Research Journal of Applied Sciences, Engineering and Technology 6(1): 171-174, 2013
DOI:10.19026/rjaset.6.4054
ISSN: 2040-7459; e-ISSN: 2040-7467

Submitted: January 10, 2013 Accepted: January 31, 2013 Published: June 05, 2013

Research Article

A New Algorithm for Stochastic Variational Inequality with an Application

[1]Bing Liang, [1]Chunsheng Shi, [2]Xue Tong and [2]Tianhua Song
[1]School of Management, Harbin Institute of Technology, Harbin 15001, China
[2]School of Humanities and Social Science, Harbin Institute of Technology, Harbin 15001, China

Abstract: Recently stochastic variational inequality has been extensively studied. However there are few methods can be effectively realized. This study considers to solve stochastic variational inequality by combining quasi-Monte Carlo approach and interior point method. The global convergence is established for the new algorithm. An application for the synergies analysis of the supply chain after M&A from the literature is discussed.

Keywords: Convergence, interior point method, quasi-monte carlo approach, supply chain, stochastic variational inequality

INTRODUCTION

This study considers the following stochastic variational inequality problem: find a vector $X^* \in \mathcal{H}$ such that:

$$(X - X^*)^T E[F(X^*, \omega)] \geq 0, \ \forall X \in \mathcal{H}, \ \omega \in \Omega, \qquad (1)$$

where, $\mathcal{H} \subset R^n$, Ω is the underlying sample space, $F : R^n \times \omega \to R^n$ is a is continuously differentiable mapping, $E[\cdot]$ is the mathematical expectation and superscript T denotes transpose.

Model (1) provides a framework for modeling of the equilibrium under uncertainty as a generalization of the stochastic complementarily problem (Chen *et al*., 2009) and has been receiving much attention in the recent literature (Lin, 2009). When $\mathcal{H} = R^n$, the (1) further reduces to a system of stochastic equations: Finding $(y, z) \in \mathcal{R}^m \times \mathcal{R}^n$ satisfying $\mathbb{E}_\omega[F(y, z, \omega)] = 0$.

In this study, we propose a Monte Carlo sampling method combined with a homogeneous interior-point method for solving (1) based on the new reformulation, which is different from all the reformulation mentioned above. That is, by the reformulation, we discretize the true problem by approximating the expected value of the underlying functions with its sample average and then solve the sample averaged approximation problem by utilizing the homogeneous interior-point methods. The Monte Carlo sampling method is a very popular method in stochastic programming and it is essentially the same as the sample path method. Our focus here is on the combination of the Monte Carlo sampling method with a homogeneous interior-point method for solving (1). This distinguishes our approach from the above two ways. We establish convergence of global optimal solutions and stationary points of approximation problems generated by the proposed method. Moreover, as quasi-Monte Carlo methods are generally faster than Monte Carlo methods, we suggest a combined quasi-Monte Carlo sampling and a homogeneous interior-point method. An application for the synergies analysis of the supply chain after M&A from the literature (Liang and Shi, 2012) is discussed.

ALGORITHM

In this section, we propose an algorithm for solving (1) by combining the homogeneous interior point method and quasi-Monte Carlo method. We first consider the following variation inequality problem. Find $X^* \geq 0$, such that:

$$< X - X^*, \ \sum_{l=1}^{L} p_l F(X^*, \omega_l) >= 0, \ \forall X \in \mathcal{H} \qquad (2)$$

which corresponds to the discrete case of (1) where $\Omega := \{\omega_1, \omega_2, \cdots, \omega_L\}$. In (2), p_l denotes the probability of the random event $\omega_l \in \Omega$. We note that problem (2) is equivalent to the following Standard Complementarily Problem (SCP):

$$\begin{cases} min \quad \langle X, S \rangle \\ s.t. \ \ S = \sum_{l=1}^{L} p_l F(X, \omega_l), \\ \qquad (X, S) \in \mathcal{H} \times R_+. \end{cases} \qquad (3)$$

(SCP) is said to be (asymptotically) feasible if and only if there is a bounded sequence $(X^t, S^t) \in \mathcal{H} \times R_+^n$, $t = 1, 2, \cdots$, such that:

Corresponding Author: Bing Liang, School of Management, Harbin Institute of Technology, Harbin 15001, China

$$lim_{t\to\infty} S^t - \sum_{l=1}^{L} p_l F(X^t, \omega_l) = 0,$$

where any limit point $(\hat{X}, \hat{S})$ of the sequence is called an (asymptotically) feasible point for (SCP). (SCP) has an interior feasible if it has an (asymptotically) feasible point $(\hat{X}, \hat{S}) \in R^n_{++} \times R^n_{++}$. (SCP) is said to be (asymptotically) solvable if there is an (asymptotically) feasible $(\hat{X}, \hat{S})$ such that $\langle \hat{X}, \hat{S} \rangle = 0$, where $(\hat{X}, \hat{S})$ is called the "optimal" or "complementary" solution for (SCP). Note even if (SCP) is feasible, it does not imply that (SCP) has a solution. If (SCP) has a solution, then it has a maximal solution (X^*, S^*) where the number of positive components in (X^*, S^*) is maximal. The indices of those positive components are invariant among all maximal solutions for (SCP).

The homogeneous model to (SCP) is (HSCP):

$$\begin{cases} \min \ \langle X, S\rangle + \tau\kappa \\ S.t. \begin{pmatrix} S \\ \kappa \end{pmatrix} = \begin{pmatrix} \tau \sum_{l=1}^{L} p_l F(X/\tau, \omega_l) \\ -\left\langle X, \ \sum_{l=1}^{L} p_l F(X/\tau, \omega_l)\right\rangle \end{pmatrix} \\ (X, \tau, S, \kappa) \geq 0 \end{cases} \tag{4}$$

Let $\Phi(X, \tau) : R^{n+1}_{+} \rightarrow R^{n+1}$ be defined by:

$$\Phi(X, \tau) = \begin{pmatrix} \tau \sum_{l=1}^{L} p_l F(X/\tau, \omega_l) \\ -\left\langle X, \ \sum_{l=1}^{L} p_l F(X/\tau, \omega_l)\right\rangle \end{pmatrix} \tag{5}$$

Lemma 1: Let Φ be defined by (4), then:

- $(X; \tau)^T \Phi(X, \tau) = 0$
- $\nabla\Phi(X, \tau)(X; \tau) = \Phi(X, \tau)$
- $(X; \tau)^T \nabla\Phi(X, \tau) = -\Phi(X, \tau)^T$

We suppose that $\nabla\Phi$ is positive definite in R^{n+1}_{++}, i.e., given $(X; \tau) > 0, \ \forall R^{n+1} \ni (d_X; d_\tau) \neq 0$, $(d_X; d_\tau)^T \nabla\Phi(X, \tau)(d_X; d_\tau) > 0$.

Theorem 1: Let Φ be defined by (4) and let $(X^*, \tau^*, S^*, \kappa^*)$ be a maximal complementary solution for (HSCP). Then (I) (SCP) has a solution if and only if $\tau^* > 0$. In this case, $(X^*/\tau^*, S^*/\tau^*)$ is a complementary solution for (SCP). (II) (SCP) is (strongly) infeasible if and only if $\kappa^* > 0$. In this case, $(X^*/\kappa^*, S^*/\kappa^*)$ is a certificate to prove (strongly) infeasibility.

Homogeneous interior point algorithm for solving (SCP): Due to Theorem 1,we can solve (SCP) by finding a maximal complementary solution of (HSCP).Now we design the central path of the homogeneous model. Select $X^0 > 0$, $S^0 > 0$, $\tau^0 > 0$ and $\kappa^0 > 0$ and let the residual vectors $r_l^0 = s^0 - \tau^0 F(X^0/\tau^0, \omega_l)$, $z_l^0 = \kappa^0 + \langle X^0, F(X^0/\tau^0, \omega_l)\rangle$, $l = 1, \cdots, L$. Also let $\bar{n} = \langle X^0, S^0 \rangle + \tau^0\kappa^0$. Suggested by Andersen and Ye (1998), we set:

$$X^0 = e \in R^N, \ \tau^0 = 1, \ S^0 = e \in R^N, \ \kappa^0 = 1,$$

Then $diag(X^0)S^0 = e \ \ and \ \ \tau^0\kappa^0 = 1$,where $diag(X^0)$ is the diagonal matrix of X^0. Note that $\bar{n} = n + 1$ in this setting. About the central path of the homogeneous model, we have the following theorem.

Theorem 2: Consider (HSCP).

- For any $0 < \theta \leq 1$, there exists a strictly positive point $(X > 0, \tau > 0, S > 0, \kappa > 0)$ such that:

$$\begin{pmatrix} S \\ \kappa \end{pmatrix} - \Phi(X, \tau) = \begin{pmatrix} S - \tau \sum_{l=1}^{L} F(X/\tau, \omega_l) \\ \kappa + \sum_{l=1}^{L} X^T F(X/\tau, \omega_l) \end{pmatrix} = \theta \begin{pmatrix} \sum_{l=1}^{L} r_l^0 \\ \sum_{l=1}^{L} z_l^0 \end{pmatrix} \tag{6}$$

- Starting from $(X^0 = e \in R^N, \ \tau^0 = 1, \ S^0 = e \in R^N, \ \kappa^0 = 1)$, for any $0 < \theta \leq 1$ there is a unique strictly positive point $(X(\theta), \ \tau(\theta), \ S(\theta), \ \kappa(\theta))$ that satisfies Eq. (5) and:

$$\begin{pmatrix} diag(X)S \\ \tau\kappa \end{pmatrix} = \theta e. \tag{7}$$

- For any $0 < \theta \leq 1$, the solution $(X(\theta), \ \tau(\theta), \ S(\theta), \ \kappa(\theta))$ in (2) is bounded. Thus for any $0 < \theta \leq 1$ and any (S, κ) satisfying (6):

$$\mathcal{C}(\theta) := \left\{ \begin{pmatrix} X \\ \tau \\ S \\ \kappa \end{pmatrix} : \ \begin{pmatrix} diag(X)S \\ \tau\kappa \end{pmatrix} = \theta e, \right\} \tag{8}$$

is a continuous bounded trajectory.

- Any limit point $(X(0), \ \tau(0), \ S(0), \ \kappa(0))$ is a maximal complementary solution for (HSCP).

With the help of a central path, we now present an interior-point algorithm that generates iterates within a neighborhood of $\mathcal{C}(\theta)$. At iteration k with iterate $(X^k, \tau^k > 0, \kappa^k > 0, S^k > 0)$, the algorithm solves a system of linear equations for direction $(d_X, d_\tau, d_S, d_\kappa)$ from:

$$\begin{pmatrix} d_S \\ d_\kappa \end{pmatrix} - \nabla\Phi(X^k, \tau^k) \begin{pmatrix} d_X \\ d_\tau \end{pmatrix} = -\eta \left(\begin{pmatrix} S^k \\ \kappa^k \end{pmatrix} - \Phi(X^k, \tau^k) \right) \tag{9}$$

and,

$$diag(X^k;\tau^k)\begin{pmatrix} d_S \\ d_\kappa \end{pmatrix} + diag(S^k;\kappa^k)\begin{pmatrix} d_X \\ d_\tau \end{pmatrix} = \gamma\mu^k e - diag(X^k;\tau^k)\begin{pmatrix} S^k \\ \kappa^k \end{pmatrix}, \quad (10)$$

where, η and γ are proper given parameters in $(0, 1)$ and:

$$\mu^k = \frac{\langle (X^k;\tau^k),\ (S^k;\kappa^k)\rangle}{n+1}.$$

After the search direction is computed, the variables are updated. A simple update is the linear update $(X^{k+1};\tau^{k+1};S^{k+1};\kappa^{k+1}) = (X^k;\tau^k;S^k;\kappa^k) + \alpha_k(d_X;d_\tau;d_S;d_\kappa)$, where $\alpha_k \geq 0$ is a given step size. Actually, the step-size α_k can be computed by using a simple backtracking line-search. First, α_k^{max} is computed from $\alpha_k^{max} = argmax\{\alpha \in [0,1] : (X^k;\tau^k;S^k;\kappa^k) + \alpha(d_X;d_\tau;d_S;d_\kappa)\}$ Then by chosen some positive integer m we can set $\alpha_k = (\beta)^m \alpha_k^{max}$, where $\beta \in (0,1)$ (About the details, we recommend the reader to Ye (1998)).

Quasi-monte carlo algorithm for solving SVIP: We now develop an efficient numerical method for solving the Stochastic Variational Inequality Problem (SVIP). From the above discussion, (1) is equivalent to the following Standard Stochastic Variation Inequality Problems (SSVIP):

$$\begin{cases} min\ \langle X, S\rangle \\ s.t.\ S = \mathtt{E}\left[F(X,\omega)\right],\quad (X,S) \in \mathcal{H}\times R_+^n. \end{cases} \quad (11)$$

For an integrals function$\psi\ :\ \Omega \rightarrow R$, the Monte Carlosampling estimate for $\mathtt{E}[\phi(\omega)]$ is obtained by taking independentlyandidenticallydistributed random samples$[w_1, \cdots, w_k]$ from Ω and letting $\mathtt{E}[\psi(w)] \approx \frac{1}{k}\sum_{l=1}^{L}\phi(w_l)$. The strong law of large numbers guarantees that this procedure converges with probability one (abbreviated by "w.p.l") i.e:

$$\lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}\psi(w_l) = \mathtt{E}[\phi(w)] := \int_\Omega \psi(w)d\zeta(w) \quad w.p.l, \quad (12)$$

where, $\zeta(w)$ is the distribution function of w Birge and Louveaux (1997).

Thus, by taking independently and identically distributed random samples $\{w_1, \cdots, w_L\}$ from Ω, we obtain the following approximation of problem (10):

$$\begin{cases} min\ \langle X, S\rangle \\ s.t.\ S = \frac{1}{L}\sum_{l=1}^{L}\rho(\omega_l)F(X,\omega_l), \end{cases} \quad (13)$$

where, $(X,S) \in \mathcal{H}\times R_+^n$.

In what follows, let F and X denote the feasible regions of problems (10) and (12), respectively and we suppose *F* is nonempty. It is obvious that $\mathcal{F} \subset \mathcal{X}$. We investigate convergence properties of the Monte Carlo sampling method. From the inerrability of the function of *F* and (10), we get the following result immediately.

Lemma 2: For any fixed $x \in S$, there holds:

$$\mathtt{E}_\omega[F(X,\omega)] = \lim_{k\to\infty}\theta^L(X) := \frac{1}{L}\sum_{j=1}^{L}\rho(\omega_L)F(X,\omega_L).$$

Theorem 3: Suppose that (X^k, S^k) solves the approximate problem (12) for each k and $(\bar{X}, \bar{S})$ is an accumulation point of the sequence $\{X^k, S^k\}$. Then $(\bar{X}, \bar{S})$ is an optimal solution of problem (10) with probability one.

Extensions to Quasi-Monte Carlo approach: We have presented a Monte Carlo sampling and penalty approach for solving problem (10). Actually, the Monte Carlo Sampling methods have been proved useful in the evaluation of integration. However, the convergence of Monte Carlo methods is not fast and various techniques have been proposed to speed up the convergence. In this area, the most approach is the introduction of quasi-Monte Carlo methods, in which the integral is evaluated by using deterministic sequences rather than random sequences. These deterministic sequences have the property that they are well dispersed throughout the domain of integration. Sequences with this property are called low discrepancy sequences. Niederreiter (1992) for more details.

We may readily develop a quasi-Monte Carlo and smoothing approach for solving problem (10). In the case where *F* is affine, we can establish all the results in the above in a similar way and particularly, those convergence results are deterministic by (i).

APPLICATIONS

As an application, we use the new algorithm to discuss the synergies analysis of the supply chain after M&A proposed by Liang and Shi (2012) who proposed a stochastic equilibrium model for the supply chain integration after horizontal M&A. Because the pages are limited, the detailed information about the model and notations are all referred to Liang and Shi (2012). Similar to Nagurney (2009), we utilize a measure to capture the gains, if any, associated with a horizontal merger is as follows:

Table 1: The cost functions

A	M^A_1	f^4_1-2	$f^3_1+2f_1$	$2f^4_1+1$
A	M^A_2	$f^3_2+2f_2$	$f^2_2+2f_2$	$2f^3_2+2f_2$
M^A_1	$D^A_{1,1}$	$f^4_3+2f_3$	$f^3_3+2f_3$	$2f^2_3$
M^A_2	$D^A_{1,2}$	$f^3_4+2f_4$	$f^4_4+2f_2$	$2f^2_4+2f_4$
$D^A_{1,1}$	$D^A_{1,2}$	$f^3_5+2f_5$	$f^3_5+2f_5$	f^2_5+2
$D^A_{1,2}$	R^A_1	$f^4_6+2f_{26}$	$f^3_6+2f_6$	f^2_6+2
$D^A_{1,2}$	R^A_2	$f^3_7+2f_7$	$f^4_7+2f^2_7$	f^4_7+2
B	M^B_1	$f^3_8+2f_8+3$	f^4_8+3	$2f^2_8$
B	M^B_2	$f^3_9+2f_9$	$f^4_9+2f^2_9$	$2f^2_9+2$
M^B_1	$D^B_{1,1}$	$f^4_{10}+2f_{210}$	$f^3_{10}+2f_1$	$2f^2_{10}+3$
M^B_2	$D^B_{1,2}$	$f^3_{11}+2f_{11}$	$f^4_{11}+2f_1$	$2f^3_{11}$
$D^B_{1,1}$	$D^B_{1,2}$	$f^4_{12}+2$	$f^3_{12}+2f_1$	$f^2_{12}+2$
$D^B_{1,2}$	R^B_1	$f^4_{13}+f_{213}+2$	$f^3_{13}+2f_{12}$	$f^2_{13}+2$
$D^B_{1,2}$	R^B_2	$f^3_{14}+2f_{14}$	$f^2_{14}+2f_1$	$f^3_{14}+2f_{14}+2$

$$\Upsilon = [\frac{\mathrm{TC}^0 - \mathrm{TC}^i}{\mathrm{TC}^0}] \times 100\% \tag{14}$$

where, TC^i is the total cost associated with the value of the objective function

$$\sum_{\alpha \in Q^i} \mathrm{E}_{\omega}[\hat{\mathrm{c}}_{\alpha}(\mathrm{f}_{\alpha}*, \omega)]$$

for $i = 0,1$ valuated at the optimal solution for Case i.

We consider a sample example that consists of two firms and each with two manufactures, one distribution center and two retailers. Secondly, a class of stochastic variation inequality problems in which Ω only has finitely many elements. Let $\Omega = \{\omega_1, \cdots, \omega_R\}$ and suppose that $p_r = P\{\omega_j \in \Omega\} = 1/R, \quad j = 1, \cdots, R.$ Finally, we assume that the random variable ω is uniformly distributed on $\Omega = [0,1]$. For the quasi-Monte Carlo sampling method which has been extensively used to industry and economics, we use the classical Halton constructions method in to generate samples. Then, we employed the homogeneous interior point method to solve the sub problem (13). An important feature of the homogeneous algorithm is that it can detect a possible in feasible or unbounded status of the optimization problem. The capacities on all links in all the examples are set to $u_\alpha = 6$ for links α. The demands at the retailers, are $d_{R_1^A} = 3$, $d_{R_2^A} = 3$ and $d_{R_1^B} = 3$, $d_{R_2^B} = 3$.

We now provide additional details concerning the particular examples.

Example 1: The total cost functions for the links are reported in Table 1 with the total costs associated with each of the three merger cases are 310.5432, 271.7346, 271.7346, 210.5536. So the strategic advantage or synergy largely doubled for Case 3 relative to Case 1 and 2 going from 12.5 to 32.2%.

REFERENCES

Birge, J.R. and F. Louveaux, 1997. Introduction to Stochastic Programming. Springer, New York.

Chen, X., C. Zhang and M. Fukushima, 2009. Robust solution of monotone stochastic linear complementarity problems. Math. Programm., 117: 51-80.

Liang, B. and C.S. Shi, 2012. Quantifying operational synergies of M&A: An stochastic equilibrium approach. Appl. Mech. Mater., 174-177: 2830-2835.

Lin, G.H., 2009. Expected residual minimization method for stochastic variational inequality problems. J. Optimiz. Theory Appl., 140: 103-116.

Nagurney, A., 2009. A system-optimization perspective for supply chain network integration: The horizontal merger case. Transport. Res. Part E, 45: 1-15.

Niederreiter, H., 1992. Random Number Generation and Quasi-Monte Carlo Method. SIAM, Philadelphia.

Ye, Y.Y., 1998. A computational study of the homogeneous algorithm for large-scale convex optimization. Comp. Optimiz. Appl., 10: 243-269.