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Research Article

Analytical Approach to Some Highly Nonlinear Equations by Means of the RVIM

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Abstract: In this study, Reconstruction of Variational Iteration Method (RVIM) is used for computing the Generalized Hirota-Satsuma coupled KdV equation, Kawahara equation and FKdV equations. This new applied algorithm is a powerful and efficient technique in finding the approximate solutions for the linear and nonlinear equations. RVIM as a method that based on Laplace transform has high rapid convergence and reduces the size of calculations using only few terms, so many linear and nonlinear equations can be solved by this method. Results are compared with those of Adomian's Decomposition Method (ADM). The results of Reconstruction of Variational Iteration Method (RVIM) are of high concentration and the method is very effective and succinct.

Keywords: Generalized hirota-satsuma coupled KdV equation, kawahara equation, Reconstruction of Variational Iteration Method (RVIM), some FKdV equations

INTRODUCTION

Nonlinear phenomena play a crucial role in applied mathematics and physics. The results of solving nonlinear equations can guide authors to know the described process deeply. But it is difficult for us to obtain the exact solution for these problems. In recent decades, there has been great development in the numerical analysis (Burden and Faires, 1993) and exact solution for nonlinear partial Equations. Reaching to a high accurate approximation for linear and nonlinear equations has always been important while it challenges tasks in science and engineering. Therefore, several numbers of approximate methods have been established like Homotopy Perturbation Method (HPM) (Yildirim and Ersen, 2010; Moallemi *et al*., 2012) Variational Iteration Method (VIM) (Nikkar and Mighani, 2012; Saadati *et al*., 2009; He, 1999, 2000), Energy Balance Method (Nikkar *et al*., 2011) Homotopy Analysis Method (Khan *et al*., 2012) and so on each of which has advantages and disadvantages. We introduce a new analytical method of nonlinear problems called the reconstruction of variational iteration method, which in the case of comparing with VIM (Nikkar and Mighani, 2012; Saadati *et al*., 2009; He, 1999, 2000), HPM (Yildirim and Ersen, 2010; Moallemi *et al*., 2012) not uses Lagrange multiplier as variational methods do and not requires small parameter in equations as the perturbation techniques. RVIM has been shown to solve a large class of nonlinear problems with approximations converging to solutions rapidly, effectively, easily and accurately. The method used gives rapidly convergent successive approximations. As stated before, we aim to achieve analytic solutions to problems. We also aim to approve that the reconstruction of variational iteration method is powerful, efficient and promising in handling scientific and engineering problems. Besides the aim of this letter is to show that RVIM is strongly and simply capable of solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also is very user friend because it reduces the size of calculations by not requiring calculating Lagrange multiplier. The most sensible advantages of RVIM are using Laplace Transform and choosing initial conditions simply and easily in solving linear and nonlinear equations. Effectiveness and convenience of this method is revealed in comparisons with the exact solution. The results illustrate that the RVIM can faithfully capture the posteriori distribution in a computationally efficient way. In this study we consider RVIM to find the solution of the Generalized Hirota-Satsuma coupled KdV equation, Kawahara equation (Polat *et al*., 2006; Kaya and Al-Khaled, 2007) and FKdV equations.

DESCRIPTION OF THE METHOD

In the following section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform (Hesameddini and

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Latifizadeh, 2009; Nikkar *et al*., 2012), will be investigated a large of problems in science and engineering involve the solution of partial differential equations. Suppose *x*, *t* are 2 independent variables; consider *t* as the principal variable and *x* as the secondary variable. If *u (x, t)* is a function of 2 variables *x* and *t*, when the Laplace transform is applied with *t* as a variable, definition of Laplace transform is:

$$\mathbb{L}[u(x,t);s] = \int_0^\infty e^{-st}u(x,t)dt \quad (1)$$

We have some preliminary notations as:

$$\mathbb{L}\left[\frac{\partial u}{\partial t};s\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}dt = sU(x,s) - u(x,0) \quad (2)$$

$$\mathbb{L}\left[\frac{\partial^2 u}{\partial t^2};s\right] = s^2U(x,s) - su(x,0) - u_t(x,0) \quad (3)$$

where,

$$U(x,s) = \mathbb{L}[u(x,t);s] \quad (4)$$

We often come across functions which are not the transform of some known function, but then, they can possibly be as a product of 2 functions, each of which is the transform of a known function. Thus we may be able to write the given function as *U (x, s)*,*V (x, s)* where *U(s)* and *V(s)* are known to the transform of the functions *u(x, t)*, *v(x, t)*, respectively. The convolution of *u (x, t)* and *v (x, t)* is written *u (x, t)*(x, t)*. It is defined as the integral of the product of the 2 functions after one is reversed and shifted.

Convolution Theorem: If *U (x, s)*, *V (x, s)* are the Laplace transform of *u (x, t)*, *v(x, t)*, when the Laplace transform is applied to *t* as a variable, respectively; then *U (x, t) V (x, t)* is the Laplace Transform of $\int_0^t u(x,t-\mathcal{E}vx,\mathcal{E}d\mathcal{E}$:

$$\mathbb{L}^{-1}[U(x,s).V(x,s)] = \int_0^t u(x,t-\mathcal{E})v(x,\mathcal{E})d\mathcal{E} \quad (5)$$

To facilitate our discussion of Reconstruction of Variational Iteration Method, introducing the new linear or nonlinear function $h(u(t,x)) = f(t,x) - N(u(t,x))$ and considering the new equation, rewrite $h(u(t,x)) = f(t,x) - N(u(t,x))$ as:

$$L(u(t,x)) = h(t,x,u)) \quad (6)$$

Now, for implementation the correctional function of VIM based on new idea of Laplace transform, applying Laplace Transform to both sides of the above equation so that we introduce artificial initial conditions to zero for main problem, then left hand side of equation after transformation is featured as:

$$\mathbb{L}[L\{u(x,t)\}] = U(x,s)P(s) \quad (7)$$

where, *P(s)* is polynomial with the degree of the highest order derivative of the selected linear operator:

$$\mathbb{L}[L\{u(x,t)\}] = U(x,s)P(s) = \mathbb{L}[h\{(x,t,u)\}] \quad (8)$$

Then:

$$U(x,s) = \frac{\mathbb{L}[h\{(x,t,u)\}]}{P(s)} \quad (9)$$

Suppose that $D(s) = \frac{1}{P(s)}$ and $\mathbb{L}[h\{(x,t,u)\}] = H(x,s)$. Therefore, using the convolution theorem we have:

$$U(x,s) = D(s).H(x,s) = \mathbb{L}\{(d(t) * h(x,t,u))\} \quad (10)$$

Taking the inverse Laplace transform on both side of equation:

$$u(x,t) = \int_0^t d(t-\mathcal{E})h(x,\mathcal{E},u)d\mathcal{E} \quad (11)$$

Thus the following reconstructed method of variational iteration formula can be obtained:

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t d(t-\mathcal{E})h(x,\mathcal{E},u_n)d\mathcal{E} \quad (12)$$

And $u_0(x,t)$ is initial solution with or without unknown parameters. In absence of unknown parameters, $u_0(x,t)$ should satisfy initial/ boundary conditions.

APPLICATION OF RVIM

In this section, we will apply the RVIM to solve Generalized Hirota-Satsuma coupled KdV equation, Kawahara equation and FKdV equations.

Generalized hirota-satsuma coupled KdV equation system: We consider a generalized Hirota-Satsuma coupled KdV equation system as Kaya (2004):

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3(uw)_x \\ v_t &= -v_{xxx} + 3uv_x \\ w_t &= -w_{xxx} + 3uw_x \end{aligned} \quad (13)$$

Sucted to the initial conditions:

$$u(x,0)=\frac{1}{3}(\beta-2k^2)+2k^2\tanh^2(kx)$$
$$v(x,0)=-\frac{4k^2c_0(\beta+k^2)}{3c_1^{\,2}}+\frac{4k^2(\beta+k^2)}{3c_1^{\,2}}\tanh(kx) \quad (14)$$
$$w(x,0)=c_0+c_1\tanh(kx)$$

At first rewrite Eq. (13) based on selective linear operator as:

$$\mathbb{L}\{u(x)\} = u_t = \overbrace{(\frac{1}{2}u_{xxx}-3uu_x+3(uw)_x)}^{h(x,t,u)}$$

$$\mathbb{L}\{v(x)\} = v_t = \overbrace{(-v_{xxx}+3uv_x)}^{h(x,t,u)} \quad (15)$$

$$\mathbb{L}\{w(x)\} = w_t = \overbrace{(-w_{xxx}+3uw_x)}^{h(x,t,u)}$$

Now Laplace transform is implemented with respect to independent variable x on both sides of Eq. (15) and by using the new artificial initial condition (which all of them are zero) we have:

$$\begin{cases} s\cup(x,t)=\mathbb{L}\{h(x,t,u)\} \\ s\cup(x,t)=\mathbb{L}\{h(x,t,v)\} \\ s\cup(x,t)=\mathbb{L}\{h(x,t,w)\} \end{cases} \quad (16)$$

$$\begin{cases} \cup(x,t)=\frac{\mathbb{L}\{h(x,t,u)\}}{s} \\ \cup(x,t)=\frac{\mathbb{L}\{h(x,t,v)\}}{s} \\ \cup(x,t)=\frac{\mathbb{L}\{h(x,t,w)\}}{s} \end{cases} \quad (17)$$

And whereas Laplace inverse transform of *1/s* is as follows:

$$\mathbb{L}^{-1}[1/s]=1 \quad (18)$$

Therefore by using the Laplace inverse transform and convolution theorem it is concluded that:

$$(x,t)=\int_0^t h(x,\varepsilon,u)\,d\varepsilon$$
$$v(x,t)=\int_0^t h(x,\varepsilon,v)\,d\varepsilon \quad (19)$$
$$w(x,t)=\int_0^t h(x,\varepsilon,w)\,d\varepsilon$$

Hence, we arrive the following iterative formula for the approximate solution of subject to the initial condition (14). So, in exchange with applying recursive algorithm, following relations are achieved:

$$u_{n+1}=u_0+\int_0^t(\frac{1}{2}u_{n_{xxx}}-3u_nu_{n_x}+3(u_nw_n)_x)d\varepsilon$$
$$v_{n+1}=v_0+\int_0^t(-v_{n_{xxx}}+3u_nv_{n_x})d\varepsilon \quad (20)$$
$$w_{n+1}=w_0+\int_0^t(-w_{n_{xxx}}+3u_nw_{n_x})d\varepsilon$$

Now we start with an arbitrary initial approximation $u_0(x,t)=e^x$, that satisfies the initial condition and by using the RVIM iteration formula (20), we have the following successive approximation:

$$u_1=u_0+\int_0^t(\frac{1}{2}u_{0_{xxx}}-3u_0u_{0_x}+3(u_0w_0)_x)d\varepsilon$$
$$v_1=v_0+\int_0^t(-v_{0_{xxx}}+3u_0v_{0_x})d\varepsilon \quad (21)$$
$$w_1=w_0+\int_0^t(-w_{0_{xxx}}+3u_0w_{0_x})d\varepsilon$$
$$\vdots$$

whereas, the RVIM method admits the use of:

$$u=\lim_{n\to\infty}u_n$$
$$v=\lim_{n\to\infty}v_n$$
$$w=\lim_{n\to\infty}w_n$$

Now we study the diagrams obtained by RVIM and ADM (Kaya, 2004) (Fig. 1-4).

Kawahara equation: We consider Kawahara equation as Kaya (2003):

$$u_t-uu_x+u_{xxx}-u_{xxxxx}=0 \quad (22)$$

With the following initial conditions:

$$u(x,0)=\frac{105}{169}\sec h^4(\frac{x}{2\sqrt{13}}) \quad (23)$$

At first rewrite eq. (22) based on selective linear operator as:

$$\{u(x)\} = u_t = \overbrace{(-uu_x+u_{xxx}-u_{xxxxx})}^{h(x,t,u)} \quad (24)$$

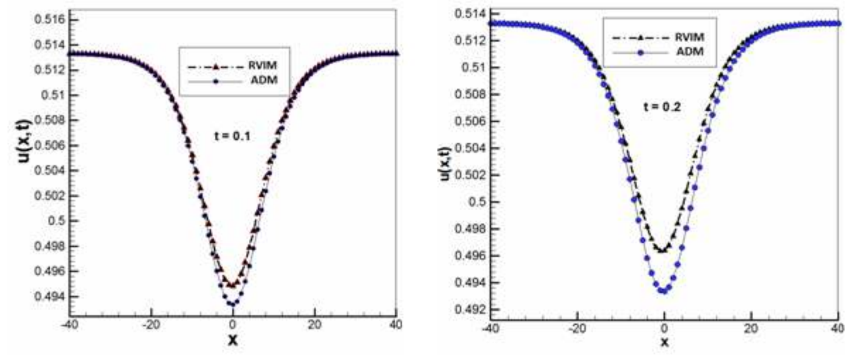


Fig. 1: The comparison of RVIM and ADM for the solution u (x, t) for different values of t

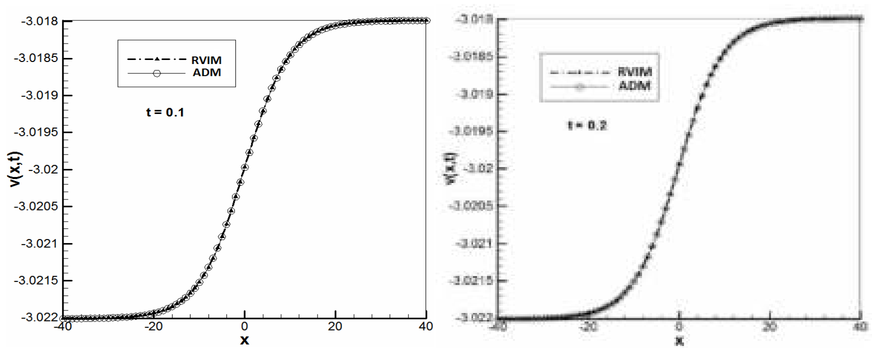


Fig. 2: The comparison of RVIM and ADM for the solution v (x, t) for different values of t

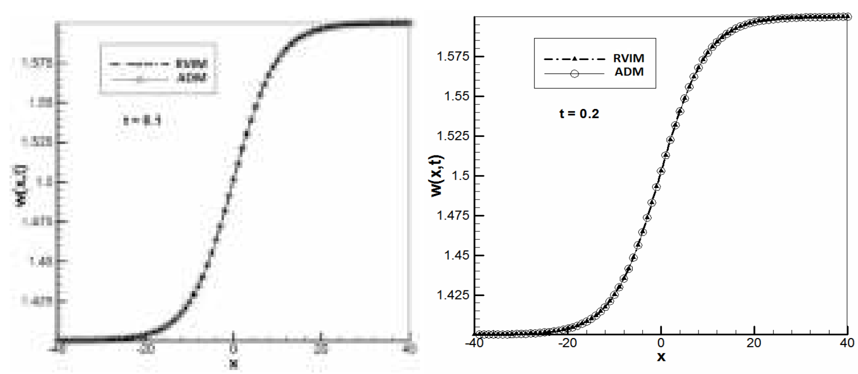
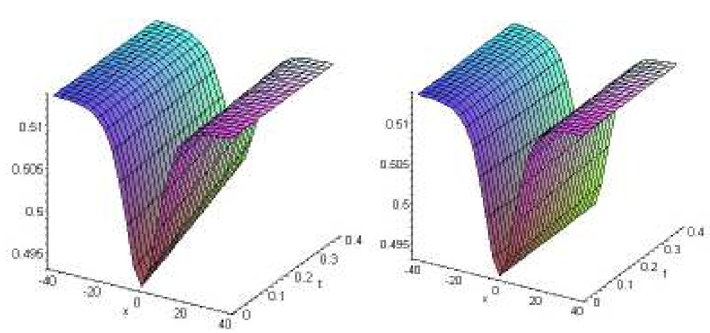


Fig. 3: The comparison of RVIM and ADM for the solution w (x, t) for different values of t



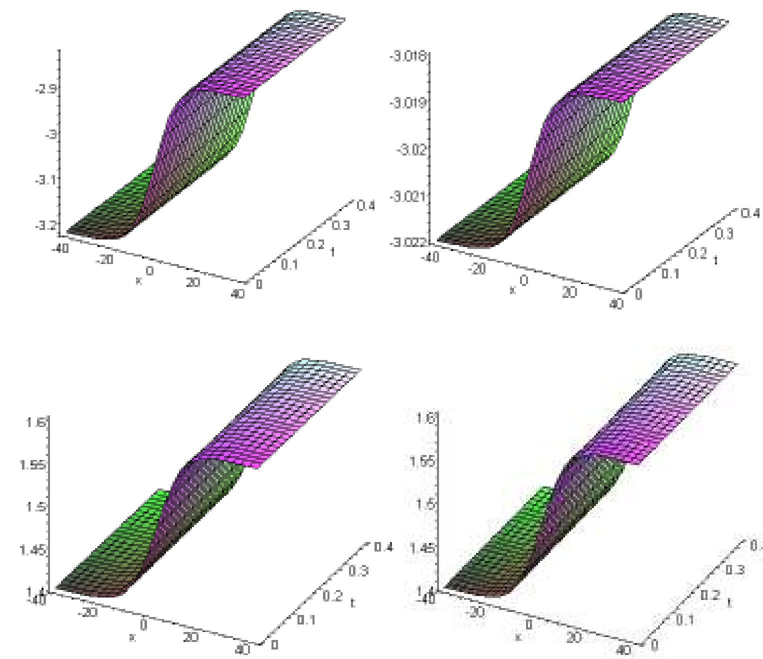


Fig. 4: The surfaces on both columns, respectively show the solutions, u (x, t), v (x, t), w(x, t), for RVIM on the right and ADM on the left

Now Laplace transform is implemented with respect to independent variable x on both sides of Eq. (24) and by using the new artificial initial condition (which all of them are zero) we have:

$$s \cup (x,t) = \mathbb{L}\,\{h(x,t,u)\} \qquad (25)$$

$$\cup\,(x,t) = \frac{\mathbb{L}\,\{h(x,t,u)\}}{s} \qquad (26)$$

And whereas Laplace inverse transform of *1/s* is as follows:

$$\mathbb{L}^{-1}[1/s] = 1 \qquad (27)$$

Therefore, by using the Laplace inverse transform and convolution theorem it is concluded that:

$$u(x,t) = \int_0^t h(x,\varepsilon,u)\, d\varepsilon \qquad (28)$$

Hence, we arrive the following iterative formula for the approximate solution of subject to the initial condition (23). So, in exchange with applying recursive algorithm, following relations are achieved:

$$u_{n+1} = u_0 + \int_0^t (-u_n u_{n_x} + u_{n_{xxx}} - u_{n_{xxxxx}})\,d\varepsilon \qquad (29)$$

Now we start with an arbitrary initial approximation $u_0(x,t) = e^x$, that satisfies the initial condition and by using the RVIM iteration formula (29), we have the following successive approximation (Fig. 5, 6):

$$u_1 = u_0 + \int_0^t (-u_0 u_{0_x} + u_{0_{xxx}} - u_{0_{xxxxx}})\,d\varepsilon \qquad (30)$$

$\vdots$

whereas, the RVIM method admits the use of:

$$u = \lim_{n\to\infty} u_n$$

Fifth order KdV equations: We consider a fifth order KdV equation as Kaya (2003):

$$u_t + uu_x - uu_{xxx} + u_{xxxxx} = 0 \qquad (31)$$

With the following initial conditions:

$$u(x,0) = e^x \qquad (32)$$

At first rewrite Eq. (31) based on selective linear operator as:

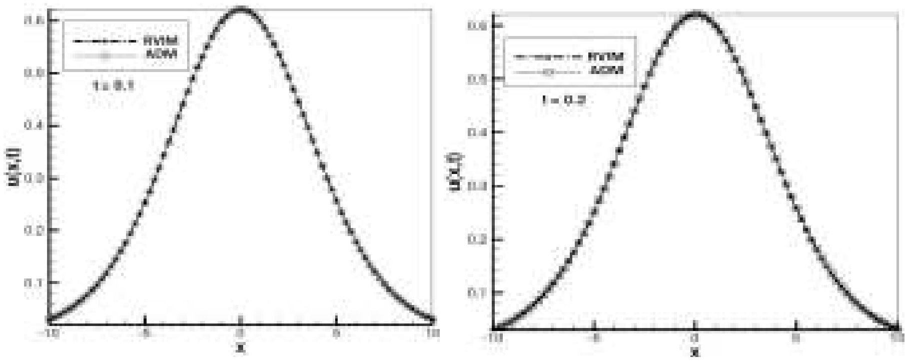


Fig. 5: The comparison of RVIM and ADM for the solution u (x, t) for different values of t

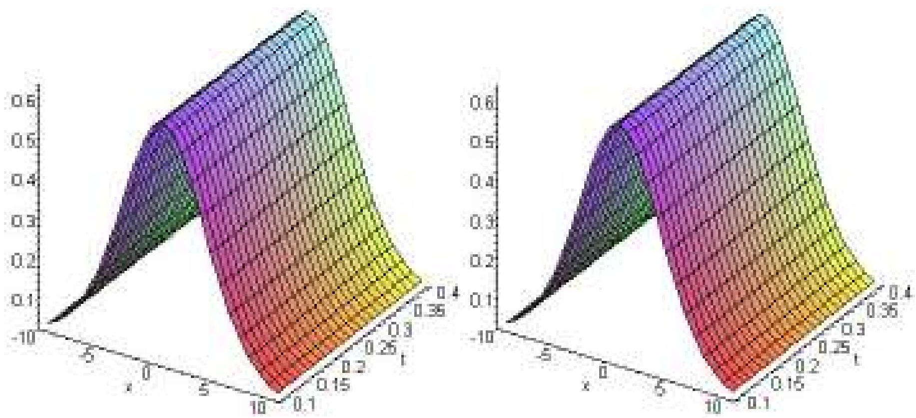


Fig. 6: The surfaces on both columns, respectively show the solutions, u (x, t), for RVIM on the right and ADM on the left

$$\mathbb{L}\,\{u(x)\} = u_t = \overbrace{(uu_x - uu_{xxx} + u_{xxxxx})}^{h(x,t,u)} \quad (33)$$

Now Laplace transform is implemented with respect to independent variable x on both sides of Eq. (33) and by using the new artificial initial condition (which all of them are zero) we have:

$$s \cup (x,t) = \mathbb{L}\,\{h(x,t,u)\} \quad (34)$$

$$\cup\,(x,t) = \frac{\mathbb{L}\,\{h(x,t,u)\}}{s} \quad (35)$$

And whereas Laplace inverse transform of *1/s* is as follows:

$$\mathbb{L}^{-1}[{}^{1}/_{s}] = 1 \quad (36)$$

Therefore, by using the Laplace inverse transform and convolution theorem it is concluded that:

$$u(x,t) = \int_0^t h(x,\varepsilon,u)\, d\varepsilon \quad (37)$$

Hence, we arrive the following iterative formula for the approximate solution of subject to the initial condition (32). So, in exchange with applying recursive algorithm, following relations are achieved:

$$u_{n+1} = u_0 + \int_0^t (u_n u_x - u_n u_{n_{xxx}} + u_{n_{xxxxx}})\, d\varepsilon \quad (38)$$

Now we start with an arbitrary initial approximation $u_0(x,t) = e^x$, that satisfies the initial condition and by using the RVIM iteration formula (38), we have the following successive approximation:

$$u_1 = u_0 + \int_0^t (u_0 u_{0_x} - u_0 u_{0_{xxx}} + u_{0_{xxxxx}})\, d\varepsilon \quad (39)$$

⋮

where, as the RVIM method admits the use of:

$$u = lim_{n\to\infty}\, u_n$$

CONCLUSION

In this study, an explicit analytical solution is obtained for Some Highly Nonlinear Equations by means of the Reconstruction of Variational Iteration Method (RVIM), which is a powerful mathematical tool in dealing with nonlinear equations. The results clearly indicate the reliability and accuracy of the proposed technique. The obtained solutions are compared with those of ADM. Simplicity and requiring less computation, rapid convergence and high accuracy are advantages of this technique

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