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Research Article

A Characterization of Sporadic Janko Group J_1

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Abstract: Let G be a group and $\omega(G)$ be the set of element orders of G. Let k∈ $\omega(G)$ and s_k be the number of elements of order k in G. Let nse (G) = {s_k| k $\omega(G)$ }. In Khatami *et al*. (2011) and Liu (2012c) the authors proved that $L_3(2)$ and $L_3(4)$ are unique determined by nse (G). In this study, we prove that if G is a group such that nse (G) = nse (J_1), then G ≅ J_1.

Keywords: Element order, Janko group, Number of elements of the same order, Thompson' problem

INTRODUCTION

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. The set of element orders of G is denoted by $\omega(G)$. Let k ∈ $\omega(G)$ and S_k be the number of elements of order k in G. Let nse (G) = {S_k| k ∈ $\omega(G)$}. Let $\pi(G)$ denote the set of prime p such that G contains an element of order p. A finite group G is called a simple *Kn* -group, if G is a simple group with $|\pi(G)|$ = n. Thompson posed a very interesting problem related to algebraic number fields as follows Shi (1989).

Thomson' problem: Let T(G) ={(n, s | n ∈ ω(G) and S_n ∈ nse (G)}, where S_n is the number of elements with order n. Suppose that T(G) = T (H). If G is a finite solvable group, is it true that H is also necessarily solvable?

It was proved that: Let G be a group and M some simple K_i -group, i = 3, 4, then $G \cong M$ if and only if |G| = |M| and nse (G) = nse (M) (Shao *et al*., 2009, 2008). And also the groups A_{12}, A_{13} and L_5 (2) are characterizable by order and nse (Guo *et al*., 2012; Liu, 2012a; 2012b), respectively). Recently, all sporadic simple groups are characterizable by nse and order (Khalili *et al*., 2013)).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in T (G) of the Thompson Problem, in other words, it remains only nse (G), whether can it characterize finite simple groups? Up to now, some groups especial for PSL (2,q), can be characterized by only the set nse (G) (Khalili *et al*., 2011; Shen *et al*., 2010), respectively).

The author has proved that the groups $L_3(4)$ and $L_2(16)$ are characterizable by nse (Liu, 2012b, 2013) respectively). In this study, it is shown that the group J_1, which the number of the set of the same order is 10, also can be characterized by nse (J_1), that is.

MAIN THEOREM

Let G be a group with nse (G) = nse (J_1).

Then G ≅ J_1. In the proof of the Main Theorem, we use the technique of Khalili *et al*. (2003) to make the proof simple. Namely, we investigate the influence of the primes 7, 19 on the set {2, 3, 5, 7, 11, 19} and the order of elements of the group. Hence we only consider the sets {2}, {2, 3}, {2, 5}, {2, 11}, {2, 3, 5}, {2, 3, 11}, {2, 3, 5, 11} and π (G) = {2, 3, 5, 7, 11, 19}. Finally, we get the desired result by order consideration.

Some lemmas:

Lemma 1: Frobenius (1895) Let G be a finite group and m be a positive integer dividing |G|. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \| L_m(G)|$

Lemma 2: Miller (1904) Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and n = p^sm with (p,m) = 1. If P is not cyclic and s>1, then the number of elements of order n is always a multiple of P^s.

Lemma 3: Shen (2010) Let G be a group containing more than two elements. If the maximal numbers of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2-1)$

Lemma 4: Hall (1959) [Theorem 9.3.1]) Let G be a finite solvable group and |G| = mn, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, (m,n) = 1. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m

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be the number of Hall π -subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:

- $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ for some P_j
- The order of some chief factor of G is divided by $q_i^{\beta_i}$

PROOF OF THEOREM

Let G be a group such that nse (G) = nse (J_1) and s_n be the number of elements of order n. By Lemma 3 we have G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if n>2, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have:

$$\begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d|m} s_d \end{cases} \quad (1)$$

We rewrite the main theorem here to read easily.

Theorem: Let G be a group with nse (G) = nse (J_1) = {1, 1463, 5852, 11704, 15960, 23408, 25080, 27720, 29260, 35112}. Then G ≅ J_1.

Proof: We prove the theorem by first proving that $\pi(G) \subseteq$ {2, 3, 5, 7, 11, 19}, second showing that |G| = |J_1| and so G≅ J_1

By (1.1), $\pi(G) \subseteq$ {2, 3, 5, 7, 11, 13, 19, 29}. If m>2, then ϕ (m) is even, then s_2 = 1463, $2 \in \pi(G)$. In the following, we prove that 13, $29 \notin \pi(G)$.

If $13 \in \omega(G)$, then by (1.1), s_{13} = 35112. If $2 \cdot 13 \in \omega(G)$, then $s_{26} \in$ nse (G). Therefore $2 \cdot 13 \notin \omega(G)$. Now we consider Sylow 13-subgroup P_{13} acts fixed point freely on the set of elements of order 2, then $|P_{13}| \mid s_2$ (= 1463), a contradiction.

Similarly $29 \notin \omega(G)$. In fact, by (1.1), s_{29} = 29260.

If $2 \cdot 29 \in \omega(G)$, then $s_{58} \notin$ nse (G). Therefore $2 \cdot 29 \notin \omega(G)$. Now we consider Sylow 29-subgroup P_{29} acts fixed point freely on the set of elements of order 2, then $|P_{29}| \mid s_2$ (= 1463), a contradiction.

Therefore, $\pi(G) \subseteq$ {2, 3, 5, 7, 11, 19}. By (1.1), we have that S_3 = 5852 or 23408, s_5 = 11704, s_7 = 25080, S_{11} = 15960 and S_{19} = 27720.

If 3.19 ϵ ω (G), then by (1.1) $57 \mid 1 + s_3 + s_{19} + s_{57}$ and $\phi(57) \mid s_{57}$ and so we have $s_{57} \notin$ nse (G), a contradiction. Hence 3.19 ∉ ω (*G*).

If 5.19 ∈ω (*G*), then by (1.1) $95|1+s_5+s_{19}+s_{95}$ and $\phi(95) \mid s_{95}$ and so we have $s_{95} \notin$ nse (G), a contradiction. Hence $5 \cdot 19 \notin \omega(G)$.

Similarlly, we can prove that $7 \cdot 19 \notin \omega(G)$ and $11 \cdot 19 \notin \omega(G)$

If $2^a \in \omega(G)$, then $\phi(2^a) \mid s_{2^a}$ and so $0 \le a \le 5$.

Similarly if $3^a \in \omega(G)$, then $1 \le a \le 3$; if $5^a \in \omega(G)$, then $1 \le a \le 2$; if $11^a \in \omega(G)$, then $1 \le a \le 2$.

If $7^a \in \omega(G)$, then $1 \le a \le 2$. If $7^2 \in \omega(G)$, then $s_{7^2} \notin$ nse (G), therefore a = 1. Similarly, if $19^a \in \omega(G)$, then a = 1.

Therefore, we have that $2^6, 3^4 \notin \omega(G)$; $p \cdot 19 \notin \omega(G)$ with p = 2, 3, 5, 7, 11, 19; $p^2 \notin \omega(G)$ with p = 7, 19; $p^3 \notin \omega(G)$ with p = 5, 11.

If $5 \in \pi(G)$, then exp (P_5) = 5 or 25.

If exp (P_5) = 5, then $|P_5| \mid 1 + s_5$(= 11705) and so $|P_5| = 5$. It follows that $n_5 = s_5 / \phi(5) = 2 \cdot 7 \cdot 11 \cdot 19$ and so 7, 11, 19 $\in \pi(G)$.

If exp (P_5) = 25, then $|P_5| \mid 1 + s_5 + s_{25}$(= 39425) and so $|P_5| = 25$, $n_5 = 2 \cdot 3^2 \cdot 7 \cdot 11$

If $7 \in \pi(G)$, then $|P_7| \mid 1 + s_7$(= 25081) and so $|P_7| = 7$. So $n_7 = 2^2 \cdot 5 \cdot 11 \cdot 19$ and so 5, 11, 19 $\in \pi(G)$.

If $11 \in \pi(G)$, then exp (P11) = 11, 121.

If exp (P_{11}) = 11, then $|P_{11}| \mid 1 + s_{11}$(= 11705) and so $|P_{11}| = 11$. It follows that $n_{11} = 2^2 \cdot 3 \cdot 7 \cdot 19$ and so 3, 7, 19 $\in \pi(G)$.

If exp (P_{11}) = 121, then $|P_{11}| \mid 1 + s_{11} + s_{121}$(= 43681) and so $|P_{11}| = 121$. Hence $n_{11} = 2^2 \cdot 3^2 \cdot 7$ and so 3, $7 \in \pi(G)$.

If $19 \in \pi(G)$, then as $19^2 \notin \omega(G)$, we have that $|P_{19}| \mid 1 + s_{19}$ (= 27721) and $|P_{19}| = 19$. Hence $n_{19} = 2^2 \cdot 5 \cdot 7 \cdot 11$. So 5, 7, 11 $\in \pi(G)$.

Therefore we only consider that the set $\pi(G)$ is {2}, {2, 3}, {2, 5}, {2, 11}, {2, 3, 5}, {2, 3, 11}, {2, 3, 5, 11} and $\pi(G)$ = {2, 3, 5, 7, 11, 19}. So we divide the proofs into the following cases.

Case a: $\pi(G)$ = {2}. In this case, ω(G) ⊆{1, 2, 2^2, 2^3, 2^4, 2^5}. But the number of the set of nse (G) is 10, so we get a contradiction.

Case b: $\pi(G)$ = {2,3}. Since $3^4 \in \omega(G)$, then exp (P_3) = 3, 9, 27. So we consider the following subcases.

First let S_3 = 5852.

- **Subcase b.1.1:** exp (P_3) = 3. $|P_3|| 1+s_3$ (= 5853) and so $|P_3|=3$. So $n_3 = 2\cdot7\cdot11\cdot19$. It follows that 7, 11, 19 $\in \pi(G)$, a contradiction.
- **Subcase b.1.2:** exp (P_3) = 9. $|P_3|| 1+s_3+s_9$ (= 30933) and so $|P_3|$ = 9. Whence $n_3 = 2^2\cdot5\cdot11\cdot19$ and so 5, 11, 19 ∈ $\pi(G)$, a contradiction.
- **Subcase b.1.3:** exp (P_3) = 27. We know that $|P_3|\geq 3^3$. If $|P_3|=3^3$, then we have a contradiction since $s_{27} \in$ nse (G).

Second let S_3 = 23408.

- **Subcase b.2.1:** exp (P_3) = 3, $|P_3|| 1+s_3$ (= 23409) and so $|P_3|| 3^4$.

If $|P_3|$ = 3, then $n_3 = 2^3\cdot5\cdot7\cdot11\cdot19$. It follows that 5, 7, 11, 19 ∈ π(G), a contradiction. If $|P_3|$ = 9, then |G| = 175560+5852k$_1$+11704k$_2$+15960k$_3$+23408k$_4$+25080k$_5$+27720k$_6$+29260k$_7$+35112k$_8$ = $2^l.3^2$ with $k_1,\cdots,k_8$ and l are nonnegative integers and $0\leq k_1+\cdots+k_8\leq 1$. Since 175560 ≤|G| = $2^l\cdot3^2$ ≤ 175560+35112 · 1 · 8, we have l = 15. Hence |G|=175560+5852k_1+11704k_2+15960k_3+23408 k_4 +25080k_5+27720k_6+29260k_7+35112k_8 = $2^{15}\cdot3^2$, but the equation has no solution in N. If $|P_3|=3^3$ or $|P_3|=3^4$, then by Lemma 2, $s_3 = |P_3|t$ for some integer t. But the equation has no solution in N.

- **Subcase b.2.2:** exp (P_3) = 9. We have that $|P_3|| 1+s_3+s_9$(= 51129) and so $|P_3|=9$. Whence $n_3 = 2^2\cdot3\cdot5\cdot7\cdot11$ and so 5, 7, 11 $\in \pi(G)$, a contradiction.
- **Subcase b.2.3:** exp (P_3) = 27. We know that $|P_3|\geq 3^3$. If $|P_3|=3^3$, then s_{27} = 27720 and so $n_3 = 2^2\cdot5\cdot7\cdot11$. It follows that 5, 7, 11 ∈ $\pi(G)$, a contradiction.

If $|P_3|\geq 3^4$, then by Lemma 2, we also get a contradiction.

Case c: $\pi(G)$ = {2, 5}.

If exp (P_5}) = 5, then $|P_5|| 1+s_5$(= 11705) and so $|P_5|=5$. Since $n_5 = 2\cdot7\cdot11\cdot19$, 7, 11, 19 $\in\pi(G)$, a contradiction.

If exp (P_5) = 25, then $|P_5|| 1+s_5+s_{25}$(= 39425) and so $|P_5|=25$. Since $n_5 = 2^2\cdot3^2\cdot7\cdot11$, 3, 7, 11 $\in\pi(G)$, a contradiction.

Case d: $\pi(G)$ = {2, 11}.

If exp (P_{11}) = 11, then $|P_{11}|| 1+s_{11}$ (= 15961) and so $|P_{11}|=11$. So $n_{11} = 2^2\cdot3\cdot7\cdot19$ and 3, 7, 19 $\in\pi(G)$, a contradiction.

If exp (P_{11}) = 121, then $|P_{11}|| 1+s_{11}+s_{121}$ (= 43681) and so $|P_{11}|=121$. So $n_{11} = 2^2\cdot3^2\cdot7$ and 3, 7 $\in\pi(G)$, a contradiction.

Case e: $\pi(G)$ = {2, 3, 5}.

The proof is the same as Case c.

Case f: $\pi(G)$ = {2, 3, 11}.

The proof is the same as Case d.

Case g: $\pi(G)$ = {2, 3, 5, 7, 11, 19}.

We know that $|P_7|=7$ and $|P_{19}|=19$.

We first show that $|P_5|=5$ and $|P_{11}|=11$.

If 5.7∈ω (G), set P and Q are Sylow 7-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore we have that $s_{35} = \phi(35)\cdot n_7\cdot k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_7)$. As $n_7 = s_7/\phi(7) = 4180$, $100320 \mid s_{35}$ and then $s_{35} = 100320t$ for some integer t, but the equation has no solution in N, a contradiction. Hence $5\cdot7\notin\omega(G)$. It follows that the group P_5 acts fixed point freely on the set of order 7 and so $|P_5|| s_7$ (= 25080). Hence $|P_5|$ = 5.

Similarly since $11\cdot19\notin\omega(G)$, then we have that the groupP11 acts fixed point freely on the set of order 19 and so $|P_{11}|| s_{19}$ (= 27720). Hence $|P_{11}|$ = 11.

Since $2\cdot19\notin\omega(G)$, the group P_2 acts fixed point freely on the set of order 19 and so $|P_2|| s_{19}$ (= 27720). Hence $|P_2|| 2^3$.

Similarly since $3\cdot19\notin\omega(G)$, we also have $|P_3|=3$.

Therefore $|G|=2^m\cdot3\cdot5\cdot7\cdot11\cdot19$. But $\sum_{s_k\in nse(G)} s_k$ = 175560 = $2^3\cdot3\cdot5\cdot7\cdot11\cdot19 \leq 2^m\cdot3\cdot5\cdot7\cdot11\cdot19$. Thus

$|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = |J_1|$. So since nse (G) = nse (J_1), we have from (Khalili *et al*., 2013), that $G \cong J_1$. This completes the proof.

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