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Research Article

Systems Variables and Structural Controllability: An Inverted Pendulum Case

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Abstract: In order to explore the essence of structural controllability, structural controllability of inverted pendulum is analyzed in this presented study and a sufficient controllability condition of a class of perturbed linear system is obtained, which is essential to prove the structural controllability for the perturbed inverted pendulum. Two different structured models of inverted pendulum are constructed. Structural controllability of both cases are discussed and compared, which shows that the usual model used in controller design for inverted pendulum is just a special case of normal model for inverted pendulum.

Keywords: Perturbed linear system, Rational Functions Matrix with multi-parameter (RFM), structural controllability, system variable

INTRODUCTION

Here consider a linear time-invariant system denoted by differential equation:

$$\dot{x} = Ax + Bu \tag{1}$$

where, we denote by $x \in R^n$, $u \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ the system states, system inputs, coefficient matrix and input matrix, respectively. Define its controllability matrix by $Q_c = [B, AB, \dots, A^{n-1}B]$ and a composite matrix by $Q(s) = [sI - A, B]$. It is clearly that system (1) is completely controllable if and only if rank $(Q_c) = n$, which is equivalent to that for all $s \in C$ (C is the complex number field), rank $(Q(s)) = n$. Generally, controllability is a generic property (Lee and Markus, 1967), that is, if we denote all the controllable system a set by Y, then Y is a open and dense set.

In traditional systematic theory, the mathematic model (1) is described exactly, that is, all the elements of coefficient matrix and input matrix are accurately numerical numbers. However, from the view point of engineering, it is difficult to get such a accurate model, which is to say that there may exist uncertainty in elements of A and B. most of these uncertainties are resulted from the uncertainties of system variables. If we include these system variables in physical model, then it turns out to be a system model with variables, which we can say structural model.

The inverted pendulum is a multi-parameter system and most research on it is control strategy. Variable structure method is used to control the inverted pendulum, in which the sliding mold control law was obtained by Lyapunov method (Ma and Lu, 2009). An anther Lyapunov based control method was proposed to control the inverted pendulum (Ibanez *et al*., 2005). Other method such that the passive fault tolerant method was used to control a double inverted pendulum (Niemann and Stoustrup, 2005). On the other hand, the rod optimal selection problem in multi-link inverted pendulum was researched (Bei-Chen and Shuang, 2006). By using the controllability matrix Q_c which is obtained by different rods combination, the smallest Eigen values of different controllability matrices Q_c are calculated. Then the system which is corresponding to the smallest Eigen value is considered to be the optimal choice. In that study, the problem that rod is considered to be system variable is concerned, nevertheless, the calculation is still a process over the real number field. From the view point of structural model, it is worth researching whether the system variables affect the system property (such as controllability).

In this study structural controllability of inverted pendulum is analyzed. Two structural models of inverted pendulum are constructed. For a class of perturbed linear system a sufficient condition of controllability is presented, which is a basement for analyzing structural controllability of the second type of structural model. Also this study gives a research that whether system variables affect the controllability of inverted pendulum from the view of structural model point, that is, structural controllability. In order to explore the essence of structural controllability, in this study we just consider the linearized inverted pendulum model.

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SOME DEFINITIONS AND TWO INTERMEDIATE CONCLUSION

For linear time-invariant system (1), an important conception is controllability, that is, whether we can find some suitable inputs u such that for any given initial state x_0 it can be steered to any desired destination state. With the linear time-invariant systems, there exist some definitions about controllability as follow:

Definition 1: For linear time-invariant system (1) if there exist some admissible control u (t) (with the constrain:

$$\int_{t_0}^{t_a} |u(t)|^2 \, dt < \infty \;)$$

during the time interval (t_0, t_a) and transfer the initial state x (t_0) ≠ 0 to x (t_a) = 0, then state x (t) is said to be controllable over the interval (t_0, t_a), or controllable at time t_a.

Definition 2: If any state of linear time-invariant system (1) is controllable over the time interval (t_0, t_a), then the system (1) is said to be completely controllable.

Here we consider a class of linear time-invariant system (1) with perturbation denoted by:

$$\dot{x} = Ax + Bu + P \tag{2}$$

where, we denote by $x \in R^n$, $u \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $P \in R^n$ the system states, system inputs, coefficient matrix, input matrix and constant perturbation respectively.

Now define a symmetric matrix:

$$W_C(t_0, t_a) = \int_{t_0}^{t_a} e^{A(t_a-\tau)} B(\tau) B^T(\tau) e^{A^T(t_a-\tau)} d\tau \tag{3}$$

For system (1) symmetric matrix W_C (t_0, t_a) is said to be the controllability Gramian matrix. Similarly, for system (2) we can also define its symmetric controllability Gramian matrix denoted by $\overline{W}_C$ (t_0, t_a) which differ from that of system (1).

For system (2), here gives a conclusion about its controllability.

Theorem 1: Linear time-invariant system (2) is controllable if its controllability Gramian matrix $\overline{W}_C$ (t_0, t_a) is nonsingular.

Proof: Because $\overline{W}_C$ (t_0, t_a) is nonsingular, then $\overline{W}_C^{-1}$ (t_0, t_a) exists. Suppose the system input is with the form as follow:

$$u(t) = -B^T e^{-A^T t} \overline{W}_C^{-1}(t_0, t_a)(x_0 - P)$$

where, (x_0 - P) is the initial state at time t_0. Then the solution x (t_a) of linear time-invariant system (2) at time t_a can be denoted to:

$$x(t_a) = e^{At_a}(x_0 - P) - \int_{t_0}^{t_a} e^{A(t_a-t)} Bu(t) dt$$

$$= e^{At_a}(x_0 - P) - \int_{t_0}^{t_a} e^{A(t_a-t)} BB^T e^{-A^T t} \overline{W}_C^{-1}(t_0, t_a)(x_0 - P) dt$$

$$= e^{At_a}(x_0 - P) - e^{At_a} \int_{t_0}^{t_a} e^{-At} BB^T e^{-A^T t} dt \overline{W}_C^{-1}(t_0, t_a)(x_0 - P)$$

$$= e^{At_a}(x_0 - P) - e^{At_a} \overline{W}_C(t_0, t_a) \overline{W}_C^{-1}(t_0, t_a)(x_0 - P)$$

$$= e^{At_a}(x_0 - P) - e^{At_a}(x_0 - P) = 0$$

By the Definition 1, linear time-invariant system (2) is controllable.

Lemma 1: For linear time-invariant system (1), the non-singularity of controllability Gramian matrix W_C (t_0, t_a) is equivalent to that rank (Q_c) = n.

Then for system (2), we define a matrix denoted by $\bar{Q}_c$, which is also said to be the controllability matrix. Another conclusion about controllability of system (2) is immediate.

Theorem 2: Linear time-invariant system (2) is controllable if its controllability matrix $\bar{Q}_c$ is of full rank, that is, rank ($\bar{Q}_c$) = n.

Proof: By the Theorem 1 and Lemma 1, the proof is immediate.

SOME NOTATIONS OF STRUCTURAL CONTROLLABILITY

In classical control theory, the elements of matrices A and B are considered to be accurate exactly, but for some physical reasons some of these elements are approximate numerical numbers. So sometimes the variables are preserved in system mathematic model. The model with variables is considered to be more reasonable and only the known zeros are decided to be accurate.

The conception of structural controllability is presented first (Lin, 1974). During the analysis of structural controllability only the structure of matrices *A* and *B* is concerned, which means that the objective researched is not the numerically known matrices *A* and *B*, but the corresponding same dimension structured matrices denoted by $\bar{A}$ and $\bar{B}$, respectively.

Definition 3: The element of structured matrix $\bar{M}$ is ether zero or undetermined value differing from any other undetermined element. If all the elements of structured matrix $\bar{M}$ are concrete numerical numbers, then it is considered to be a admissible numerical realization. If the numerical matrices *M* and *N* are both the admissible numerical realization of structured matrix $\bar{M}$, then they are structurally equivalent.

Other structured matrices are proposed by researchers (Anderson and Clements, 1981; Yamada and Luenberger, 1985; Murota, 1999) after Lin's structural matrix. However, there exists a problem that the inverse matrix of these structured matrices is no longer to be their corresponding structured matrix. Calculating inverse matrix is frequent during exploring the properties of system (1) and (2), so all these structured matrices are not fitted.

Then a class of more general structured matrix was proposed, which is call rational functions matrix with Multi-Parameter (RFM) (Lu and Wei, 1991, 1994).

Let $E_{r=(r_1,\ldots,r_q)\in R^q}$ be the parameter vector in system (1) or (2). R^q is called the definition domain of *r* or parameter space. Let F (r) denote the field of all rational functions with real coefficients in *q* parameters $r_1, \ldots, r_q$. F (r) [λ] denote the polynomial ring in λ with coefficients in members over F (r).

Definition 4: If each member of matrix *M* is a member over F (r), then the matrix *M* is said to be a rational matrix or a matrix over F (r). If all the matrices of systems are consider to be RFM, then the systems is said to be a rational function systems with multi-parameters *r* or a systems over F (r).

The most advantage of RFM over other structured matrices is that the inverse matrix of RFM is still RFM, which is more convenient in analyzing system properties. The structured matrix used in this presented study is RFM defined with Definition 4.

Now we can define the mathematic model of linear time-invariant system over F (r):

$$\dot{x} = A(r)x + B(r)u \tag{4}$$

or linear time-invariant system with constant perturbation over F (r):

$$\dot{x} = A(r)x + B(r)u + P \tag{5}$$

where, r denotes the parameter vector in systems (4) and (5).

Theorem 3: Linear time-invariant system with constant perturbation (5) is structural controllable if its controllability matrix $\bar{Q}_c$ (r) is of full rank, that is, rank ($\bar{Q}_c$ (r)) = n.

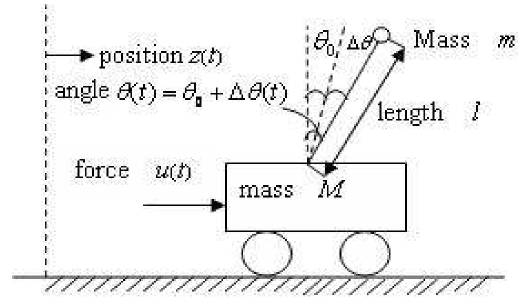

Fig. 1: One stage inverted pendulum system

Proof: By Theorem 2 and Definition 4, the proof is immediate.

STRUCTURAL CONTROLLABILITY ANALYSES FOR INVERTED PENDULUM

The inverted pendulum now is considered to be a normal system, so here we just give the simple illustration figure shown by Fig. 1, if any doubt may refer to Shen *et al.* (2003). In order to exploring the structural properties here the one stage linearized inverted pendulum is concerned. In this section two different linearized cases are considered as follow:

Linearization at up-vertical equilibrium point: This is the usual case studied. The dynamic behavior of one stage inverted pendulum is described by 4 states, that is, position of car, velocity of car, angular of rod and velocity of rod angular, denoted by z, $\dot{z}$, θ and $\dot{\theta}$ respectively.

In order to determine the mathematic model governing this system, we first concern the horizontal position *z* of the car, meanwhile the position of the pendulum is $z + l\sin\theta$. Then by Newton's second law, that is, mass×acceleration = force, in the horizontal direction yields:

$$M\frac{d^2z}{dt^2} + m\frac{d^2(z + l\sin\theta)}{dt^2} = u \tag{6}$$

Again applying the Newton's second law to pendulum mass, in the up-vertical direction to the rod, we can get:

$$m\ddot{z}\cos\theta + ml\ddot{\theta} = mg\sin\ddot{\theta} \tag{7}$$

where, notation *g* is the acceleration of gravity in the downward direction.

The differential Eq. (6) and (7) are nonlinear, we need a further simplification.

Now let $\theta(t) = \theta_0 + \Delta\theta(t)$, where θ_0 is the equilibrium point, which we call it the static operating point. θ_0 is a constant, $\Delta\theta(t)$ is the deviation that departs from θ_0. When $\theta_0 = 0$ and $\theta(t) = \Delta\theta(t)$ is not large, we can get its linearized mathematic model described by state space form:

$$\dot{X} = A(r)X + B(r)U \tag{8}$$

where,

$$A(r) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} B(r) = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix} X = \begin{bmatrix} z \\ \dot{z} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

Let parameter r = (M, m, l, g) $\in R^4$, then the controllability matrix Q_C (r) is:

$$Q_c(r) = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{M} & 0 & \frac{mg}{M^2 l} \\ \frac{1}{M} & 0 & \frac{mg}{M^2 l} & 0 \\ 0 & -\frac{1}{Ml} & 0 & \frac{-(M+m)g}{M^2 l} \\ -\frac{1}{Ml} & 0 & \frac{-(M+m)g}{M^2 l} & 0 \end{bmatrix}$$

Since determinant of Q_C (r) is:

$$\frac{g^2}{M^2 l^2}\left[(M+m)l - m\right]^2$$

and when (M + m) l = m, Q_c (r) has full rank and the system is completely controllable. It is known that relation (M + m) l = m is an algebraic variety. Let set S be S = {r|detQ$_c$ (r) = 0}, then the Lebesgue measurement of inverted pendulum (8) being nonstructural controllable is zero, which means that inverted pendulum (8) is structural controllable.

Linearization at any up operating point: What we have discussed in the above subsection is the usual linerized case, under which condition that the static operating point is $\theta_0 = 0$. Now we will discuss large range that the angular θ_0 takes value from the interval [-60°, 60°]. If we do this, then we are sure that "dynamic model described by differential Eq. (6) and (7) is controllable at the operating point $\theta_0 = 0$" is exactly reliable when $|\theta(t)| \leq 60°$.

Because the equilibrium point $\theta_0 \neq 0$, now we apply $\theta(t) = \theta_0 + \Delta\theta(t)$, where $\Delta\theta(t)$ is a small perturbation, which is close to zero. According to the fact that $\cos\Delta\theta \approx 1$ and $\sin\Delta\theta \approx \Delta\theta$, then applying these to the trigonometric expansions, we have:

$$\sin\theta = \sin(\theta_0 + \Delta\theta) = \sin\theta_0 \cos\Delta\theta + \cos\theta_0 \sin\Delta\theta_0\Delta\theta$$

$$\approx \sin\theta_0 + \cos\theta \tag{9}$$

$$\cos\theta = \cos(\theta_0 + \Delta\theta) = \cos\theta_0 \cos\Delta\theta - \sin\theta_0 \sin\Delta\theta_0$$

$$\approx \cos\theta_0 - \sin\theta_0\Delta\theta \approx \cos\theta \tag{10}$$

The effectiveness of Eq. (10) depends on the assumption that $|\theta_0| \leq 60°$ and $\Delta\theta \leq 0.1$ because of $\sin\theta_0\Delta\theta \ll cos\theta_0$. According to Eq. (9), we have:

$$\frac{d\sin\theta}{dt} = \cos\theta_0\Delta\dot{\theta} \tag{11}$$

$$\frac{d^2\sin\theta}{dt} = \cos\theta_0\Delta\ddot{\theta} \tag{12}$$

Substituting the Equalities (11) and (12) into Eq. (6), we can have an approximate differential equation for the inverted pendulum system:

$$(M+m)\ddot{z} + ml\cos\theta_0\Delta\ddot{\theta} = u \tag{13}$$

Again applying the same method to Eq. (7), we have:

$$\cos\theta_0\ddot{z} + l\ddot{\theta} - g\cos\theta_0\Delta\theta = g\sin\theta_0 \tag{14}$$

Because of:

$$\ddot{\theta} = \frac{d^2(\theta_0 + \Delta\theta)}{dt^2} = \Delta\ddot{\theta}$$

Equation (14) takes the form:

$$\cos\theta_0\ddot{z} + l\Delta\ddot{\theta} - g\cos\theta_0\Delta\theta = g\sin\theta_0 \tag{15}$$

According to Eq. (13) and (14), we have:

$$\Delta\ddot{\theta} = \frac{(M+m)g\cos\theta_0}{(M+m-m\cos^2\theta_0)l}\Delta\theta - \frac{\cos\theta_0}{(M+m-m\cos^2\theta_0)l}u + \frac{(M+m)g\sin\theta_0}{(M+m-m\cos^2\theta_0)l}$$

$$= \frac{M+m}{(M+m-m\cos^2\theta_0)l}(g\cos\theta_0\Delta\theta + g\sin\theta_0 - \frac{\cos\theta_0}{M+m}u) \tag{16}$$

$$\ddot{z} = \frac{-mg\cos^2\theta_0}{M+m-m\cos^2\theta_0}\Delta\theta + \frac{1}{M+m-m\cos^2\theta_0}u - \frac{mg\cos\theta_0\sin\theta_0}{M+m-m\cos^2\theta_0}$$

$$= \frac{-mg\cos^2\theta_0}{M+m-m\cos^2\theta_0}\Delta\theta + \frac{1}{M+m-m\cos^2\theta_0}u - \frac{mg\cos\theta_0\sin\theta_0}{M+m-m\cos^2\theta_0} \tag{17}$$

Now we denote the state vector to being with the form:

$$X = \begin{bmatrix} z & \dot{z} & \Delta\theta & \Delta\dot{\theta} \end{bmatrix}^T$$

Then we may write Eq. (14) and (17) in state space form:

$$\dot{X} = \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \Delta\dot{\theta} \\ \Delta\ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg\cos^2\theta_0}{M+m-m\cos^2\theta_0} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g\cos\theta_0}{(M+m-m\cos^2\theta_0)l} & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \\ \Delta\theta \\ \Delta\ddot{\theta} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \dfrac{1}{M+m-m\cos^2\theta_0} \\ 0 \\ \dfrac{-\cos\theta_0}{(M+m-m\cos^2\theta_0)l} \end{bmatrix} u + \begin{bmatrix} 0 \\ \dfrac{-mg\cos\theta_0\sin\theta_0}{M+m-m\cos^2\theta_0} \\ 0 \\ \dfrac{(M+m)g\sin\theta_0}{(M+m-m\cos^2\theta_0)l} \end{bmatrix} \quad (18)$$

Here we consider the case in subsection A that $\theta_0 = 0$ and substitute it into Eq. (18). Then it turns out Eq. (6), which means that Eq. (18) is reliable.

Supposing that $\Delta\theta = 0,\ \Delta\dot{\theta} = 0,\ \Delta\ddot{\theta} = 0$ i.e., this inverted pendulum runs at the static point $\theta = \theta_0$. Then by Eq. (16) we can get that:

$$u_0 = \frac{(M+m)g\sin\theta_0}{\cos\theta_0} = (M+m)gtg\theta_0 \quad (19)$$

Substituting Eq. (19) into (17) yields:

$$\ddot{z} = \frac{1}{M+m-m\cos^2\theta_0}\left[\frac{(M+m)g\sin\theta_0}{\cos\theta_0} - mg\cos\theta_0\sin\theta_0\right]$$
$$= gtg\theta_0 \quad (20)$$

It is clearly that (M + m) $\ddot{z}$ = (M + m) gtg $\theta_0 = u_0$, which is in fact the Newton's second law.

Let u = u_0 + Δu where u_0 = (M + m) gtgθ_0, then by substituting them into Eq. (17) we have:

$$\ddot{z} = \frac{-mg\cos^2\theta_0}{M+m-m\cos^2\theta_0}\Delta\theta$$

$$+ \frac{\left[(M+m)gtg\theta_0 + \Delta u - mg\cos\theta_0\sin\theta_0\right]}{M+m-m\cos^2\theta_0}$$

$$= \frac{-mg\cos^2\theta_0}{M+m-m\cos^2\theta_0}\Delta\theta + \frac{1}{M+m-m\cos^2\theta_0}\Delta u + \ddot{z}_0 \quad (21)$$

where $\ddot{z}_0 = gtg\theta_0$.

Then by substituting u = u_0 + Δu and u_0 = (M + m) gtg θ_0 into Eq. (16) we get:

$$\Delta\ddot{\theta} = \frac{(M+m)g\cos\theta_0}{(M+m-m\cos^2\theta_0)l}\Delta\theta$$

$$-\frac{\cos\theta_0}{(M+m-m\cos^2\theta_0)l}\Delta u \quad (22)$$

According to Eq. (21) and (22) we can get another state space form with the form as follow:

$$\dot{X} = \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \Delta\dot{\theta} \\ \Delta\ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg\cos^2\theta_0}{M+m-m\cos^2\theta_0} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g\cos\theta_0}{(M+m-m\cos^2\theta_0)l} & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \\ \Delta\theta \\ \Delta\ddot{\theta} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \dfrac{1}{M+m-m\cos^2\theta_0} \\ 0 \\ \dfrac{-\cos\theta_0}{(M+m-m\cos^2\theta_0)l} \end{bmatrix} \Delta u + \begin{bmatrix} 0 \\ \ddot{z}_0 \\ 0 \\ 0 \end{bmatrix} \quad (23)$$

It is clearly that above system (23) is the same form of system (5). Now let us test the rank of its controllability matrix $\bar{Q}_c$ (r). First let parameter vector *r* be (M, m, l, g, θ_0). Then for calculating easily, we let N denote (M + m - m $\cos^2$ θ_0), then we get the controllability matrix $\bar{Q}_c$ (r) with the form:

$$\bar{Q}_c(r) = \begin{bmatrix} 0 & \dfrac{1}{N} & 0 & \dfrac{mg\cos^2\theta_0\cos\theta_0}{N^2 l} \\ \dfrac{1}{N} & 0 & \dfrac{mg\cos^2\theta_0\cos\theta_0}{N^2 l} & 0 \\ 0 & \dfrac{-\cos\theta_0}{Nl} & 0 & \dfrac{-(M+m)g\cos^2\theta_0\cos\theta_0}{N^2 l} \\ \dfrac{-\cos\theta_0}{Nl} & 0 & \dfrac{-(M+m)g\cos^2\theta_0\cos\theta_0}{N^2 l} & 0 \end{bmatrix}$$

It is clear that rank $\bar{Q}_c$ (r) = 4, which means that by Theorem 3 system (23) is structural controllable.

CONCLUSION

From the structural model of view point, the structural controllability of linearized one-stage inverted pendulum is analyzed in this presented study. For the two cases that linearization at up-vertical equilibrium point and linearization at any up operating point, structural controllability of the two cases are discussed, which proves that the former case is the special form of later case. The linear structural model obtained in the second case is not the standard linear structural model (4), but it is a standard linear structural model with constant perturbation. So the controllability condition of structural model (5) is explored. However, the condition obtained in this presented study is only a sufficient condition, so exploring the necessary condition is the future work.

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