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Research Article

Exact Solutions to Some Nonlinear Partial Differential Equations in Mathematical Physics Via the (G′/G) -Expansion Method

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Abstract: The (G'/G)-expansion method is a powerful tool for the direct analysis of contender nonlinear equations. In this study, we search new exact traveling wave solutions to some nonlinear partial differential equations, such as, the Kuramoto-Sivashinsky equation, the Kawahara equation and the Carleman equations by means of the (G'/G)-expansion method which are very significant in mathematical physics. The solutions are presented in terms of the hyperbolic and the trigonometric functions involving free parameters. It is shown that the novel (G'/G)-expansion method is a competent and influential tool in solving nonlinear partial differential equations in mathematical physics.

Keywords: Homogeneous balance method, nonlinear partial differential equations, the (G'/G)-expansion method, traveling wave solution

INTRODUCTION

Nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus the issue becomes more complicated and hence needs ultimate solution. Therefore, the studies of exact solutions of Nonlinear Partial Differential Equations (NLPDEs) play a crucial role to understand the physical mechanism of nonlinear phenomena. Advance nonlinear techniques are significant to solve inherent nonlinear problems, particularly those are involving dynamical systems and related areas. In recent years, significant improvements have been made for searching exact solutions of NLPDEs. Many effective and powerful methods have been established to handle the NLPDEs, such as, the Backlund transformation method (Rogers and Shadwick, 1982), the inverse scattering method (Ablowitz and Clarkson, 1991), the F-expansion method (Zhou *et al*., 2003; Wang and Zhou, 2003), the Darboux transformation method (Rogers and Schief, 2003), the variational iteration method (He, 1997; Mohiud-Din *et al*., 2010), the homogeneous balance method (Wang, 1995; Wang *et al*., 1996), the tanh-function method (Parkes and Duffy, 1996), the extended tanh-function method (Fan, 2000), the Jacobi elliptic function expansion method (Liu *et al*., 2001), the auxiliary equation method (Sirendaoreji, 2004), the Exp-function method (He and Wu, 2006; Akbar and Ali, 2011; Naher *et al*., 2011a; Naher *et al*., 2012), the homotopy perturbation method (Jafari and Aminataei, 2010; Yildirim *et al*., 2011), Adomian decomposition method (Adomian, 1994) and so on.

Very lately, Wang *et al*. (2008) developed a new method called the (G'/G) -expansion method to look for travelling wave solutions of nonlinear evolution equations. The (G'/G) -expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in (G'/G) where G satisfies the second order linear Ordinary Differential Equation (ODE), $G'' + \lambda G" + \mu$ G = 0, λ and μ are arbitrary constants. The degree of the polynomial can be determined by balancing the highest order derivative and nonlinear terms appearing in the given nonlinear equations. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the (G'/G)-expansion method, Wang *et al*. (2008) successfully obtained travelling wave solutions of four nonlinear partial differential equations.

Later, works have been done on the application of the (G'/G)-expansion method to various nonlinear partial differential equations. For example, Xiong and Xia (2010) employed this method for searching exact solutions to the (2+1)-dimensional Boiti-Leon-Pempinelle equation. Naher *et al*. (2011b) found some new exact solutions to the Caudrey-Dodd-Gibbon equation by using this method. Feng and Zheng (2010a) sought the travelling wave solution of the fifth order

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Sawada-Kotera equation and Feng (2010) sought the solution of the seventh order Sawada-Kotera equation by using the (G'/G) -expansion method. Feng and Zheng (2010b) also investigated the exact solutions of the Drinfel'd-Sokolov-Satsuma-Hirota (DSSH) equation, Kodomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation and (3+1)-dimensional Yu-Toda-Sasa-Fukuyama (YTSF) equation via this method. By means of the same method, Zayed and Khaled (2009) constructed solutions of the Broer-Kaup equation, breaking soliton equation, the Nizhnik-Nokikov-Vesselov equation and the Boussinesq-Kodomtsev-Petviashvili (BKP) equations. Sousaraie and Bagheri (2010) applied the (G'/G) -expansion method to the nonlinear Klein-Gordon equation while Neyrame *et al*. (2010) used this method to seek solutions of the Boussinesq equation and the Benjamin-Ono equations. Using the (G'/G)-expansion method, Zheng (2011) obtained the travelling wave solutions of the sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation. Feng *et al*. (2011) also used this method to solve the Kolmogorov-Petrovskii-Piskunov equation. Zhou *et al*. (2009) solved the osmosis K (2, 2) equation. Gomez and Salas (2010) investigated solutions of the generalized Benjamin-Bona-Mahony with variable coefficients. Akbar *et al*. (2012a) obtained abundant traveling wave solutions of the Generalized Bretherton equation by employing the (G'/G)-expansion method. Akbar *et al*. (2012b) also obtained more new exact solutions of some NLPDEs by a generalized and improved (G'/G) -expansion method. Still, substantial work has to be done in order for the (G'/G)-expansion method to be well established, since every nonlinear equation has its own physically significant rich structure.

Our aim in this study is to present an application of the (G'/G)-expansion method to some nonlinear PDEs in mathematical physics namely, the Kuramoto-Sivashinsky equation, the Kawahara equation and the Carleman equations.

METHODOLOGY

Let us consider the nonlinear partial differential equation of the form:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt} ...) = 0 \tag{1}$$

where u = u(x, t) is an unknown function, P is a polynomial in u = u(x, t) and its partial derivatives, wherein the highest order derivatives and the nonlinear terms are involved. The main steps of the (G'/G)-expansion method are as follows:

Step 1: Introducing a new complex variable ξ in terms of the independent variable x and t, we suppose that

$$u(x,t) = u(\xi)\,, \xi = x - V\,t\,. \tag{2}$$

where,
V is the speed of the traveling wave

The traveling wave variable (2), allows us to transform the nonlinear partial differential equation (1) into an ODE for u(ξ):

$$Q(u, -Vu', u', V^2u'', -V\,u'', u'', \cdots) = 0 \tag{3}$$

Step 2: Suppose the solution of the ODE (3) can be expressed by a polynomial in (G'/G') as follows:

$$u(\xi) = \sum_{n=0}^{m} \alpha_n (G'/G)^n\,,\ \alpha_m \neq 0 \tag{4}$$

where, G = G(ξ) satisfies the second order linear ODE:

$$G'' + \lambda\, G' + \mu\, G = 0, \tag{5}$$

where α_n, (n = 0, 1, 2,…, m), λ and μ are constants to be determined later.

Step 3: The value of m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms of the highest order appearing in Eq. (3). More clearly, if the degree of u (ξ) is D[u(ξ)]=n, then the degree of the other expressions are as follows:

$$D[\frac{d^p u(\xi)}{d\xi^p}] = n + p,\ D[u^p (\frac{d^q u(\xi)}{d\xi^q})^s] = np + s(n+q)$$

Step 4: Substituting Eq. (4), together with Eq. (5) into Eq. (3) and collecting all terms of the same power of (G'/G'), the left hand side of Eq. (3) transforms to another polynomial in (G'/G'). Equating each coefficient to this polynomial to zero, yields a set of algebraic equations for α_n, (n = 0, 1, 2, …, m), V, λ and μ.

Step 5: Suppose that the constants α_n, V, λ and μ can be found by solving the algebraic equations obtained in step 4. Substituting the solutions of Eq. (5) and the values of α_n, V, λ and μ into Eq. (4), we obtain the traveling wave solutions of the nonlinear evolution Eq. (1).

APPLICATIONS

In this section, we will bring to bear the (G'/G)-expansion method to some nonlinear evolution equations.

The kuramoto-sivashinsky equation: We begin with the Kuramoto-Sivashinsky equation:

$$u_t = -u\,u_x - u_{xx} - u_{xxxx} \tag{6}$$

This equation has applications in water waves and one-dimensional evolution of small amplitude long waves in several problems arising in fluid dynamics.
The traveling wave transformation (2), transforms the Eq. (6) into an ODE in u (ξ):

$$u^{(4)} + u'' + u\,u' - Vu' = 0 \tag{7}$$

Integrating Eq. (7) with respect to ξ, yields:

$$u''' + u' + (1/2)\,u^2 - Vu + C = 0 \tag{8}$$

where,
C is an integral constant to be determined later
The solution of Eq. (8) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \alpha_0 + \alpha_1 (\frac{G'}{G}) + \alpha_2 (\frac{G'}{G})^2 + \cdots + \alpha_m (\frac{G'}{G})^m, \alpha_m \neq 0 \tag{9}$$

where, α_n, (n = 0, 1, 2,…, m) are constants to be determined and G = G(ξ) satisfies the Eq. (5). Now balancing the highest order linear tem u''' with the highest order nonlinear term u^2, we obtain, m = 3.
Therefore, Eq. (9) takes the form:

$$u(\xi) = \alpha_0 + \alpha_1 (\frac{G'}{G}) + \alpha_2 (\frac{G'}{G})^2 + \alpha_3 (\frac{G'}{G})^3,\ \alpha_3 \neq 0 \tag{10}$$

Substituting Eq. (10) into Eq. (8) along with Eq. (5) and then collecting all terms with the same power of (G'/G') , the left hand side of Eq. (8) is converted into another polynomial in(G'/G') . Setting each coefficient of this polynomial to zero, we obtain a set of simultaneous algebraic equations for $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, C V, λ and μ as follows:

$$\begin{gathered}
(1/2)\alpha_3^2 - 60\alpha_3 = 0,\ \alpha_2\alpha_3 - 144\alpha_3\lambda - 24\alpha_2 = 0, \\
(1/2)\alpha_2^2 + \alpha_1\alpha_3 - 54\alpha_2\lambda - 114\alpha_3\mu - 111\alpha_3\lambda^2 - 6\alpha_1 - 3\alpha_3 = 0, \\
\alpha_1\alpha_2 + \alpha_0\alpha_3 - V\alpha_3 - 12\alpha_1\lambda - 40\alpha_2\mu - 38\alpha_2\lambda^2 - 27\alpha_3\lambda^3 - 168\alpha_3\lambda\mu - 2\alpha_2 - 3\alpha_3\lambda = 0 \\
(1/2)\alpha_1^2 + \alpha_0\alpha_2 - V\alpha_2 - 7\alpha_1\lambda^2 - 8\alpha_1\mu - 8\alpha_2\lambda^3 - 60\alpha_3\mu^2 - 52\alpha_2\lambda\mu - 57\alpha_3\lambda^2\mu - \alpha_1 \\
-2\alpha_2\lambda - 3\alpha_3\mu = 0 \\
\alpha_0\alpha_1 - V\alpha_1 - \alpha_1\lambda^3 - 16\alpha_2\mu^2 - 8\alpha_1\lambda\mu - 14\alpha_2\lambda^2\mu - 36\alpha_3\lambda\mu^2 - \alpha_1\lambda - 2\alpha_2\mu = 0 \\
-\alpha_1\lambda^2\mu - 6\alpha_2\lambda\mu^2 - 2\alpha_1\mu^2 - 6\alpha_3\mu^3 - \alpha_1\mu + (1/2)\alpha_0^2 + C - V\alpha_0 = 0
\end{gathered} \tag{11}$$

Solving the set of algebraic equations given in Eq. (11), yields:

Case 1: $\alpha_3 = 120,\quad \alpha_2 = 0,\quad \alpha_1 = (-270/19),\quad \alpha_0 = \alpha_0,\quad V = \alpha_0,\quad \lambda = 0,\quad \mu = (-11/76),\quad C = (4950/6859) + (1/2)\alpha_0^2$ (12)

Case 2: $\alpha_3 = 120,\ \alpha_2 = 0,\ \alpha_1 = (90/19),\ \alpha_0 = \alpha_0,\ V = \alpha_0,\ \lambda = 0,\ \mu = (1/76),\ C = (450/6859) + (1/2)\alpha_0^2$ (13)

where,
α_0 is an arbitrary constant

Since in case 1, $\lambda = 0$ and $\mu = \frac{-11}{76} < 0$, therefore, substituting Eq. (12) into Eq. (10), we obtain the following travelling wave solution:

$$u(x,t)=\alpha_0-\frac{270}{19}\sqrt{\frac{11}{76}}\left(\frac{C_1\sinh(\sqrt{11/76}\,\xi)+C_2\cosh(\sqrt{11/76}\,\xi)}{C_1\cosh(\sqrt{11/76}\,\xi)+C_2\sinh(\sqrt{11/76}\,\xi)}\right)+\frac{330}{19}\sqrt{\frac{11}{76}}\left(\frac{C_1\sinh(\sqrt{11/76}\,\xi)+C_2\cosh(\sqrt{11/76}\,\xi)}{C_1\cosh(\sqrt{11/76}\,\xi)+C_2\sinh(\sqrt{11/76}\,\xi)}\right)^3 \tag{14}$$

where, $\xi = x - \alpha_0 t$, α_0, C_1, C_2 are arbitrary constants. If C_1 and C_2 take particular values many known solutions in the literature are revealed and some new solutions are constructed. Suppose $C_1 \neq 0$ but $C_2 = 0$, then solution (14) transforms to the soliton solution:

$$u(x,t)=\alpha_0-\frac{270}{19}\sqrt{\frac{11}{76}}\tanh(\sqrt{11/76}\,\xi)+\frac{330}{19}\sqrt{\frac{11}{76}}\tanh^3(\sqrt{11/76}\,\xi) \tag{15}$$

On the other hand, if $C_1 \neq 0$ and $C_1^2 > C_2^2$, solution (14) reduces to the soliton solution:

$$u(x,t)=\alpha_0-\frac{270}{19}\sqrt{\frac{11}{76}}\tanh(\sqrt{11/76}\,\xi+\xi_0)+\frac{330}{19}\sqrt{\frac{11}{76}}\tanh^3(\sqrt{11/76}\,\xi+\xi_0) \tag{16}$$

where, $\xi_0 = \tanh^{-1}(C_2/C_1)$.

Again for case 2, $\lambda = 0$ and $\mu = \frac{1}{76} > 0$, substituting Eq. (13) into Eq. (10), we obtain the following travelling wave solution:

$$u(x,t)=\alpha_0+\frac{90}{19}\sqrt{\frac{1}{76}}\left(\frac{-C_1\sin(\sqrt{1/76}\,\xi)+C_2\cos(\sqrt{1/76}\,\xi)}{C_1\cos(\sqrt{1/76}\,\xi)+C_2\sin(\sqrt{1/76}\,\xi)}\right)+\frac{30}{19}\sqrt{\frac{1}{76}}\left(\frac{-C_1\sin(\sqrt{1/76}\,\xi)+C_2\cos(\sqrt{1/76}\,\xi)}{C_1\cos(\sqrt{1/76}\,\xi)+C_2\sin(\sqrt{1/76}\,\xi)}\right)^3 \tag{17}$$

where, $\xi = x - \alpha_0 t$; α_0, C_1 and C_2 are arbitrary constants. If C_1 and C_2 are taken particular values, many known solutions in the literature can be revealed. In particular, if $C_1 = 0$ but $C_2 \neq 0$, solution (17) reduces to the solitary wave solution:

$$u(x,t)=\alpha_0+\frac{90}{19}\sqrt{\frac{1}{76}}\cot(\sqrt{1/76}\,\xi)+\frac{30}{19}\sqrt{\frac{1}{76}}\cot^3(\sqrt{1/76}\,\xi) \tag{18}$$

If $C_2 \neq 0$ and $C_2^2 > C_1^2$, then solution (17) reduces to the soliton solution:

$$u(x,t)=\alpha_0+\frac{90}{19}\sqrt{\frac{1}{76}}\cot(\sqrt{1/76}\,\xi+\xi_0)+\frac{30}{19}\sqrt{\frac{1}{76}}\cot^3(\sqrt{1/76}\,\xi+\xi_0) \tag{19}$$

where $\xi_0 = \tan^{-1}(C_1/C_2)$.

Remark 1: To the best of our consciousness the above solutions (14)-(19) have not been reported in the previous research.

The kawahara equation: Now we will consider the Kawahara equation:

$$u_t + u_{xxx} + \eta\, u_{(5x)} + u\, u_x = 0\,, \tag{20}$$

where,
η is a real parameter

The Kawahara equation occurs in the theory of magneto-acoustic waves in plasma and in the theory of shallow water waves with surface tension.
The traveling wave transformation (2) transforms the Eq. (20) into the following ODE:

$$-Vu' + u''' + \eta\, u^{(5)} + u\, u' = 0 \tag{21}$$

where, $u^{(5)}$ and primes denote the derivative with respect to ξ. Eq. (21) is integrable, therefore, integrating with respect toξ, yields:

$$C - V\,u + u'' + \eta\,u^{(4)} + (1/2)u^2 = 0\,, \tag{22}$$

where,

C is an integral constant. Balancing the order of $u^{(4)}$ and u^2 in Eq. (22), we obtain: m = 4. Therefore, the solution of the Kawahara Eq. (20) takes the form:

$$u(\xi) = \alpha_0 + \alpha_1\,(\frac{G'}{G}) + \alpha_2\,(\frac{G'}{G})^2 + \alpha_3\,(\frac{G'}{G})^3 + \alpha_4\,(\frac{G'}{G})^4,\;\; \alpha_4 \neq 0 \tag{23}$$

Substituting Eq. (23) into Eq. (22), using Eq. (5) and then collecting all terms with the same power of (G'/G), the left hand side of Eq. (22) is converted into a polynomial in (G'/G). Setting each coefficient of this polynomial to zero, we obtain the following set of algebraic equations for α_0, α_1, α_2, α_3, α_4 C V, λ and μ:

$$\begin{aligned}
&840\eta\,\alpha_4 + (1/2)\alpha_4^2 = 0,\\
&2640\eta\,\alpha_4\,\lambda + 360\eta\,\alpha_3 + \alpha_3\alpha_4 = 0\,,\\
&3020\eta\alpha_4\,\lambda^2 + \alpha_2\,\alpha_4 + 1080\eta\,\alpha_3\,\lambda + 2080\eta\,\alpha_4\mu + 20\alpha_4 + (1/2)\alpha_3^2 + 120\eta\alpha_2 = 0\,,\\
&24\eta\,\alpha_1 + \alpha_2\,\alpha_3 + 1476\,\eta\alpha_4\lambda^3 + \alpha_1\,\alpha_4 + 12\,\alpha_3 + 1164\eta\;\alpha_3\,\lambda^2 + 816\eta\;\alpha_3\,\mu + 336\eta\,\alpha_2\,\lambda\\
&+4608\eta\,\alpha_4\,\lambda\,\mu + 360\,\alpha_4\,\lambda = 0,\\
&60\eta\,\alpha_1\,\lambda + \alpha_1\,\alpha_3 - V\alpha_4 + 525\eta\,\alpha_3\,\lambda^3 + 256\eta\alpha_4\lambda^4 + 1680\eta\;\alpha_3\,\lambda\,\mu + 330\eta\,\alpha_2\,\lambda^2\\
&+240\eta\;\alpha_2\mu + (1/2)\,\alpha_2^2 + 1696\eta\,\alpha_4\,\mu^2 + 6\,\alpha_2 + 3232\eta\,\alpha_4\,\lambda^2\,\mu + 21\alpha_3\,\lambda + \alpha_0\,\alpha_4\\
&+32\,\alpha_4\,\mu + 16\eta\;\alpha_4\,\lambda^2 = 0,\\
&1062\eta\,\alpha_3\,\lambda^2\mu + 10\,\alpha_2\,\lambda + 130\eta\,\alpha_2\,\lambda^3 + 40\eta\,\alpha_1\mu + 28\,\alpha_4\,\lambda\,\mu + \alpha_1\,\alpha_2 + 576\eta\alpha_3\mu^3\\
&+50\eta\;\alpha_1\,\lambda^2 + 81\eta\,\alpha_3\lambda^4 + 700\eta\,\alpha_4\,\lambda^3\mu + 18\,\alpha_3\mu + 2\,\alpha_1 + \alpha_0\,\alpha_3 - V\,\alpha_3\\
&+440\eta\,\alpha_3\,\lambda\,\mu + 2240\eta\;\alpha_4\lambda\,\mu^2 + 9\alpha_3\,\lambda^2 = 0,\\
&12\,\alpha_4\mu^2 + 8\,\alpha_2\,\mu + 16\eta\;\alpha_2\,\lambda^4 + 480\eta\,\alpha_4\mu^3 + 136\eta\;\alpha_2\,\mu^2 + 232\eta\,\alpha_2\,\lambda^2\mu\\
&+15\,\alpha_3\,\lambda\,\mu + (1/2)\;\alpha_1^2 + 4\,\alpha_2\,\lambda^2 + 60\eta\alpha_1\,\lambda\,\mu + 660\eta\;\alpha_4\,\lambda^2\mu^2 + 15\eta\,\alpha_1\lambda^3\\
&+3\;\alpha_1\,\lambda + 696\eta\;\alpha_3\,\lambda\,\mu^2 + 195\eta\,\alpha_3\,\lambda^3\mu - V\,\alpha_2 + \alpha_0\alpha_2 = 0,\\
&16\eta\alpha_1\,\mu^2 + \alpha_1\,\lambda^2 + \alpha_0\,\alpha_1 + 2\alpha_1\,\mu + 6\alpha_2\,\lambda\,\mu + \eta\alpha_1\lambda^4 + 150\eta\alpha_3\,\lambda^2\,\mu^2 + 120\eta\alpha_3\,\mu^3\\
&+22\eta\alpha_1\,\lambda^2\mu + 30\eta\;\alpha_2\,\lambda^3\mu + 440\eta\alpha_4\,\lambda\,\mu^3 - V\,\alpha_1 + 6\alpha_3\,\mu^2 + 120\eta\;\alpha_2\lambda\,\mu^2 = 0,\\
&C + \alpha_1\,\lambda\,\mu + 2\,\alpha_2\,\mu^2 + 16\eta\,\alpha_2\,\mu^3 + \eta\,\mu\,\alpha_1\,\lambda^3 + (1/2)\,\alpha_0^2 - V\,\alpha_0 + 36\eta\,\alpha_3\,\lambda\,\mu^3\\
&+8\eta\,\alpha_1\lambda\,\mu^2 + 24\eta\,\alpha_4\,\mu^4 + 14\eta\,\alpha_2\,\lambda^2\mu^2 = 0.
\end{aligned} \tag{24}$$

Solving the set of algebraic equations given in Eq. (24) for α_0, α_1, α_2, α_3, α_4, C V, λ and μ, yields:

$$\begin{aligned}
&\alpha_4 = -1680\eta,\;\; \alpha_3 = 0,\;\;\; \alpha_2 = -(840/13),\;\;\; \alpha_1 = 0,\;\; \alpha_0 = \alpha_0,\;\;\; V = (69 + 169\,\alpha_0\,\eta)/(169\eta),\;\;\; \lambda = 0,\;\;\; \mu = (1/52\eta),\\
&C = (3465 + 28561\,a_0^2\eta^2 + 23322\,a_0\eta)/(57122\,\eta^2)
\end{aligned} \tag{25}$$

where,

α_0 is an arbitrary constant.

Now, if η>0 then μ>0. In this case substituting Eq. (25) into Eq. (23), we obtain the following travelling wave solution:

$$\begin{aligned}
u(x,t) = \alpha_0 - (210/169\,\eta)&\left(\frac{-C_1\sin(\sqrt{1/52\eta}\;\xi) + C_2\cos(\sqrt{1/52\,\eta}\;\xi)}{C_1\cos(\sqrt{1/52\eta}\;\xi) + C_2\sin(\sqrt{1/52\eta}\;\xi)}\right)^2\\
-(105/169\,\eta)&\left(\frac{-C_1\sin(\sqrt{1/52\,\eta}\;\xi) + C_2\cos(\sqrt{1/52\,\eta}\;\xi)}{C_1\cos(\sqrt{1/52\,\eta}\;\xi) + C_2\sin(\sqrt{1/52\,\eta}\;\xi)}\right)^4
\end{aligned} \tag{26}$$

where, ξ= x-{(69+169$\alpha_0\eta$)/(169η)}t, α_0, C_1 and $C_2$are arbitrary constants.
Again if η<0 then μ<0, therefore, substituting Eq. (25) into Eq. (23), we obtain the following soliton solution:

$$u=\alpha_0+(210/169\delta)\left(\frac{C_1\sinh(\sqrt{1/52\delta}\,\xi)+C_2\cosh(\sqrt{1/52\delta}\,\xi)}{C_1\cosh(\sqrt{1/52\delta}\,\xi)+C_2\sinh(\sqrt{1/52\delta}\,\xi)}\right)^2 +(105/169\delta)\left(\frac{C_1\sinh(\sqrt{1/52\delta}\,\xi)+C_2\cosh(\sqrt{1/52\delta}\,\xi)}{C_1\cosh(\sqrt{1/52\delta}\,\xi)+C_2\sinh(\sqrt{1/52\delta}\,\xi)}\right)^4 \tag{27}$$

where, ξ = x-{(69+169$\alpha_0\eta$)/(169η)}t, η = - δ, δ>0 and α_0, C_1 and $C_2$are arbitrary constants. When C_1 and C_2 are taken special values many known solutions are obtained from solutions (26) and (27):
In particular, if $C_2\neq 0$ and $C_1^2>C_2^2$, solutions (26) and (27) respectively reduce to:

$$u=\alpha_0-(210/169\eta)\cot^2(\sqrt{1/52\eta}\,\xi+\xi_0)-(105/169\eta)\cot^4(\sqrt{1/52\eta}\,\xi+\xi_0) \tag{28}$$

and,

$$u=\alpha_0+(210/169\delta)\coth^2(\sqrt{1/52\delta}\,\xi+\xi_1)+(105/169\delta)\coth^4(\sqrt{1/52\delta}\,\xi+\xi_1) \tag{29}$$

where, $\xi_0=\tan^{-1}(\frac{C_1}{C_2})$ and $\xi_1=\tanh^{-1}(\frac{C_1}{C_2})$.

Remark 2: To our knowledge the above solutions (26-29) have not been found in the preceding study.

The carleman equation: Lastly, we consider the Carleman equation:

$$u_t+u_x=v^2-u^2 \tag{30}$$

$$v_t-v_x=u^2-v^2 \tag{31}$$

The Carleman equation has applications in the field nonlinear dynamical systems.
The wave transformation (2), transforms the Eq. (30) and (31) into the following ODE:

$$(1-V)u'=v^2-u^2 \tag{32}$$

$$-(1+V)v'=u^2-v^2 \tag{33}$$

According to the (G'/G)-expansion method, the solutions of the Eq. (32) and (33) are of the form:

$$u(\xi)=\alpha_0+\alpha_1(G'/G)+\alpha_2(G'/G)^2+\cdots+\alpha_m(G'/G)^m,\ \alpha_m\neq 0 \tag{34}$$

and

$$v(\xi)=\beta_0+\beta_1(G'/G)+\beta_2(G'/G)^2+\cdots+\beta_n(G'/G)^n,\ \beta_n\neq 0 \tag{35}$$

where, the values of m and n have to be determined. Balancing the order of u' and u^2 in Eq. (32), we obtain m = 1 and balancing the order of v' and v^2 in Eq. (33), we obtain n=1.
Therefore, Eq. (34) and (35) take the form:

$$u(\xi)=\alpha_0+\alpha_1(\frac{G'}{G}),\ \alpha_1\neq 0 \tag{36}$$

and

$$v(\xi)=\beta_0+\beta_1(\frac{G'}{G}),\ \beta_1\neq 0 \tag{37}$$

Substituting Eq. (36) and (37) into Eq. (32) and (33), using Eq. (5) and then collecting all terms with the same power of (G'/G'), the left hand side of Eq. (32) and (33) transform into polynomials in (G'/G'). Setting each coefficient of this polynomial to zero, we obtain the following set of algebraic equations for α_0, α_1, β_0, β_1, V, λ and μ:

$$\alpha_1^2 + V\alpha_1 - \alpha_1 - \beta_1^2 = 0,\ V a_1 \lambda - a_1 \lambda - 2 b_0 b_1 + 2 a_0 a_1 = 0,$$
$$-a_1 \mu - b_0^2 + a_0^2 + V a_1 \mu = 0,\ a_1^2 - b_1 V - b_1 - b_1^2 = 0, \tag{38}$$
$$-V b_1 \lambda - b_1 \lambda - 2 b_0 b_1 + 2 a_0 a_1 = 0,\ -b_1 \mu - b_0^2 + a_0^2 + V b_1 \mu = 0.$$

Solving the set of algebraic equations given in Eq. (38), yields:

$$\alpha_1 = (-2\alpha_0 \beta_0^2)/(\alpha_0^3 + \alpha_0^2 \beta_0 - \alpha_0 \beta_0^2 - \beta_0^3),\ \beta_1 = (-2\alpha_0^2 \beta_0)/(\alpha_0^3 + \alpha_0^2 \beta_0 - \alpha_0 \beta_0^2 - \beta_0^3),\ \lambda = 0,$$
$$\mu = -(\alpha_0^6 + 2\alpha_0^5 \beta_0 - \alpha_0^4 \beta_0^2 - 4\alpha_0^3 \beta_0^3 - \alpha_0^2 \beta_0^4 + 2\alpha_0 \beta_0^5 + \beta_0^6)/(4\alpha_0^2 \beta_0^2),\ V = -(\alpha_0 - \beta_0)/(\alpha_0 + \beta_0), \tag{39}$$

where, α_0 and β_0 are arbitrary constants.

It is noted that, for all values of α_0 and β_0, μ is negative. Suppose μ = - δ therefore, δ>0. Now substituting Eq. (39) into Eq. (36) and (37), we obtain the following travelling wave solutions

$$u = \alpha_0 - \{(2\alpha_0 \beta_0^2)/(\alpha_0^3 + \alpha_0^2 \beta_0 - \alpha_0 \beta_0^2 - \beta_0^3)\} \sqrt{\delta}\, \frac{C_1 \sinh(\sqrt{\delta}\,\xi) + C_2 \cosh(\sqrt{\delta}\,\xi)}{C_1 \cosh(\sqrt{\delta}\,\xi) + C_2 \sinh(\sqrt{\delta}\,\xi)} \tag{40}$$

$$v = \beta_0 - \{(2\alpha_0^2 \beta_0)/(\alpha_0^3 + \alpha_0^2 \beta_0 - \alpha_0 \beta_0^2 - \beta_0^3)\} \sqrt{\delta}\, \frac{C_1 \sinh(\sqrt{\delta}\,\xi) + C_2 \cosh(\sqrt{\delta}\,\xi)}{C_1 \cosh(\sqrt{\delta}\,\xi) + C_2 \sinh(\sqrt{\delta}\,\xi)} \tag{41}$$

where, $\xi = x + \{(\alpha_0 - \beta_0)/(\alpha_0 + \beta_0)\}t$,
$\delta = (\alpha_0^6 + 2\alpha_0^5\beta_0 - \alpha_0^4\beta_0^2 - 4\alpha_0^3\beta_0^3 - \alpha_0^2\beta_0^4 + 2\alpha_0\beta_0^5 + \beta_0^6)/(4\alpha_0^2\beta_0^2)$
and α_0, β_0, C_1 and C_2 are arbitrary constants.

In particular, if $C_2 \neq 0$ and $C_1^2 > C_2^2$, solutions (40) and (41) reduce to respectively:

$$u = \alpha_0 - \{(2\alpha_0 \beta_0^2)/(\alpha_0^3 + \alpha_0^2 \beta_0 - \alpha_0 \beta_0^2 - \beta_0^3)\} \sqrt{\delta} \coth(\sqrt{\delta}\,\xi + \xi_0) \tag{42}$$

and

$$v = \beta_0 - \{(2\alpha_0^2 \beta_0)/(\alpha_0^3 + \alpha_0^2 \beta_0 - \alpha_0 \beta_0^2 - \beta_0^3)\} \sqrt{\delta} \coth(\sqrt{\delta}\,\xi + \xi_0) \tag{43}$$

where, $\xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2}\right)$

Remark 3: To the best of our awareness the solutions (40)-(43) have not been found in the past literature.

CONCLUSION

In this study, by means of the (G'/G)-expansion method some new exact solitary wave solutions of the Kuramoto-Sivashinsky equation, the Kawahara equation and the Carleman equation are successfully achieved. The solutions include the hyperbolic and the trigonometric functions. These types of solutions have many potential applications in water waves, fluid dynamics, plasma physics and nonlinear dynamics. It is important to note that the (G'/G)-expansion method is direct, concise, elementary and comparing to other methods such as tanh-coth method, Jacobi elliptic function method, Exp-function method etc., it is easier, effective and powerful. We have verified all solutions by putting them back into the original equations and found correct. The method can be further applied to other nonlinear equations to establish new reliable solutions.

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