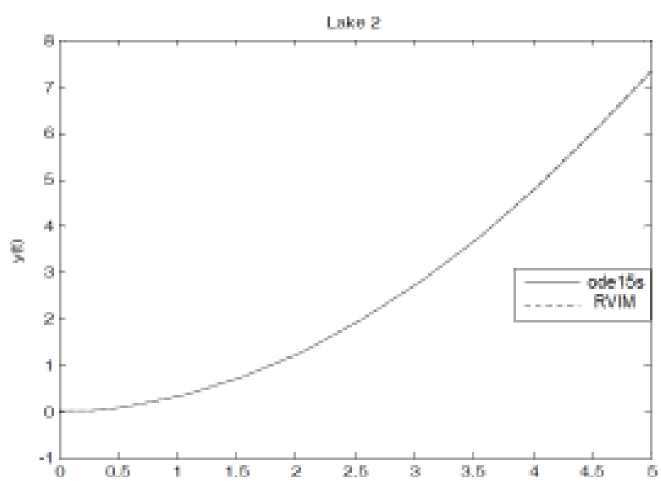

Fig. 3: The comparison of the results of y(t) via the two methods for a system of lakes (1)

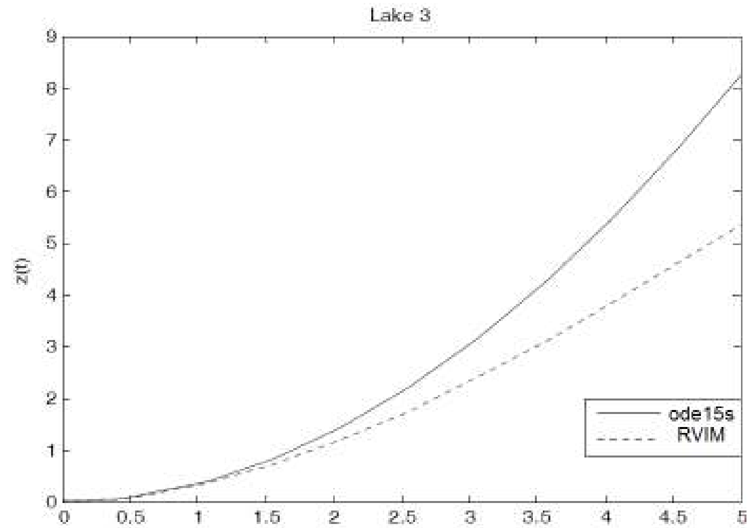

Fig. 4: The comparison of the results of z(t) via the two methods for a system of lakes (1)

CONCLUSION

In the present study we have applied the Reconstruction of Variational Iteration Method (RVIM) for finding the solutions of nonlinear ordinary differential equation systems such as non-linear oscillatory systems. Comparison of results in graphical schemes proved that RVIM can be used in applied mathematics as a trustworthy and explicit method. According to results RVIM is capable of reducing the size of calculation and it omits the difficulty arising in calculating nonlinear intricately terms. When compared with other methods, it is clear that RVIM provides highly accurate analytic solutions for nonlinear problems and the solution is quite elegant and fully acceptable in accuracy.

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