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Research Article

A Novel Validity Index for Evaluating the Item Ordering Structure Based on Q-matrix Theory

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Abstract: Before then, no validity index can be used for evaluating the item ordering structure for cognition diagnosis. In this study, based on the Q-matrix theory, an ideal item ordering structure without slip and guess for an efficient test is proposed by using the set containing relation operation. It can be viewed as a criterion of the item ordering structure for a given test. Furthermore, a novel criterion-related validity index for evaluating the item ordering structure of any item ordering algorithm is proposed, it is more useful for cognition diagnosis.

Keywords: Efficient item, item relational structure theory, ordering theory, Q-matrix, validity index

INTRODUCTION

For any given test, the item ordering theory was first proposed by Airasian and Bart (1973), since if any two items have ordering relation, then these two items must also have correlation relation, for overcoming this drawback, Takeya (1980) proposed an improved method, called item relational structure theory. Since the threshold limit value of each of above mentioned methods is fixed at constant, the OT is $\varepsilon \in [0.02, 0.04]$ and the IRS is 0.5, both of them lack of statistical meaning. Based on the empirical distribution of the critical values, Liu *et al*. (2011) proposed an further improved method, called the improved item relational structure theory. It is a more effective way to construct the item ordering structure of examinees for cognition diagnosis. However, all of above mentioned methods do not concern to consider the most important issue that each item of the given test whether it is efficient or not. It is obviously that if any item in the test is inefficient, then the ordering relation about this item with other one is also inefficient, in other words, before to use the item ordering theory, we must make sure, first that each cognition skills of the test is efficient to fit the structure of the given cognition skills. Furthermore, for the given cognition skills, if all items of the given test are efficient, then how to judge the item ordering structure of the results by using a specific ordering theory is the most important problem, since before then, there is not any validity index can be used for evaluating the item ordering structure for cognition diagnosis. For tunately, Q-matrix theory Tatsuoka (1983) can be used to decide a test-blueprint to make sure that each item of the test is efficient to fit the structure of the given cognition skills.

In this study, based on the Q-matrix theory, the reduce Q-matrix can be obtained, in which, each item of the test is efficient to fit the structure of the given cognition skills and then, by using the set containing relation operation, an ideal item ordering structure without slip and guess for an efficient test is proposed, it can be viewed as a criterion of the examinees' item ordering structure after taking the efficient test and then, according to the proposed criterion, a novel criterion-related validity index for evaluating the item ordering structure of any item ordering algorithm is proposed.

Q-MATRIX THEORY FOR COGNITION DIAGNOSIS

Attribute structure and its matrix representation:
Definition 1: Prerequisite attributes: Let A = $\{a_1, a_2, \ldots, a_m\}$ be the set of m cognitive skills, called attributes $a_1, a_2, \ldots, a_m$:

- If the examinee before master the attribute a_j, he must master the attribute a_i first, then a_j is called the prerequisite attribute of attribute a_j and denoted as $a_i \rightarrow a_j$, otherwise dented as $a_i \nrightarrow a_j$.
- The graph of the prerequisite relations among all of the attributes is called the structure graph of the attributes set.

Theorem 1: If A = $\{a_1, a_2, \ldots, a_m\}$ is the attributes set and then:

$$a_i \rightarrow a_j, a_j \rightarrow a_k \Rightarrow a_i \rightarrow a_k \tag{1}$$

where, i, j, k = 1, 2,…, m

Definition 2: The adjacency matrix of the attributes set: If A = {a_1, a_2, …, a_m} is the attributes set, then Boolean matrix A_m is called the adjacency matrix of A where,

$$A_m = [a_{ij}]_{m\times m}$$

$$a_{ij} = \begin{cases} 1 & if\ a_i \to a_j \\ 0 & if\ a_i \not\to a_j \end{cases} \tag{2}$$

Definition 3: The reachability matrix of the attributes set: If A_m is the adjacency matrix of the attributes set A satisfying (3), then the Boolean matrix R is called the reachability matrix of A, where the addition operator is Boolean addition operation:

$$R_m = \left[r_{ij}\right]_{m\times m} = \left(A_m + I_m\right)^{k+1} = \left(A_m + I_m\right)^k, k \in N \tag{3}$$

Attribute-item incident matrix, Q-matrix:

Definition 4: A prerequisite attribute of an item: Let A = {a_1, a_2, …,a_m} be the attributes set and $I_{\{p\}}$ = {$\underline{I}_1$, $\underline{I}_2$,…, $\underline{I}_{2m-1}$} be the set of all possible item categories each of them masters at least one attribute in A.

If any examinee can answer the item $\underline{I}_j$ correctly, he must master the attribute a_i first, then we call a_l is a prerequisite attribute of item $\underline{I}_j$, denoted $a_j \to \underline{I}_j$, otherwise, denoted $a_i \not\to \underline{I}_j$.

Definition 5: Attribute-possible item incident matrix: If A = {a_1, a_2, …, a_m} is the attributes set and $I_{\{p\}}$ = {$\underline{I}_1$, $\underline{I}_2$,…, $\underline{I}_{2m-1}$} is the set of all possible item categories, each of them contains at least one attribute in A, then the matrix $Q_{m\times(2^m-1)} = [q_{ij}]_{m\times(2^m-1)}$ is called the incident matrix of A and $I_{\{P\}}$, or Q- matrix of A and $I_{\{P\}}$, where,

$$q_{ij} = \begin{cases} 1 & if\ a_i \to \underline{I}_j \\ 0 & if\ a_i \not\to \underline{I}_j \end{cases} \tag{4}$$

Attribute-efficient item incident matrix, reduced Q-matrix:

Definition 6: Efficient item, inefficient item: If $\underline{I}_k$ is a possible item satisfying:

$$a_i \to a_j, a_j \to \underline{I}_k \Rightarrow a_i \to \underline{I}_k \tag{5}$$

where, i, j = 1, 2, …, m, k = 1, 2, …, (2^m - 1).

Then $\underline{I}_k$ is called an efficient item fitted in with the attributes structure, otherwise, it is an inefficient item.

Definition 7: Attribute-efficient item incident matrix, Reduced Q-matrix: Let $Q_{m\times(2^m-1)} = [q_{ij}]_{m\times(2^m-1)}$ be Q-matrix of A and the possible item set $I_{\{P\}}$, then Q-matrix of A = {a_1, a_2,…, a_m} and the efficient item set $I_{\{E\}}$ = {$\underline{I}$ | $\underline{I} \in I_{\{P\}}$, $\underline{I}$ is an efficient item}, is called the reduced Q-matrix.

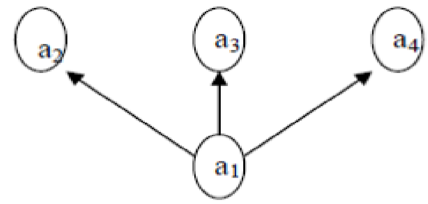

Fig 1: The structure graph of A = {a_1, a_2, a_3, a_4}

Example 1: Consider an addition test of two proper fractions, let A = {a_1, a_2, a_3, a_4} be the attributes set, its attributes are defined as follows, we can obtain its structure graph as Fig. 1, then its adjacency matrix, reachability matrix, Q-matrix and reduced Q-matrix can be found as follows:

- a_1 : Add the numerators of the fractions with the same denominator
- a_2 : Find a common denominator of the fractions
- a_3 : Reduce the sum of the fractions such that the denominator and numerator are relative prime
- a_4 : Covert the improper fraction into a mixed fraction

$$A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$Q_{4\times15} = \begin{bmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,15} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,15} \\ q_{3,1} & q_{3,2} & \cdots & q_{3,15} \\ q_{4,1} & q_{4,2} & \cdots & q_{4,15} \end{bmatrix} = \begin{bmatrix} 1&0&0&0&1&1&1&0&0&0&1&1&1&0&1 \\ 0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&0&1&0&0&1&0&1&0&1&1&0&1&1&1 \\ 0&0&0&1&0&0&1&0&1&1&0&1&1&1&1 \end{bmatrix}$$

$$Q_R = \begin{bmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&0&0&1&1&0&1 \\ 0&0&1&0&1&0&1&1 \\ 0&0&0&1&0&1&1&1 \end{bmatrix} \tag{7}$$

Ideal item ordering structure theory based on reduced Q-matrix:

Definition 8: Ideal item ordering structure: Let:

$$Q_R = Q_{m\times n} = \left[q_{ij} \right]_{m\times n}$$
$$= \left[\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_n \right] = \left[\underline{I}_1, \underline{I}_2, \ldots, \underline{I}_n \right] \quad (8)$$

$$Set\left(\underline{I}_j\right) = \left\{ a \in A \mid a \rightarrow \underline{I}_j \right\},\ j = 1, 2, \ldots, n \quad (9)$$

If $\underline{I}_j \neq \underline{I}_k$, Set ($I_j$) $\subset$ Set (I_k) then we say that $\underline{I}_j$ is a prerequisite item to $\underline{I}_k$, denoted as $\underline{I}_j \rightarrow \underline{I}_k$. Otherwise, we say that $\underline{I}_j$ is not a prerequisite item to $\underline{I}_k$, denoted as $\underline{I}_j \not\rightarrow \underline{I}_k$.

Definition 9: The adjacency matrix and reachability matrix of I{E}:

- Let the Boolean matrix M_A ($I_{\{E\}}$) be the adjacency matrix of $I_{\{E\}}$ satisfying where,

$$M_A\left(I_{\{E\}}\right) = \left[b_{ij} \right]_{n\times n}$$

$$b_{ij} = \begin{cases} 1 & if\ \underline{I}_i \rightarrow \underline{I}_j \\ 0 & if\ \underline{I}_i \not\rightarrow \underline{I}_j \end{cases} \quad (10)$$

- Let the Boolean matrix M_A ($I_{\{E\}}$) be the reachability matrix of $I_{\{E\}}$ satisfying:

$$\left[M_R\left(I_{\{E\}}\right) \right] = \left[M_A\left(I_{\{E\}}\right) + I_n \right]^{k+1}$$
$$= \left[M_A\left(I_{\{E\}}\right) + I_n \right]^{k},\ k \in N$$
(11)

where the addition operator is a Boolean addition operation.

Example 2: Let the data be the same as example 1. Then,

$$Q_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$
$$= \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right] \quad (12)$$
$$= I_{\{E\}} = \left[\underline{I}_1, \underline{I}_2, \underline{I}_3, \underline{I}_4, \underline{I}_5, \underline{I}_6, \underline{I}_7, \underline{I}_8 \right]$$

From the definition 8, we can obtain the graph of item ordering structure as Fig. 2.

From Fig. 2 and the definition 9, we can the adjacency matrix and reachability matrix of the items as follows:

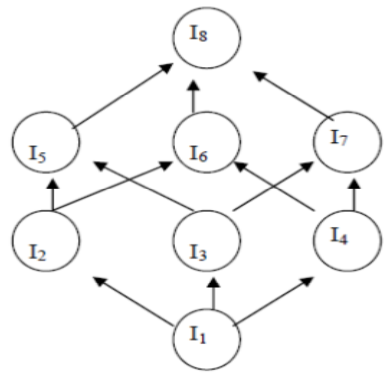

Fig. 2: The structure graph of $I_{\{E\}} = \left[\underline{I}_1, \underline{I}_2, \underline{I}_3, \underline{I}_4, \underline{I}_5, \underline{I}_6, \underline{I}_7, \underline{I}_8 \right]$

$$M_A\left(I_{\{E\}}\right) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_R\left(I_{\{E\}}\right) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (13)$$

Item ordering structure theories:

Improved item relational structure theory: Since the threshold limit value of Airasian and Bart's Ordering Theories is fixed at a constant $\varepsilon \in [0.02, 0.04]$ (Airasian and Bart, 1973) and the threshold limit value of Takeya's Item Relational Structure Theory is 0.5 (Takeya, 1980), both of them lack statistical meaning. Based on the empirical distribution of the critical values, Liu *et al*. (2011) proposed an improved Item Relational Structure Theory, as follows:

Definition 10: Improved item relational structure theory based on the empirical distribution of the critical values: For any given test, let X = (X_1, X_2,…, X_n) denote a vector containing n binary item scores variables. Each individual taking n-item test produces a vector x = (x_1, x_2, …, x_n) containing ones and zeros.

Let P = (X_j, = 0, X_k = 1) be the joint probability of X_j = 0, X_k = 1 and P (X_j = 0), P (X_k = 1) be the marginal probabilities of X_j = 0, X_k = 1, respectively and:

$$r_{jk}^{*} = 1 - \frac{P(X_j = 0, X_k = 1)}{P(X_j = 0)P(X_k = 1)} \quad (14)$$

Let n be number of items, m be number of examinees, therefore, the number of all ordering index r^*_{jk} is n (n-1), then we can obtain a distribution of all ordering index r^*_{jk} and let the threshold limit value of IRS be defined as:

$$r_c = \arg_x \left[1 - \int_{-\infty}^{x} f\left(r_{jk}^{*}\right) dr_{jk}^{*} = 0.05 \right] \quad (15)$$

where $f(r_{jk}^*)$ is the probability density function of random variable r_{jk}^*.

If $r_{jk}^* > r_c$, then we say that item X_j is a prerequisite to X_k denoted as $X_j \rightarrow X_k$, otherwise, we say that item X_j is not a prerequisite to X_k denoted as $X_j \not\rightarrow X_k$.

Validity index for item ordering structure based on Q-matrix theory:

Definition 11: Let M_R $(I_{\{E\}}) = [e_{ij}]_{n\times n}$ be the reach ability matrix of $I_{\{E\}}$, M_R $(I'_{\{E\}}) = [e'_{ij}]_{n\times n}$, be a reachability matrix of $I'_{\{E\}}$, which is obtained by using an item ordering structure algorithm from a given test for some specific examinees, then based on the correlation relation, the validity index of $I'_{\{E\}}$, denoted as Val $(I'_{\{E\}} | I_{\{E\}})$, is defined as below:

$$V_{al}\left(I'_{\{E\}} \mid I_{\{E\}}\right) = \frac{1}{2}\left[1 + \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(e_{ij} - \overline{e}\right)\left(e'_{ij} - \overline{e}'\right)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(e_{ij} - \overline{e}\right)^2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(e'_{ij} - \overline{e}'\right)^2}}\right]$$

$$\overline{e} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}, \ \overline{e}' = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} e'_{ij} \quad (16)$$

where, Val $(I'_{\{E\}} | I_{\{E\}}) \in [0, 1]$, the larger the value is, the better the validity is.

Example 3: Let the data be the same as example 2. We can take Q_R as a test blueprint to formulate the all efficient items of a test as below:

$$\begin{aligned} &I_1: \ \frac{1}{5} + \frac{3}{5} = \qquad I_2: \ \frac{2}{3} + \frac{1}{5} = \\ &I_3: \ \frac{1}{12} + \frac{5}{12} = \qquad I_4: \ \frac{4}{7} + \frac{5}{7} = \\ &I_5: \ \frac{3}{7} + \frac{1}{14} = \qquad I_6: \ \frac{3}{7} + \frac{3}{5} = \\ &I_7: \ \frac{11}{12} + \frac{5}{12} = \qquad I_8: \ \frac{6}{7} + \frac{9}{14} = \end{aligned} \quad (17)$$

To use this test to some fifth grade students in an elementary school, by using above mentioned improved item relational structure algorithm, we can obtain the reachability matrix of their item relational structure, M_R $(L'_{\{E\}})$, as follows.

Then we can obtain the value of criterion-related validity is Val $(l'_{\{E\}}|I_{\{E\}}) = 0.9271$:

$$M_R(I'_{\{E\}}) = \begin{bmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&0&1&1&1&1&1 \\ 0&0&1&0&1&1&1&1 \\ 0&0&0&1&0&1&1&1 \\ 0&0&0&0&1&0&1&1 \\ 0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \quad (18)$$

CONCLUSION

Before then, there is no other validity index can be used for evaluating the item ordering structure for cognition diagnosis. In this study, based on the Q-matrix theory, a novel criterion-related validity index for evaluating the item ordering structure of any item ordering algorithm is proposed. It is very useful for cognition diagnosis.

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